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Corso di Laurea Magistrale in Matematica

Tesi di Laurea

A VANISHING RESULT FOR THE ℓ^2-BETTI NUMBERS IN TERMS OF THE INTEGRAL FOLIATED SIMPLICIAL VOLUME

Relatore:
Prof. *Roberto Frigerio*

Controrelatore:
Prof. *Carlo Petronio*

Studente:
Andrea Tamburelli

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Contents

Introduction

The main motivation of this thesis lies in the following conjecture by Gromov, which suggests a connection between the ℓ^2-Betti numbers $b_k^{(2)}(M)$ and the simplicial volume $\|M\|$ of a closed connected oriented aspherical manifold M:

Conjecture. Let M be a closed connected oriented aspherical manifold with $\|M\| = 0$. Then

$$b_k^{(2)}(M) = 0 \qquad \forall\, k \geq 0\ .$$

The ℓ^2-Betti numbers for a Riemannian manifold M endowed with a free cocompact isometric action of a group G were first defined by Atiyah in terms of the heat kernel

$$b_k^{(2)}(M, G) = \lim_{t \to 0} \int_F \mathrm{trace}_{\mathbb{R}}(e^{-t\Delta_k}(x, x)) d\mathrm{Vol}$$

where F is a fundamental domain for the action of G and $e^{-t\Delta_k}(x, y)$ is the heat kernel on k-forms on M. Nowadays there is an algebraic and more general approach due to Lück, who proved that ℓ^2-Betti numbers can be computed as the dimensions of certain homology modules, like ordinary Betti numbers. More precisely, the k-th ℓ^2-Betti number of a CW-complex Z endowed with a free cocompact action of a group G is the Von Neumann dimension of the k-th cellular reduced homology group of Z with coefficients in $\ell^2(G)$. This algebraic definition allows to compare ℓ^2-Betti numbers with their classical counterparts: they share some basic properties such as homotopy invariance, the Euler-Poincaré formula, Poincaré duality and Künneth formula, but there are also differences. One important feature of ℓ^2-Betti numbers is that they are multiplicative under finite coverings in the following sense: if $p : M \to N$ is a d-sheeted covering, then $b_p^{(2)}(M) = d \cdot b_p^{(2)}(N)$. This forces the ℓ^2-Betti numbers of a CW-complex X to vanish if X admits non-trivial self-coverings.

The simplicial volume $\|M\|$ is a real valued homotopy invariant for closed oriented connected topological manifolds M, defined as the infimum of the ℓ^1-norm of real singular cycles representing the fundamental class. In some sense, it measures the complexity of the manifold, as it is always bounded from above by the minimum number of simplices in a triangulation of M. At first sight the definition of ℓ^2-Betti numbers and of the simplicial volume do not indicate a relationship between them, but in certain situations these invariants behave similarly, for example the simplicial volume is multiplicative under finite coverings, as well.

Gromov gave a suggestion of a possible strategy to prove his conjecture relating ℓ^2-Betti numbers and the simplicial volume of aspherical manifolds. The starting point was the upper bound

$$\sum_{j=0}^{n} b_j^{(2)}(M) \leq 2^{n+1}\|M\|_{\mathbb{Z}} \tag{1}$$

where $\|M\|_{\mathbb{Z}}$ denotes the integral simplicial volume, which is defined as the minimum of the ℓ^1 norms of integral cycles representing the fundamental class. Unfortunately, the integral simplicial volume is a rough estimate of its real counterpart, for example $\|M\|_{\mathbb{Z}} \geq 1$ for every manifold, whereas $\|M\|$ can vanish. Thus, Gromov introduced a more sophisticated version, called integral foliated simplicial volume, combining the rigidity of the integral coefficients with the flexibility of measure spaces. More precisely, if G is the fundamental group of a closed, connected and oriented manifold M with universal covering $\tilde{M}$ and X is a standard probabiliy space endowed with a measure-preserving action of G, we can consider the singular chain complex with twisted coefficients $L^\infty(X, \mathbb{Z}) \otimes_G C_*(\tilde{M})$. This can be endowed with an ℓ^1-norm by taking the integral of the absolute values of the coefficient functions. The X-parameterized simplicial volume is the infimum of the ℓ^1 norm of cycles representing the fundamental class and the integral foliated simplicial volume $| M |$ is the infimum of the X-parameterized simplicial volume over all possible standard probability spaces X. This version of simplicial volume fits into the sandwich

$$\|M\| \leq | M | \leq \|M\|_{\mathbb{Z}}$$

and shares the same property of multiplicativity over finite coverings, like the simplicial volume. Using this tool, Schmidt ([3]) was able to improve Inequality (1) obtaining that

$$\sum_{j=0}^{n} b_j^{(2)}(M) \leq 2^{n+1} | M | \,, \tag{2}$$

thus giving a positive answer to Gromov conjecture if the simplicial volume is replaced by the integral foliated simplicial volume. Actually, little is known about the relationship between these two quantities: the integral foliated simplicial volume has been calculated explicitly only for simply-connected manifolds, product manifolds which split an S^1-factor, or Seifert 3-manifolds. Recently, C. Löh and C.Pagliantini proved that it coincides with the simplicial volume for hyperbolic 3-manifolds, also proving a suitable version of proportionality principle, which reminds the classical proportionality principle of simplicial volume proved by Gromov and Thurston. Of course, a positive answer to Gromov's conjecture might be obtained by extending this result to every closed connected oriented aspherical manifold.

The thesis is organized as follows: in the first chapter, we will deal with the integral foliated simplicial volume, following a recent paper by C. Löh and C. Pagliantini. By inspecting the behaviour of the parameterized simplicial volume when the probability space changes, we will prove that the integral foliated simplicial volume is multiplicative under finite coverings. In the last section, we will prove the main result of the chapter,

which states that integral foliated simplicial volume and simplicial volume coincide for closed connected oriented hyperbolic 3-manifolds. This result is obtained by constructing a sequence of hyperbolic 3-manifolds, whose ratio between the simplicial volume and the stable integral simplicial volume tends to zero, by means of tools coming from the 3-dimensional topology, in particular Dehn-filling.
In Chapter 2 we will define the ℓ^2-Betti numbers for CW-complexes endowed with a cocompact cellular action of a group G, according to Lück's algebraic approach. This requires a concept of dimension in a non-commutative setting, thus the first section is devoted to the exposition of this theory for Hilbert G-modules. In the last section we will prove the main properties of ℓ^2-Betti numbers and we will calculate them explicitly for 1- and 2-dimensional compact manifolds.
In Chapter 3 we will prove Inequality (2). This will require a more general dimension theory for modules over a Von Neumann algebra, which will be developed in the first section. The main idea introduced by Schmidt is that the ℓ^2-Betti numbers can be computed via singular homology with coefficients in the Von Neumann algebra of the orbit equivalence relation of the fundamental group acting on the universal covering. We will explain these concepts in the last section.

Chapter 1

The integral foliated simplicial volume

Integral foliated simplicial volume is a version of simplicial volume combining the rigidity of integral coefficients with the flexibility of measure spaces. It was introduced by Gromov as an instrument to estimate ℓ^2-Betti numbers, as we will see in the next chapters.
After a brief review of the main properties of the simplicial volume, we are going to define the integral foliated simplicial volume of connected, closed and oriented manifolds, illustrating some basic results. Following a recent article by C.Löh and C. Pagliantini ([10]) we will calculate it for hyperbolic and Seifert 3-manifolds: to this aim we will introduce the notion of stable integral simplicial volume, as well.

1.1 Simplicial volume and stable integral simplicial volume

In this section we recall briefly the main properties of the simplicial volume and give the definition of the stable integral simplicial volume, trying to underline their differences.

Let X be a topological space and R be a normed ring, usually $R = \mathbb{Z}, \mathbb{R}$. We denote with $S_i(X)$ the set of all i-singular simplices of X and with $C_i(X, R)$ the free R-module generated by $S_i(X)$. We indicate with $(C_*(X, R), d_*)$ the singular chain complex of X and with $H_*(X, R)$ its homology.
We endow the R-module $C_i(X, R)$ with an ℓ^1-norm, defining

$$\Big\| \sum_{\sigma \in S_i(X)} a_\sigma \sigma \Big\|_1^R = \sum_{\sigma \in S_i(X)} |a_\sigma| \ .$$

This norm induces a semi-norm on homology in the following way: if $\alpha \in H_i(X, R)$

$$\|\alpha\|_1^R = \inf\{\|c\|_1^R \mid c \in C_i(X, R),\ d_i(c) = 0,\ [c] = \alpha\} \ .$$

Remark 1.1. Actually, $\|\cdot\|_1^{\mathbb{Z}}$ is not a proper semi-norm, as it is not multiplicative.

