# Nonlinear solutions to Cauchy's functional equation

Federico Volpe

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# Linear solutions

*Functional equations* are equations in which a function appears as an unknown.

Cauchy's functional equation is the equation:
$f(x+y) = f(x) + f(y)$, where $f$ is a function from $\mathbb{R}$ to itself.

It is trivial to verify that every function of the form $f(x) = cx$ with $c \in \mathbb{R}$ is a valid solution to the functional equation. In fact, a stroger result is true.

# Solving over $\mathbb{Q}$

**Theorem**

*Let $f$ be a solution to Cauchy's functional equation. Then $f|_{\mathbb{Q}}$ is a linear function.*

Proof sketch: Let $c = f(1)$ and let $\frac{p}{q}$ be a generic rational number. It's easy to show by induction that $qf(\frac{p}{q}) = f(\frac{p}{q}) + ... + f(\frac{p}{q}) =$

$f(p) = f(1) + ... + f(1) = cp \Rightarrow f(\frac{p}{q}) = c(\frac{p}{q})$, as desired.

# $\mathbb{R}$ as a $\mathbb{Q}$-vector space

These are however not the only solutions to the functional equation if we accept the *axiom of choice*, which implies the existence of a basis for any vector space.

Viewing $\mathbb{R}$ as a $\mathbb{Q}$-vector space, we can thus consider a basis $B$.

With the same reasoning used in the proof of the previous theorem, we can show that for any $b \in B, r \in \mathbb{Q}$, $f(br) = rf(b)$.

# $\mathbb{Q}$-linear functions

The aforementioned observation proves that the functions we are looking for are not only additive, but must be linear functions over $\mathbb{R}$ (viewed as a vector space $\mathbb{Q}$).
In turn, each of these functions will be a solution to Cauchy's equation.

Using basic linear algebra, we can now observe that there's precisely one linear function, and thus one distinct solution to Cauchy's equation, for every choice of $f(B)$ where $B$ is a fixed $\mathbb{Q}$-basis of $\mathbb{R}$.

# Cardinality of the set of nonlinear solutions

Since for any basis element there are $|\mathbb{R}|$ possible choices, the cardinality of the set of solutions is $|\mathbb{R}|^{|B|}$.

We now observe, using cardinal arithmetic, that $|\mathbb{R}| = |\bigcup_{S\in\mathcal{P}_{\text{fin}}(B)} \mathbb{Q}^{|S|}|$[1] $= |\mathcal{P}_{\text{fin}}(B)||\mathbb{Q}| = |B||\mathbb{Q}| = \max(|B|; |\mathbb{Q}|) \Rightarrow |B| = |\mathbb{R}|$

Thus we can conclude that the set of solutions has cardinality $|\mathbb{R}|^{|\mathbb{R}|} = 2^{|\mathbb{R}|} > |\mathbb{R}|$, which is the cardinality of the set of the *linear* ones. So in a sense "most" solutions are nonlinear.

[1] The copies of $\mathbb{Q}^{|S|}$ we are considering the union of are to be understood as always distinct from each other

# Graph of a nonlinear solution

We now prove a remarkable result about the graph of a nonlinear solution.

**Theorem**

*Let $f$ be a nonlinear solution to Cauchy's functional equation and let $G_f = \{(x; f(x)), x \in \mathbb{R}\}$ be its graph. Then $G$ is dense in $\mathbb{R}^2$.*

We begin by proving a lemma.

# Lemma

**Theorem**

*For any real function $f$, if $G_f$ is dense in $\mathbb{R}^2$ so is $G_{f+cx}$*

Proof sketch: For any point $(X, Y)$, we can consider find a point of $G_f$, $(X', f(X'))$ arbitrarily close to $(X, Y - cX)$. Letting $\epsilon$ be this distance and using the triangle inequality multiple times we obtain

$$|(X, Y) - (X', f(X') + cX')| \leq$$
$$|X - X'| + |Y - cX - f(X')| + |cX - cX'| \leq (c + 2)\epsilon.$$

Since we can choose $\epsilon$ to be arbitrarily small and $(X', f(X') + cX') \in G_{f+cx}$, the lemma is proved.

# Main proof

Let $f$ be a nonlinear solution to Cauchy's equation. Then by the lemma we just proved it suffices to prove the graph of $g = f - f(1)x$ is dense in $\mathbb{R}^2$.

Since $f$ is nonlinear there is $z \in \mathbb{R}$ such that $g(z) = w \neq 0$.

We then observe that $g(q + rz) = rw$ for any $r, q \in \mathbb{Q}$.

# Main proof

Now let $(X, Y)$ be a generic point in $\mathbb{R}^2$, then for any $\epsilon$, since $w\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $r$ such that for any $q \in \mathbb{Q} |g(q + rz) - Y| \leq \epsilon$.

Similarly, since $rz + \mathbb{Q}$ is also dense in $\mathbb{R}$, we can choose a $q$ such that $|q + rz - X| \leq \epsilon$.

We thus found a point $(q + rz, g(q + rz))$ which is arbitrarily close to $(X, Y)$, as desired, and the main result is therefore proven.

# Corollaries

This result implies various weaker ones, such that

1. Every nonlinear solution of Cauchy's functional equation is nowhere continuous
2. Every nonlinear solution of Cauchy's functional equation is unbounded on any interval
3. Every nonlinear solution of Cauchy's functional equation is non-monotonic on any interval