Let M be a closed, connected and oriented n-manifold. Let $[M]_{\mathbb{Z}}$ be the integral fundamental class of M, i.e. the positive generator of $H_n(M, \mathbb{Z}) \cong \mathbb{Z}$. The real fundamental class $[M]_{\mathbb{R}}$ is the image of the integral fundamental class under the morphism of change of coefficients $\iota : H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{R})$.

Definition 1.2. The simplicial volume of M is

$$\|M\| := \|[M]_{\mathbb{R}}\|_1^{\mathbb{R}} := \|[M]_{\mathbb{R}}\|_1 \ .$$

The integral simplicial volume of M is

$$\|M\|_{\mathbb{Z}} := \|[M]_{\mathbb{Z}}\|_1^{\mathbb{Z}} \ .$$

If M is a connected and oriented n-manifold with non-empty boundary, then $H_n(M, \mathbb{Z})$ vanishes. In order to define the simplicial volume, we have to consider the relative homology $H_n(M, \partial M, \mathbb{Z}) \cong \mathbb{Z}$. We denote with $[M, \partial M]_{\mathbb{Z}}$ its positive generator and with $[M, \partial M]_{\mathbb{R}}$ its image under the morphism of change of coefficients. We can endow the R-module of the relative singular chain $C_n(M, \partial M, R) = C_n(M, R)/C_n(\partial M, R)$ with the quotient norm, which descends in homology inducing a semi-norm, as described above.

Definition 1.3. The simplicial volume of M is

$$\|(M, \partial M)\| := \|[M, \partial M]_{\mathbb{R}}\|_1 \ .$$

Remark 1.4. By the very definition, it is obvious that $\|M\| \leq \|M\|_{\mathbb{Z}}$, but in general we do not have the equality. For instance $\|M\|_{\mathbb{Z}} \geq 1$, whereas $\|M\|$ can vanish, as it will be clear later.

Lemma 1.5. *Let $f : X \to Y$ be a continous map between topological spaces. For every $\alpha \in H_i(X, \mathbb{R})$ the following relation holds*

$$\|f_*(\alpha)\|_1 \leq \|\alpha\|_1 \ .$$

Proof. Let $\sum_{\sigma \in S_i(X)} c_\sigma \sigma \in \alpha$. A representative of $f_*(\alpha)$ is $\sum_{\sigma \in S_i(X)} c_\sigma (f \circ \sigma)$. Therefore,

$$\begin{aligned} \|f_*(\alpha)\|_1^{\mathbb{R}} &= \Big\| \Big[\sum_{\sigma \in S_k(X)} c_\sigma f \circ \sigma \Big] \Big\|_1^{\mathbb{R}} \leq \Big\| \sum_{\sigma \in S_k(X)} c_\sigma f \circ \sigma \Big\|_1^{\mathbb{R}} \\ &\leq \sum_{\sigma \in S_k(X)} |c_\sigma| = \Big\| \sum_{\sigma \in S_k(X)} c_\sigma \sigma \Big\|_1^{\mathbb{R}} \ . \end{aligned}$$

Taking the infimum over all representatives of α we have the thesis. □

Lemma 1.6. *Let $f : M \to N$ be a continuos function between closed, connected and oriented n-manifolds with degree* $\deg(f)$. *Then*

$$\|M\| \geq |\deg(f)| \|N\| \ .$$

Proof. By the previous lemma we have

$$\|M\| = \|[M]\|_1 \geq \|f_*([M])\|_1 = |\deg(f)| \|N\| \ .$$

□

Corollary 1.7. *The simplicial volume is a homotopy invariant.*

Corollary 1.8. *If a manifold M admits self-maps of degree ≥ 2, then $\|M\| = 0$.*

For example, $\|S^n\| = 0$ for every $n \geq 1$.

Remark 1.9. This result does not hold for the integral simplicial volume, as, for example, S^1 admits self-maps of arbitrarily high degree, but $\|S^1\|_{\mathbb{Z}} = 1$.

However, the main difference between the simplicial volume and its integral version is their behaviour with respect to finite coverings.

Proposition 1.10. *Let $\phi : M \to N$ be a d-sheeted covering between closed, connected and oriented n-manifolds. Then*

$$\|M\| = d \cdot \|N\| \ .$$

Proof. Since a d-sheeted covering is a continous map of degree d, we have $\|M\| \geq d \cdot \|N\|$. For the other inequality, let $\sigma : \Delta^n \to N$ be an n-singular simplex of N. Since Δ^n is simply connected, σ can be lifted to $\tilde{\sigma} : \Delta^n \to M$ obtaining an n-singular simplex of M. Since ϕ is a d-sheeted covering, σ has exaclty d liftings, which we denote with $\tilde{\sigma}_j$ for $j \in \{1, \dots, d\}$. Let $c = \sum_{\sigma \in S_i(N)} c_\sigma \sigma$ be a representative of the fundamental class of N. We notice that

$$\tilde{c} = \sum_{\sigma \in S_k(N)} c_\sigma \sum_{j=1}^{d} \tilde{\sigma}_j$$

is a representative of $[M]_{\mathbb{R}}$, because it is a cycle, the map $\phi_* : H_n(M) \to H_n(N)$ is injective and

$$\begin{aligned}
\phi_*([\tilde{c}]) &= \Big[\sum_{\sigma \in S_k(N)} c_\sigma \sum_{j=1}^{d} \phi \circ \tilde{\sigma} \Big] = \Big[\sum_{\sigma \in S_k(N)} c_\sigma \sum_{j=1}^{d} \sigma \Big] \\
&= \Big[\sum_{\sigma \in S_k(N)} d \cdot c_\sigma \sigma \Big] = d \cdot \Big[\sum_{\sigma \in S_k(N)} c_\sigma \sigma \Big] \\
&= d \cdot [c] = d \cdot [N]_{\mathbb{R}} = \phi_*([M]_{\mathbb{R}}) \ .
\end{aligned}$$

As a consequence,

$$\|M\| = \|[M]_{\mathbb{R}}\|_1 \leq \|\tilde{c}\|_1 \leq d \sum_{\sigma \in S_k(N)} |c_\sigma| = d \cdot \|c\|_1 \ ,$$

and taking the infimum over all representatives of the fundamental class of N we obtain $\|M\| \leq d\|N\|$. □

On the other hand, the integral simplicial volume cannot be multiplicative under finite coverings, as this would imply, for example, that $\|S^1\|_{\mathbb{Z}} = 0$. Therefore, we introduce its stable version.

Definition 1.11. The stable integral simplicial volume of a closed, connected and oriented manifold M is

$$\|M\|_{\mathbb{Z}}^{\infty} = \inf\Big\{\frac{\|\tilde{M}\|_{\mathbb{Z}}}{d} \mid d \in \mathbb{N},\ \tilde{M} \to M \text{ d-sheeted covering}\Big\} .$$

Lemma 1.12. *The stable integral simplicial volume is multiplicative under finite coverings and the inequality $\|M\| \leq \|M\|_{\mathbb{Z}}^{\infty}$ still holds.*

Proof. Let $N \to M$ be a d-sheeted covering and $\tilde{N} \to N$ be an n-sheeted covering. Then the composition $\tilde{N} \to M$ is a dn-sheeted covering. Therefore,

$$\|M\|_{\mathbb{Z}}^{\infty} \leq \frac{\|\tilde{N}\|_{\mathbb{Z}}}{dn} = \frac{1}{d}\frac{\|\tilde{N}\|_{\mathbb{Z}}}{n} .$$

Taking the infimun over all possible finite coverings $\tilde{N} \to N$, we get the inequality

$$\|M\|_{\mathbb{Z}}^{\infty} \leq \frac{1}{d}\|N\|_{\mathbb{Z}}^{\infty} .$$

By definition of $\|M\|_{\mathbb{Z}}^{\infty}$, there exists a sequence of m_j-sheeted coverings $\tilde{M}_j \to M$ such that $\frac{\|\tilde{M}_j\|_{\mathbb{Z}}}{m_j} \to \|M\|_{\mathbb{Z}}^{\infty}$. By replacing $\tilde{M}_j$ with $\tilde{M}_j \times_M N$, we obtain a dm_j-sheeted covering over M which factorizes on N and

$$\|M\|_{\mathbb{Z}}^{\infty} \leq \frac{\|\tilde{M}_j \times_M N\|_{\mathbb{Z}}}{dm_j} \leq \frac{\|\tilde{M}_j\|_{\mathbb{Z}}}{m_j} \to \|M\|_{\mathbb{Z}}^{\infty} .$$

This means that, in order to calculate $\|M\|_{\mathbb{Z}}^{\infty}$, we can consider only coverings factorizing over N. As a consequence,

$$\|N\|_{\mathbb{Z}}^{\infty} \leq \frac{\|\tilde{N}\|_{\mathbb{Z}}}{n} = d\frac{\|\tilde{N}\|_{\mathbb{Z}}}{nd}$$

and taking the infimum we obtain the thesis.
By the multiplicativity of the simplicial volume we get

$$\|M\| = \frac{\|N\|}{d} \leq \frac{\|N\|_{\mathbb{Z}}}{d}$$

ad taking the infimum we have the inequality $\|M\| \leq \|M\|_{\mathbb{Z}}^{\infty}$. □

However, these two quantities are not equal in general, as the following theorem proved recently by S. Francaviglia, R.Frigerio and B.Martelli ([6]) implies:

Theorem 1.13. *For every $n \geq 4$ there exists a constant $C_n < 1$ such that for every closed, connected and oriented hyperbolic n-manifold, we have*

$$\|M\| \leq C_n\|M\|_{\mathbb{Z}}^{\infty} .$$

It is well known that a hyperbolic manifold of finite volume M is the interior part of a compact manifold, whose boundary components carry an Euclidean metric. We will denote with $\overline{M}$ such a manifold.
We will make widely use of the following result, known as proportionality principle of the simplicial volume for hyperbolic manifolds.

Theorem 1.14 (Gromov, Thurston). *Let M be a connected, oriented, complete hyperbolic n-manifold of finite volume and v_n be the volume of the ideal and regular geodesic simplex in $\overline{\mathbb{H}^n}$, then*

$$\|\overline{M}\| = \frac{Vol(M)}{v_n} \ .$$

We recall a vanishing result for the simplicial volume due to Gromov:

Definition 1.15. A group G is amenable if there exists an $\mathbb{R}$-linear map $m : \ell^\infty(G, \mathbb{R}) \to \mathbb{R}$ satisfying the following properties:

1. $m(1) = 1$;
2. if $f \geq 0$, then $m(f) \geq 0$
3. for every $g \in G$ and $f \in \ell^\infty(G, \mathbb{R})$ we have $m(gf) = m(f)$, where the G-action on $\ell^\infty(G, \mathbb{R})$ is induced by the left translation of G on G.

Theorem 1.16. *Let M be a closed, connected and oriented manifold with amenable fundamental group. Then $\|M\| = 0$.*

1.1.1 Comparison between simplicial volume and stable integral simplicial volume for hyperbolic 3-manifolds

We have seen that simplicial volume and stable integral simplicial volume are different for closed, connected and oriented hyperbolic n-manifolds when $n \geq 4$. A similar result does not hold in dimension 3, as C. Löh and C. Pagliantini built (in [10]) a sequence of closed, connected and oriented hyperbolic 3-manifolds, whose ratio between the simplicial volume and the stable integral simplicial volume tends to 1. In this section we will describe how to obtain such a family. This result will also be used further for the calculation of the integral foliated simplicial volume of hyperbolic 3-manifolds.

Definition 1.17. A triangulation of a closed 3-manifold M is the realization of M as the gluing of finitely many tetrahedra via some simplicial pairing of their faces. If $\partial M \neq \emptyset$, an ideal triangulation is a decomposition of $\text{Int}(M)$ into tetrahedra with their vertices removed. A (ideal) triangulation is semi-simplicial if every edge has distinct vertices.

Remark 1.18. When we have a closed, connected and oriented manifold M which has a triangulation consisting of k simplices σ_i, we are tempted to say that the sum of some parametrization of the simplices represents the integral fundamental class of M. Actually, this is not true, as the chain obtained with this procedure may not be a cycle. If we work with real coefficients, we can overcome this problem by considering

$$alt(\sigma) = \frac{1}{(k+1)!} \sum_{\tau \in S^{k+1}} (-1)^{sgn(\tau)} \sigma \circ \bar{\tau}$$

where $\bar{\tau}$ is the unique affine diffeomorphism of the standard n-simplex corresponding to the permutation τ of the vertices. In this way, the chain $z = alt(\sigma_1) + \cdots + alt(\sigma_k)$ is a cycle representing the fundamental class with norm less than k. Otherwise, we need some further information about the combinatorial structure of the triangulation. The notion of semi-simplicial triangulation is sufficient to ensure that the sum of a suitable parametrization of the simplices of the triangulation $\sigma_1 + \cdots + \sigma_k$ is a cycle with integral coefficients representing the integral fundamental class of M. Thus an upper-bound for the integral simplicial volume is the minimun number of tetrahedra in a semi-simplicial triangulation of M.

Definition 1.19. A compact polyhedron P is the support of a finite simplicial complex. It is simple if the link of every point $x \in P$ is homeomorphic to a circle (regular point), a circle with a diameter (triple point) or a circle with three radii (vertex). A simple polyhedron is naturally stratified:

- a 2-dimensional stratum is one connected component of the set of the regular points;
- a 1-dimensional stratum is a triple line;
- a 0-dimensional stratum is a vertex.

A simple polyhedron is special if the stratification is cellular, i.e. each n-dimensional stratum is an n-cell. A special spine for a compact 3-manifold with boundary M is a special polyhedron such that there exists a collapse of M onto P, or, equivalently, $M \setminus P$ consists of a disjoint union of open balls and collars of the boundary components; if M is closed, a special spine for M is, by definition, a special spine for $M \setminus B$, where B is an open ball.

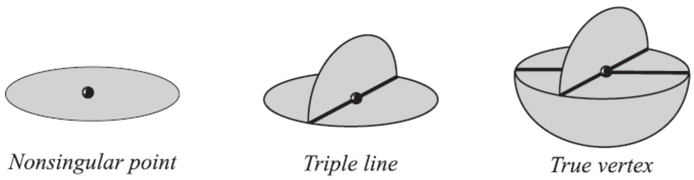

Figure 1.1: Allowable neighborhoods in a simple polyhedron

Definition 1.20. Let M be a compact 3-manifold. The special complexity $c_S(M)$ of M is the minimum number of vertices of a special spine for M.

For our purposes we need to recall the following result proved by Matveev ([14]):

Proposition 1.21. *Let M be a compact 3-manifold. Then there exists a bijection between special spines and ideal triangulations of M such that the number of vertices in the special spine is equal to the number of tetrahedra in the corresponding triangulation.*

Proof. Each tetrahedron contains a butterfly, which is a special spine with exactly one vertex (Figure 1.2). Gluing the tetrahedra of the triangulation, we get a special spine for

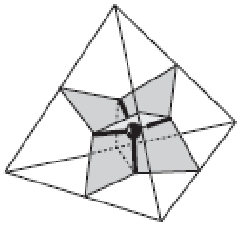

Figure 1.2: Butterfly in a tetrahedron

M. Viceversa, every special spine of M can be decomposed into a collection of butterflies: if we replace each butterfly with a tetrahedron, we get a triangulation of M. □

Proposition 1.22. *Let $\overline{M}_{(5)}$ be the compactification of the complement of the 5-chain link in S^3. Then $c_S(\overline{M}_{(5)}) = \|\overline{M}_{(5)}\| = 10$. Moreover, $c_S(\overline{M}_{(5)})$ is realised by a special spine dual to an ideal semi-simplicial triangulation.*

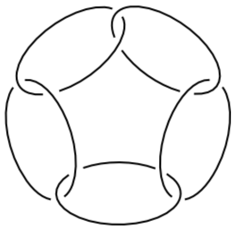

Figure 1.3: Diagram of the 5-chain link

Proof. It can be proved that $M_{(5)}$ admits a complete finite volume hyperbolic structure, triangulated by 10 regular ideal tetrahedra whose vertices lie in different cusps. This means that $\overline{M}_{(5)}$ has an ideal semi-simplicial triangulation consisting of 10 tetrahedra. Thus, $c_S(M_{(5)}) \leq 10$. By theorem 1.14 we have that $\|\overline{M}_{(5)}\| = \frac{Vol(\overline{M}_{(5)})}{v_3} = 10$. The equality follows because for every oriented connected finite volume hyperbolic 3-manifold M with compactification $\overline{M}$ the inequality $\|\overline{M}\| \leq c_S(\overline{M})$ holds, as the volume of M can be computed by straightening any ideal triangulation of M and summing the volume of the straight version of the tetrahedra ([5]). □

Proposition 1.23. *Let N be the compactification of a connected, oriented and hyperbolic 3-manifold of finite volume. Suppose N admits a semi-simplicial triangulation dual to a* *spine realizing $c_S(N)$. Let M be a manifold obtained from N via Dehn-filling. Then*

$$\|M\|_{\mathbb{Z}}^{\infty} \leq c_S(N)$$

Proof. Let P be a special spine of N realizing $c_S(N)$ dual to a semi-simplicial triangulation. Let $T_1, \ldots, T_k$ be the boundary tori of N. For every $i = 1, \ldots, k$ let V_i be the open solid torus in $M \setminus P$ corresponding to the component T_i. Let D_i^1 and D_i^2 be two parallel meridian discs in V_i. If D_i^1 and D_i^2 are in general position with respect to the spine P, then $Q = P \bigcup_{i=1}^{k} D_i^1 \cup D_i^2$ is a special spine for M, dual to a semi-simplicial triangulation (because we have added two meridian discs). If we colour in green the components of Q coming from the discs D_i, we can distinguish three kinds of vertices in Q, illustrated in the figure: the vertices of type A are the vertices of P, the vertices of type B originate

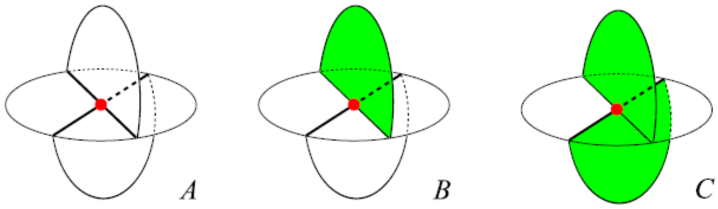

Figure 1.4: The green inserted portions of Q produce three different kinds of vertices

from the intersection of P with one meridian disc and the vertices of type C come from the intersections of P with meridian discs of different solid tori. We indicate with v_A, v_B and v_C the number of vertices of each type. In particular $v_A = c_S(N)$.
Since the fundamental group of M is residually finite, $\forall\ n > 0\ \exists n_0 > n$ and $\exists h > 0$ such that there exists a regular covering $p : \overline{M} \to M$ of degree hn_0 such that for every $i = 1, \ldots k$ the fiber $p^{-1}(V_i)$ consists of h solid tori $\overline{V}_i^1, \ldots \overline{V}_i^h$ each winding n_0 times along V_i via p. Then $\overline{Q} = p^{-1}(Q)$ is a special spine of $\overline{M}$ with $hn_0v_A + hn_0v_B + hn_0v_C$ vertices. Each meridian disc D_i lifts to n_0 copies of meridian discs in each $\overline{V}_i^j$. Therefore, we can decrease the number of vertices of this special spine by removing $n_0 - 2$ discs from each $\overline{V}_i^j$: we obtain another special spine $\overline{Q}'$ for $\overline{M}$ dual to a semi-simplicial triangulation with $hn_0v_A + hv_B + hv_C$ vertices. As a consequence,

$$\|M\|_{\mathbb{Z}}^{\infty} \leq \frac{\|\overline{M}\|_{\mathbb{Z}}}{hn_0} \leq v_A + \frac{v_B + v_C}{n_0} \xrightarrow{n \to \infty} v_A = c_S(N)\ .$$

□

Theorem 1.24. *There exists a sequence $(M_n)_{n \in \mathbb{N}}$ of closed, connected and oriented hyperbolic 3-manifolds such that*

$$\lim_{n \to \infty} \frac{\|M_n\|_{\mathbb{Z}}^{\infty}}{\|M_n\|} = 1\ .$$

Proof. Let $(M_n)_{n\in\mathbb{N}}$ be a family of hyperbolic 3-manifolds, obtained from $\overline{M}_{(5)}$ via Dehn-filling, in such a way that the coefficients of the Dehn-fillings tend to infinity. By Thurston's Dehn filling theorem ([3],[19]) $\lim_{n\to\infty} \mathrm{Vol}(M_n) = \mathrm{Vol}(M_{(5)})$. By Theorem 1.14 we obtain $\lim_{n\to\infty} \|M_n\| = \|\overline{M}_{(5)}\|$. Therefore,

$$1 \leq \frac{\|M_n\|_{\mathbb{Z}}^{\infty}}{\|M_n\|} \leq \frac{c_S(\overline{M}_{(5)})}{\|M_n\|} \xrightarrow{n\to\infty} \frac{c_S(\overline{M}_{(5)})}{\|\overline{M}_{(5)}\|} = 1 \ . \qquad \square$$

1.2 Integral foliated simplicial volume: first properties

In this section we define the integral foliated simplicial volume of a closed, connected and oriented n-manifold, illustrating its basic properties and providing some easy examples.

Let M be a closed, connected and oriented n-manifold with fundamental group Γ and universal covering $\tilde{M}$. Notice that Γ is countable because it is finitely presented. A standard Γ-space is a Borel probability space (X,μ) endowed with a left μ-preserving action of Γ. If (X,μ) is a standard Γ-space, we can define a right action of Γ on $L^\infty(X,\mu,\mathbb{Z})$ by

$$\begin{aligned} L^\infty(X,\mu,\mathbb{Z}) \times \Gamma &\to L^\infty(X,\mu,\mathbb{Z}) \\ (f,g) &\mapsto f\cdot g(x) = f(gx) \ . \end{aligned}$$

We consider the singular homology of M with coefficients in $L^\infty(X,\mu,\mathbb{Z})$, i.e the homology of the chain-complex $C_*(M, L^\infty(X,\mathbb{Z})) := (L^\infty(X,\mathbb{Z})\otimes_\Gamma C_*(\tilde{M},\mathbb{Z}), id_{L^\infty}\otimes d_*)$, where $C_*(\tilde{M},\mathbb{Z})$ is the usual singular chain-complex of $\tilde{M}$, which has a structure of left $\mathbb{Z}[\Gamma]$-module induced by the action of the fundamental group of M on its univeral covering, and d_* is its usual boundary operator.
We indicate with ι_M^X the change of cofficients omomorphism

$$\iota_M^X : C_*(M,\mathbb{Z}) \cong \mathbb{Z}\otimes_\Gamma C_*(\tilde{M},\mathbb{Z}) \to L^\infty(X,\mu,\mathbb{Z}) \otimes_\Gamma C_*(\tilde{M},\mathbb{Z})$$

induced by the inclusion $\mathbb{Z} \hookrightarrow L^\infty(X,\mu,\mathbb{Z})$ as constant functions, where $\mathbb{Z}$ is endowed with the trivial action of Γ.
We can define an ℓ^1-norm on the groups $C_i(M, L^\infty(X,\mathbb{Z}))$: given a chain $\sum_{j=1}^k f_j \otimes \sigma_j \in C_i(M, L^\infty(X,\mathbb{Z}))$ written in its canonical form, i.e. $\pi\circ\sigma_i \neq \pi\circ\sigma_j$ if $i\neq j$ where $\pi : \tilde{M} \to M$ is the universal covering, we put

$$\Big| \sum_{j=1}^k f_j\otimes\sigma_j \Big|^X = \sum_{j=1}^k \int_X |f_j| d\mu \ .$$

Definition 1.25. Let (X,μ) be a standard Γ-space. We define the X-parametrised fundamental class of M as the image of the integral fundamental class under the change of coefficients map

$$[M]^X := H_n(\iota_M^X)([M]_\mathbb{Z}) \in H_n(M, L^\infty(X,\mathbb{Z})) = H_n(L^\infty(X,\mathbb{Z})\otimes_\Gamma C_*(\tilde{M},\mathbb{Z})) \ .$$

An X-parametrised fundamental cycle is any cicle $c \in C_n(M, L^\infty(X,\mathbb{Z}))$ representing $[M]^X$.

Definition 1.26. The X-parametrised simplicial volume is

$$\left| M \right|^X = \inf\{ \left| c \right|^X \mid c \in C_n(M, L^\infty(X,\mathbb{Z})),\ [c]^X = [M]^X \}\ .$$

The integral foliated simplicial volume of M is

$$\left| M \right| = \inf_X \left| M \right|^X\ ,$$

where X ranges over the standard Γ-spaces.

Proposition 1.27. *Let M be a closed, connected and oriented n-manifold with fundamental group Γ and let (X,μ) be a standard Γ-space. Then*

$$\|M\| \leq \left| M \right|^X \leq \|M\|_{\mathbb{Z}}\ .$$

Proof. The inclusion $\mathbb{Z} \hookrightarrow L^\infty(X,\mathbb{Z})$ is isometric and the induced map sends integral fundamental cycles into X-parametrised fundamental cycles. Therefore, $\left| M \right|^X \leq \|M\|_{\mathbb{Z}}$. Consider the integration map

$$\begin{aligned} p_1 : L^\infty(X,\mathbb{Z}) \otimes_\Gamma C_n(\tilde{M},\mathbb{Z}) &\to C_n(M,\mathbb{R}) \\ f \otimes \sigma &\mapsto \Big(\int_X f d\mu \Big)(\pi \circ \sigma) \quad : \end{aligned}$$

p_1 does not increase the norm and fits into the commutative diagram

$$\begin{array}{ccc} C_n(M,\mathbb{Z}) & \xrightarrow{\ \iota_M^X\ } & L^\infty(X,\mathbb{Z}) \otimes_\Gamma C_n(\tilde{M},\mathbb{Z}) \\ & {\scriptstyle \iota} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & C_n(M,\mathbb{R}) & \end{array}$$

In particular p_1 sends X-parametrised fundamental cycles into real fundamental cycles. Therefore, $\|M\| \leq \left| M \right|^X$. ☐

Remark 1.28. If X is the standard Γ-space consisting of a single point, then $L^\infty(X,\mathbb{Z}) \otimes_\Gamma C_n(\tilde{M},\mathbb{Z}) \cong \mathbb{Z} \otimes_\Gamma C_n(\tilde{M},\mathbb{Z}) \cong C_n(M,\mathbb{Z})$. Therefore, $\left| M \right|^X = \|M\|_{\mathbb{Z}}$.

Proposition 1.29 (Integral foliated simplicial volume for simply connected manifolds)**.** *Let M be a closed, connected and simply connected manifold. Then*

$$\left| M \right| = \|M\|_{\mathbb{Z}}\ .$$

Proof. Let (X,μ) be a standard probability space. Since $\Gamma = \pi_1(M) = 0$, X is a standard Γ-space. Moreover, we have the isomorphism $L^\infty(X,\mathbb{Z}) \otimes_\Gamma C_n(\tilde{M},\mathbb{Z}) \cong L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M,\mathbb{Z})$. Let $c = \sum_{i=1}^k f_i \otimes \sigma_i \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M,\mathbb{Z})$ an X-parametrised fundamental cycle and $c_{\mathbb{Z}}$ an integral fundamental cycle. By definition, there exists a singular chain $s \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_{n+1}(M,\mathbb{Z})$ such that $c - c_{\mathbb{Z}} = d_{n+1}(s)$. For μ-almost every x, c induces an integral singular chain $c_x = \sum_{i=1}^{k} f_i(x) \otimes \sigma_i$. Since the relation $c_x - c_{\mathbb{Z}} = d_{n+1}(s)$ still holds, c_x is an integral fundamental cycle. Therefore,

$$\lvert c \rvert = \Big\lvert \sum_{i=1}^{k} f_i \otimes \sigma_i \Big\rvert = \sum_{i=1}^{k} \int_X |f_i(x)| d\mu = \int_X \sum_{i=1}^{k} |f_i(x)| d\mu$$
$$\int_X \|c_x\| d\mu \geq \int_X \|M\|_{\mathbb{Z}} d\mu = \|M\|_{\mathbb{Z}} \ .$$

Taking the infimum over all possible representatives of $[M]^X$ we have $\lvert M \rvert^X \geq \|M\|_{\mathbb{Z}}$. Taking the infimum over all standard Γ-spaces X we obtain the inequality $\lvert M \rvert \geq \|M\|_{\mathbb{Z}}$, which leads to the thesis together with Proposition 1.27. □

We are going to analyse what happens when the parameter space (X, μ) changes.

Proposition 1.30 (Comparison between parameter spaces). *Let M be a closed, connected and oriented n-manifold with fundamental group Γ. Let (X, μ) and (Y, ν) be two standard Γ-spaces. Suppose $\varphi : X \to Y$ is a measurable Γ-equivariant map such that for every measurable set $A \subset Y$*

$$\mu(\varphi^{-1}(A)) \leq \nu(A) \ .$$

Then $\lvert M \rvert^X \leq \lvert M \rvert^Y$.

Proof. Consider the chain morphism

$$\begin{aligned} \phi : L^\infty(Y, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z}) &\to L^\infty(X, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z}) \\ f \otimes \sigma &\mapsto (f \circ \varphi) \otimes \sigma \ . \end{aligned}$$

The hypothesis implies that for every integrable function $f \in L^1(Y, \mathbb{Z})$ the inequality $\int_X (f \circ \varphi) d\mu \leq \int_Y f d\nu$ holds. In particular for every chain $c \in L^\infty(Y, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z})$ we have $\lvert \phi(c) \rvert^X \leq \lvert c \rvert^Y$. Since $\mu(\mathrm{Im}(\varphi)) \geq \mu(X) = 1$, φ is ν-almost surjective. This means that ϕ sends ν-almost constant functions into μ-almost constant functions and the following diagram commutes

$$\begin{array}{ccc} L^\infty(Y, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z}) & \xrightarrow{\ \phi\ } & L^\infty(X, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z}) \\ & {\scriptstyle \iota_{\mathbb{Z}}^Y} \nwarrow \quad \nearrow {\scriptstyle \iota_{\mathbb{Z}}^X} & \\ & C_n(M, \mathbb{Z}) & \end{array}$$

Therefore, ϕ sends Y-parametrised fundamental cycles into X-parametrised fundamental cycles, so $\lvert M \rvert^X \leq \lvert M \rvert^Y$. □

Proposition 1.31 (Product of parameter spaces). *Let M be a closed, connected and oriented n-manifold with fundamental group Γ.*

(a) Let $I \neq \emptyset$ be a countable set. If $(X_i, \mu_i)_{i\in I}$ is a family of standard Γ-spaces, then the product

$$(Z,\zeta) = (\prod_{i\in I} X_i, \bigotimes_{i\in I} \mu_i)$$

with the diagonal action of Γ is a standard Γ-space and $|M|^Z \leq \inf_{i\in I} |M|^{X_i}$.

(b) Let (X,μ) be a standard Γ-space and (Y,ν) a Borel probability space. Defining $Z = X \times Y$ endowed with the action of Γ on the factor X and with the probability measure $\zeta = \mu \otimes \nu$, we have $|M|^Z = |M|^X$.

Proof. (a) It follows by applying Proposition 1.30 to the projections $\pi_i : \prod_{i\in I} X_i \to X_i$.

(b) We can see (Y,ν) as a standard Γ-space with trivial action of Γ. By the previous point, $|M|^Z \leq |M|^X$.
Consider a Z-parametrised fundamental cycle $c = \sum_{j=0}^k f_j \otimes \sigma_j$. By definition, if $c_\mathbb{Z}$ is an integral fundamental cycle of M, there exists a singular chain $s \in L^\infty(Z,\mathbb{Z}) \otimes_\Gamma C_{n+1}(\tilde{M},\mathbb{Z})$ such that

$$c - c_\mathbb{Z} = d_{n+1}(s) \ .$$

For ν-almost every $y \in Y$, the element

$$c_y = \sum_{j=0}^k (x \mapsto f_j(x,y)) \otimes \sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_\Gamma C_n(\tilde{M},\mathbb{Z})$$

is a well-defined X-parametrised fundamental cycle, because the action of Γ on Y is trivial and the relation $c_y - c_\mathbb{Z} = d_{n+1}(s)$ holds.
By Fubini's theorem we have

$$\begin{aligned} |c|^Z &= \int_{X\times Y} \sum_{j=0}^k |f_j| d(\mu\otimes\nu) = \int_Y \int_X \sum_{j=0}^k |f_j(x,y)| d\mu d\nu \\ &= \int_Y |c_y|^X d\nu \ . \end{aligned}$$

As a consequence, there exists $y \in Y$ such that c_y is an X-parametrised fundamental cycle and $|c_y|^X \leq |c|^Z$. Taking the infimum over all possible Z-parametrised fundamental cycles, we obtain the inequality $|M|^X \leq |M|^Z$.

□

Definition 1.32. Let (X,μ) be a standard Γ-space. The action of Γ on X is essentially free if μ-almost every point of X has trivial stabilizer.

Remark 1.33. If Γ is a countable group, it is always possible to find a standard Γ-space with essentially free action. We will illustrate how to construct such a space in Section 1.4.

Corollary 1.34. *Let M be a closed, connected and oriented n-manifold with fundamental group Γ. There exists a standard Γ-space (X,μ) with essentially free action of Γ such that $|M| = |M|^X$.*

Proof. Let (X_0,μ_0) be a standard Γ-space with essentially free action of Γ. Let $(X_n,\mu_n)_{n\in\mathbb{N}}$ be a family of standard Γ-spaces such that $|M|^{X_n} \leq |M| + \frac{1}{n}$. Then the diagonal action on $(X,\mu) = (\prod_{n\in\mathbb{N}} X_n, \bigotimes_{n\in\mathbb{N}} \mu_n)$ is essentially free and by the previous proposition $|M| \leq |M|^X \leq \inf_{n\in\mathbb{N}} |M|^{X_n} = |M|$. □

Proposition 1.35 (Convex combination of parameter spaces)**.** *Let M be a closed, connected and oriented n-manifold with fundamental group Γ. Let (X,μ) and (Y,ν) be two standard Γ-spaces. Let $t \in [0,1]$. Define $Z = X \cup Y$ with the probability measure $\zeta = t\mu + (1-t)\nu$ and the induced action of Γ. Then*

$$|M|^Z = t\,|M|^X + (1-t)\,|M|^Y .$$

Proof. We notice that $L^\infty(Z,\zeta,\mathbb{Z}) \cong L^\infty(X,\mu,\mathbb{Z}) \oplus L^\infty(Y,\nu,\mathbb{Z})$ via the maps

$$\begin{aligned} \phi : L^\infty(Z,\zeta,\mathbb{Z}) &\to L^\infty(X,\mu,\mathbb{Z}) \oplus L^\infty(Y,\nu,\mathbb{Z}) \\ f &\mapsto (f_{|X}, f_{|Y}) \quad ; \end{aligned}$$

$$\begin{aligned} \psi : L^\infty(X,\mu,\mathbb{Z}) \oplus L^\infty(Y,\nu,\mathbb{Z}) &\to L^\infty(Z,\zeta,\mathbb{Z}) \\ (f,g) &\mapsto f\chi_X + g\chi_Y \quad . \end{aligned}$$

Denote with Φ and Ψ the morphism induced between chain complexes. If $c = \sum_{j=1}^k f_j \otimes \sigma_j \in L^\infty(Z,\mathbb{Z}) \otimes_\Gamma C_n(\tilde{M},\mathbb{Z})$ is a Z-parametrised fundamental cycle, then both components of $\Phi(c) = (\Phi(c)_X, \Phi(c)_Y)$ are parametrised fundamental cycles. Therefore,

$$\begin{aligned} |c|^Z &= \sum_{j=1}^k \int_Z |f_j| d\zeta = t\sum_{j=1}^k \int_X |f_{j|X}| d\mu + (1-t)\sum_{j=1}^k \int_Y |f_{j|Y}| d\nu \\ &= t\,|\Phi(c)_X|^X + (1-t)\,|\Phi(c)_Y|^Y \geq t\,|M|^X + (1-t)\,|M|^Y . \end{aligned}$$

Taking the infimum over all Z-parametrised fundamental cycles we obtain

$$|M|^Z \geq t\,|M|^X + (1-t)\,|M|^Y .$$

Similarly, if $c_X = \sum_{j=1}^k f_j \otimes \sigma_j$ is an X-parametrised fundamental cycle and $c_Y = \sum_{j=1}^k g_j \otimes \sigma_j$ is a Y-parametrised fundamental cycle, then $\Psi(c_X, c_Y)$ is a Z-parametrised fundamental cycle, and

$$\begin{aligned} |M|^Z \leq |\Psi(c)|^Z &= \sum_{j=1}^k \int_Z |f_j\chi_X + g_j\chi_Y| d\zeta \\ &\leq t\sum_{j=1}^k \int_X |f_j| d\mu + (1-t)\sum_{j=1}^k \int_Y |g_j| d\nu \\ &= t\,|c_X|^X + (1-t)\,|c_Y|^Y . \end{aligned}$$

Taking the infimum over all the X- and Y-parametrised fundamental cycles, we obtain

$$|M|^Z \leq t\,|M|^X + (1-t)\,|M|^Y \ ,$$

which gives the thesis. □

Corollary 1.36. *Let M be a closed, connected and oriented n-manifold with fundamental group Γ. Then*

$$\{\,|M|^X \mid (X,\mu) - \textit{standard } \Gamma\textit{-space}\} = [\,|M|\,, \|M\|_{\mathbb{Z}}] \subset \mathbb{R} \ .$$

Let M be a closed, connected and oriented n-manifold with fundamental group Γ. Let $p : N \to M$ be a d-sheeted covering and $\Lambda < \Gamma$ the fundamental group of N. Then Λ has index d in Γ. Fix a system of representatives of Λ in Γ, i.e. a set $\{g_1, \ldots, g_d\}$ such that

$$\{g_1\Lambda, \ldots, g_d\Lambda\} = \Gamma/\Lambda \ .$$

Definition 1.37. Let (Y,ν) a standard Λ-space. The induction $(\Gamma \times_\Lambda Y, \mu)$ of (Y,ν) from Λ to Γ is the standard Γ-space defined as follows:

- $\Gamma \times_\Lambda Y = (\Gamma \times Y)/\sim$, where $(g,y) \sim (g',y')$ if and only if there exists $h \in \Lambda$ such that $g = g'h$ and $y' = hy$.
- the probability measure on $\Gamma \times_\Lambda Y$ is the one induced by the bijection

  $$\begin{aligned} \varphi : \Gamma \times_\Lambda Y &\to \Gamma/\Lambda \times Y \\ [g_j, y] &\mapsto (g_j\Lambda, y) \end{aligned}$$

  i.e. μ is the pull-back via φ of the probability measure $\frac{1}{d}\delta \otimes \nu$ on $\Gamma/\Lambda \times Y$, where δ is the counting measure.
- the action of Γ on $\Gamma \times_\Lambda Y$ is

  $$\begin{aligned} \Gamma \times (\Gamma \times_\Lambda Y) &\to \Gamma \times_\Lambda Y \\ (g, [g', y]) &\mapsto [gg', y] \ . \end{aligned}$$

Theorem 1.38 (Induction of parameter spaces)**.** *In the setup described above*

$$|M|^{\Gamma\times_\Lambda Y} = \frac{1}{d}\,|N|^Y \ .$$

Proof. Consider the following mutually inverse homomorphisms of $\mathbb{Z}[\Gamma]$-modules

$$\begin{aligned} \phi : L^\infty(\Gamma \times_\Lambda Y, \mathbb{Z}) &\to L^\infty(Y,\mathbb{Z}) \otimes_\Lambda \mathbb{Z}[\Gamma] \\ f &\mapsto \sum_{j=1}^{d} f([g_j, \cdot]) \otimes g_j \\ \psi : L^\infty(Y,\mathbb{Z}) \otimes_\Lambda \mathbb{Z}[\Gamma] &\to L^\infty(\Gamma \times_\Lambda Y, \mathbb{Z}) \\ f \otimes g_j &\mapsto \left([g_k, y] \mapsto \begin{cases} f(y) & j = k \\ 0 & \textit{otherwise} \end{cases} \right) \ . \end{aligned}$$

Since the universal covering $\tilde{N}$ of N coincides with the universal covering $\tilde{M}$ of M with the action restricted to the subgroup $\Lambda < \Gamma$, we have $C_*(\tilde{N}, \mathbb{Z}) \cong \mathbb{Z}[\Gamma] \otimes_\Gamma C_*(\tilde{M}, \mathbb{Z})$. The maps ϕ and ψ induce morphisms of chain complexes

$$\begin{aligned}\Phi : L^\infty(\Gamma \times_\Lambda Y, \mathbb{Z}) \otimes_\Gamma C_*(\tilde{M}, \mathbb{Z}) &\to L^\infty(Y, \mathbb{Z}) \otimes_\Lambda \mathbb{Z}[\Gamma] \otimes_\Gamma C_*(\tilde{M}, \mathbb{Z}) \\ f \otimes c &\mapsto \phi(f) \otimes c \ . \\ \Psi : L^\infty(Y, \mathbb{Z}) \otimes_\Lambda C_*(\tilde{N}, \mathbb{Z}) &\to L^\infty(\Gamma \times_\Lambda Y, \mathbb{Z}) \otimes_\Gamma C_*(\tilde{M}, \mathbb{Z}) \\ f \otimes c &\mapsto \psi(f \otimes 1) \otimes c \ .\end{aligned}$$

If $c_\mathbb{Z} = \sum_{j=1}^k a_j \sigma_j \in C_n(M, \mathbb{Z})$ is an integral fundamental cycle of M, we have that

$$\Phi \circ \iota_M^{\Gamma \times_\Lambda Y}(c_\mathbb{Z}) = \iota_N^Y \Big(\sum_{j=1}^k a_j \sum_{i=1}^d g_i \sigma_j \Big) \ .$$

Therefore, Φ and Ψ send parametrised fundamental cycles into parametrised fundamental cycles.
If $c = \sum_{i=1}^k f_i \otimes \sigma_i$ is a $(\Gamma \times_\Lambda Y)$-parametrised fundamental cycle, then

$$\begin{aligned}| \Phi(c) |^{Y} &= \sum_{i=1}^k \int_Y |\sum_{j=1}^d f_i([g_j, y])| d\nu \leq \sum_{i=1}^k \sum_{j=1}^d \int_Y |f_i([g_j, y])| d\nu \\ &= d \sum_{i=1}^k \int_{\Gamma \times_\Lambda Y} |f_i| d(\delta \otimes \mu) = d \, | c |^{\Gamma \times_\Lambda Y} \ .\end{aligned}$$

Similarly, if $c = \sum_{i=1}^k \sum_{j=1}^d (f_{i,j} \otimes g_j) \otimes \sigma_i$ is a Y-parametrised fundamental cycle, then

$$\begin{aligned}| \Psi(c) |^{\Gamma \times_\Lambda Y} &\leq \frac{1}{d} \int_{\Gamma \times_\Lambda Y} \sum_{i=1}^k \sum_{j=1}^d |\psi(f_{i,j} \otimes g_j)| d(\delta \otimes \nu) \\ &= \frac{1}{d} \int_Y |f_{i,j}(y)| d\nu = \frac{1}{d} | c |^{Y} \ .\end{aligned}$$

Taking the infimum over all parametrised fundamental cycles we have the claim. □

Corollary 1.39. *Let Γ/Λ be the set of the left lateral classes of Λ in Γ endowed with the left translatory action of Γ and with the normalised counting measure. It is a standard Γ-space and*

$$| M |^{\Gamma/\Lambda} = \frac{1}{d} \|N\|_\mathbb{Z} \ .$$

Proof. If Y is the standard Λ-space consisting of only one point, then $\Gamma/\Lambda = \Gamma \times_\Lambda Y$. Therefore, by the previous theorem and Remark 1.28, we have $| M |^{\Gamma/\Lambda} = \frac{1}{d} | N |^{X} = \frac{1}{d}\|N\|_\mathbb{Z}$. □

Definition 1.40. Let Γ be a group and (X, μ) a standard Γ-space. Let $\Lambda < \Gamma$ be a subgroup. The restriction $\mathrm{res}_\Lambda^\Gamma(X, \mu)$ of (X, μ) from Γ to Λ is the standard Λ-space obtained by restricting the action of Γ on X to Λ.

Proposition 1.41 (Restriction of parameter spaces). *In the setup described above,*

$$\frac{1}{d} \lvert N \rvert^{res^{\Gamma}_{\Lambda} X} \leq \lvert M \rvert^{X} .$$

Proof. It is sufficient to prove that $\lvert M \rvert^{\Gamma \times_{\Lambda} \mathrm{res}^{\Gamma}_{\Lambda} X} \leq \lvert M \rvert^{X}$. The map

$$\begin{aligned} \Gamma \times_{\Lambda} \mathrm{res}^{\Gamma}_{\Lambda} X &\to X \\ [g, x] &\mapsto g \cdot x \end{aligned}$$

satisfies the properties of Proposition 1.30, hence the thesis. □

Theorem 1.42 (Multiplicativity of the integral foliated simplicial volume). *Let M be a closed, connected and oriented n-manifold and $p : N \to M$ be a d-sheeted covering. Then*

$$\lvert M \rvert = \frac{1}{d} \lvert N \rvert .$$

Proof. By Proposition 1.41, we have $\frac{1}{d} \lvert N \rvert \leq \lvert M \rvert$. By Proposition 1.38 we obtain that $\lvert M \rvert \leq \frac{1}{d} \lvert N \rvert$. □

Corollary 1.43. *Let M be a closed, connected and oriented n-manifold. Then*

$$\|M\| \leq \lvert M \rvert \leq \|M\|^{\infty}_{\mathbb{Z}} .$$

Proof. By Proposition 1.27 we know that $\|M\| \leq \lvert M \rvert \leq \|M\|_{\mathbb{Z}}$. Let N be a d-sheeted covering of M. Then

$$\lvert M \rvert = \frac{1}{d} \lvert N \rvert \leq \frac{1}{d} \|N\|_{\mathbb{Z}} .$$

Taking the infimum over all finite coverings of M we obtain the inequality $\lvert M \rvert \leq \|M\|^{\infty}_{\mathbb{Z}}$. □

1.3 Some explicit computations

Using the multiplicativity property of the integral foliated simplicial volume and the inequalities which link it to the simplicial volume and the stable integral simplicial volume, it is possible to calculate it explicitly for surfaces, Seifert 3-manifolds and manifolds with finite fundamental group.

Proposition 1.44. *Let S be a closed, connected and orientable surface of genus g. Then*

$$\lvert S \rvert = \begin{cases} 2 & S \cong S^2 \\ \|S\| = \|S\|^{\infty}_{\mathbb{Z}} & \textit{otherwise} \end{cases}$$

Proof. If $S \cong S^2$, then S is simply connected, hence $\mathbf{|}\, S\, \mathbf{|} = \|S\|_{\mathbb{Z}} = \|S\|_{\mathbb{Z}}^{\infty} = 2$.
If S is a torus, then it admits self-coverings of arbitrarily high degree, hence $\|S\| = \mathbf{|}\, S\, \mathbf{|} = \|S\|_{\mathbb{Z}}^{\infty} = 0$.
If $g \geq 2$, a straightening process of the singular simplices ([3]) produces the inequality

$$\|S\| \geq \frac{\mathrm{Vol}(S)}{v_2} = 4g - 4 \ .$$

For the other inequality, notice that it is possible to realise S as a quotient of a polygon with $4g$-sides. Using this, one can construct a triangulation with $4g - 2$ triangles such that the sum of the simplices represents the fundamental class. Hence $\|S\|_{\mathbb{Z}} \leq 4g - 2$. Since $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, for every $d > 0$ there exists a surjective homomorphism from $\pi_1(S)$ to $\mathbb{Z}_d$, which corresponds to a d-sheeted covering $\tilde{S} \to S$. Since the euler characteristic is multiplicative on finite coverings, $\tilde{S}$ has genus $d(g - 1) + 1$. Therefore,

$$\|S\|_{\mathbb{Z}}^{\infty} \leq \frac{\|\tilde{S}\|_{\mathbb{Z}}}{d} \leq \frac{4d(g-1)+2}{d} \xrightarrow{d \to \infty} 4g - 4 \ .$$

As a consequence, we obtain the inequalities $4g - 4 \leq \|S\| \leq \mathbf{|}\, S\, \mathbf{|} \leq \|S\|_{\mathbb{Z}}^{\infty} \leq 4g - 4$. □

Proposition 1.45. *Let M be a closed, connected and oriented n-manifold with finite fundamental group. Let $\tilde{M}$ be its universal covering. Then*

$$\mathbf{|}\, M\, \mathbf{|} = \frac{1}{|\pi_1(M)|} \|\tilde{M}\|_{\mathbb{Z}} = \|M\|_{\mathbb{Z}}^{\infty} \ .$$

By contrast $\|M\| = 0$.

Proof. Since the fundamental group of M is finite, $\tilde{M}$ is a finite $|\pi_1(M)|$-sheeted covering. Therefore

$$\|M\|_{\mathbb{Z}}^{\infty} \geq \mathbf{|}\, M\, \mathbf{|} = \frac{1}{|\pi_1(M)|} \mathbf{|}\, \tilde{M}\, \mathbf{|} = \frac{1}{|\pi_1(M)|} \|\tilde{M}\|_{\mathbb{Z}} \geq \|M\|_{\mathbb{Z}}^{\infty} \ .$$

On the other hand, $\|M\| = 0$ because its fundamental group is amenable (Theorem 1.16). □

Proposition 1.46. *Let M be a closed, connected and oriented Seifert 3-manifold with infinite fundamental group. Then*

$$\|M\| = \mathbf{|}\, M\, \mathbf{|} = \|M\|_{\mathbb{Z}}^{\infty} = 0 \ .$$

Proof. A Seifert 3-manifold admits a finite covering, which is an S^1-bundle over an orientable surface Σ with Euler number $e \geq 0$. We indicate this manifold with (Σ, e). Moreover, it is well-known that (Σ, e) is either irreducible or $S^1 \times S^2$ or $\mathbb{P}^3(\mathbb{R}) \# \mathbb{P}^3(\mathbb{R})$. We analyse each case separately:

- if $(\Sigma, e) = S^1 \times S^2$, then it admits non-trivial finite self-coverings, hence $\|M\|_{\mathbb{Z}}^{\infty} \leq \|S^1 \times S^2\|_{\mathbb{Z}}^{\infty} = 0$;

- if $(\Sigma, e) = \mathbb{P}^3(\mathbb{R})\#\mathbb{P}^3(\mathbb{R})$, then it is double-covered by $S^1 \times S^2$, thus $\|M\|_{\mathbb{Z}}^{\infty} \leq \|\mathbb{P}^3(\mathbb{R})\#\mathbb{P}^3(\mathbb{R})\|_{\mathbb{Z}}^{\infty} = \frac{1}{2}\|S^1 \times S^2\|_{\mathbb{Z}}^{\infty} = 0$;
- if (Σ, e) is irreducible, the estimate $c_S((\Sigma, e)) \leq e + 6\chi_-(\Sigma) + 6$ holds ([13]), where $\chi_-(\Sigma) = \max\{-\chi(\Sigma), 0\}$. Let T be a triangulation of (Σ, e) dual to a special spine realizing its special complexity. It is not necessarily semi-simplicial, but its first baricentric subdivision T' is. Since a baricentric subdivision of a tetrahedron consists of 24 tetrahedra, T' is made of $24c_S((\Sigma, e))$ tetrahedra. Therefore, $\|(\Sigma, e)\|_{\mathbb{Z}} \leq 24(e + 6\chi_-(\Sigma) + 6)$. If $\overline{\Sigma} \to \Sigma$ is a d-sheeted covering between surfaces, it induces a covering of degree d^2 between the S^1-bundles $(\overline{\Sigma}, e) \to (\Sigma, e)$: this can be obtained by composing the d-sheeted coverings $(\overline{\Sigma}, de) \to (\Sigma, e)$ and $(\overline{\Sigma}, e) \to (\overline{\Sigma}, de)$, the latter being the result of unwrapping the fibers. As a consequence,

$$\begin{aligned} \|M\|_{\mathbb{Z}}^{\infty} &\leq \|(\Sigma, e)\|_{\mathbb{Z}}^{\infty} \leq \frac{1}{d^2}\|(\overline{\Sigma}, e)\|_{\mathbb{Z}} \\ &\leq \frac{24(e + 6\chi-(\overline{\Sigma}) + 6)}{d^2} = \frac{24(e + 6d\chi_-(\Sigma) + 6)}{d^2} \xrightarrow{d\to\infty} 0 \end{aligned}$$

Thanks to Proposition 1.43 we have the thesis.

□

1.4 Ergodic parameters

Among all possible standard Γ-spaces, the ergodic actions have better properties, which allow us to obtain further results about the integral foliated simplicial volume. In particular, we will see that the simplicial volume parametrised over an ergodic space approximates with arbitrary precision the integral foliated simplicial volume and that ergodic actions can be used to calculate the integral foliated simplicial volume of S^1.

Definition 1.47. A standard Γ-space (X, μ) is ergodic if every measurable and Γ-invariant set $E \subset X$ satisfies $\mu(E) = 0$ or $\mu(E) = 1$.

Lemma 1.48. *Let (X, μ) be a standard Γ-space. The following conditions are equivalent:*

(a) (X, μ) is ergodic;

(b) every Γ-invariant function $f : X \to \mathbb{R}$ is constant μ-a.e.

Proof. $(a) \Rightarrow (b)$ For each $c \in \mathbb{R}$ consider the set

$$A_c = \{x \in X \mid f(x) \leq c\} .$$

It is Γ-invariant, hence $\mu(A_c) = 1$ or $\mu(A_c) = 0$. Define $c_0 = \inf\{c \in \mathbb{R} \mid \mu(A_c) = 1\}$. We show that $f = c_0$ μ-a.e.. For every $n \in \mathbb{N}$ the sets $A_{c_0+\frac{1}{n}}$ have full measure and decrease to A_{c_0}, hence $\mu(A_{c_0}) = 1$. On the other hand, for every $n \in \mathbb{N}$ the sets $A_{c_0-\frac{1}{n}}$ have measure

zero and converge increasingly to $\{x \in X \mid f(x) < c_0\}$. Thus $\mu(\{x \in X \mid f(x) = c_0\}) = 1$.
$(b) \Rightarrow (a)$ Let E be a Γ-invariant measurable set. Its characteristic function χ_E is Γ-invariant, hence constant a.e.. This implies that E has measure 1 or 0. □

Definition 1.49. A standard Γ-space is mixing if for every measurable sets $A, B \subset X$ and for every divergent sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of elements of Γ (i.e. γ_n leaves every finite subset definitely) we have

$$\mu(A \cap \gamma_n \cdot B) \xrightarrow{n\to\infty} \mu(A)\mu(B) \ .$$

Lemma 1.50. *If Γ is infinite, a mixing Γ-space is ergodic.*

Proof. Let E be a Γ-invariant set and choose any infinite sequence $\{\gamma_n\} \subset \Gamma$. By definition, $\mu(E) = \mu(E \cap \gamma_n \cdot E) \xrightarrow{n\to\infty} \mu(E)^2$. Therefore, $\mu(E) = 1$ or $\mu(E) = 0$. □

It is not obvious from the definition that every group Γ admits a standard ergodic Γ-space. We build explicitly an example when Γ is infinite and countable, known as Bernoulli shift. This action will be essentially free, as well. Consider the standard Borel space $X = \{0,1\}^\Gamma$ with the product probability measure $\mu = \bigotimes_{n\in\mathbb{N}} \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)$ where δ_j is the atomic probability measure concentrated on the point j. For every finite set $F \subset \Gamma$ and $A_f \subset \{0,1\}$, define

$$A_F = \{(x_\gamma)_{\gamma\in\Gamma} \mid x_\gamma \in A_f \ \forall\ f \in F\} :$$

it is well-known that the sets A_F generate the product σ-algebra on X let F vary on all possible finite subsets of Γ. Moreover, Γ acts on X by left transations, i.e.

$$\begin{gathered} \Gamma \times X \to X \\ (\gamma \cdot (x_{\gamma'})_{\gamma'\in\Gamma}) \mapsto (x_{\gamma\gamma'}) \ . \end{gathered}$$

Using the definition of the product probablity measure, it is easy to verify that this action preserves the measure of the sets A_F. Moreover, for every divergent sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ and for every pair of finite subsets $F, F' \subset \Gamma$ we have

$$\mu(A_F \cap \gamma_n \cdot A_{F'}) \xrightarrow{n\to\infty} \mu(A_F)\mu(A_{F'})$$

because $\gamma_n \cdot F'$ and F are definitely disjoint. It follows that (X, μ) is a mixing, whence ergodic, standard Γ-space.

The importance of ergodic actions for the calculus of integral foliated simplicial volume is underlined in the following result:

Lemma 1.51 (ergodic decomposition)**.** *Let Γ be a countable group and let (X, μ) be a standard Γ-space. There exist a probability space (P, ν) and a family $(\mu_p)_{p\in P}$ of ergodic probability measures on X with the following property: for each Borel subset $A \subset X$, the function*

$$\begin{aligned} P &\to [0,1] \\ p &\mapsto \mu_p(A) \end{aligned}$$

is measurable and

$$\mu(A) = \int_P \mu_p(A) d\nu \ .$$

Proposition 1.52 (ergodic parameters)**.** *Let M be a closed, connected and oriented n-manifold with fundamental group Γ. Let (X, μ) be a standard Γ-space and $\epsilon > 0$. There exists an ergodic Γ-invariant probability measure μ' on X such that*

$$| M |^{(X,\mu')} \leq | M |^{(X,\mu)} + \epsilon \ .$$

In particular, $\forall \ \epsilon > 0$ there exists an ergodic Γ-space X such that

$$| M |^{X} \leq | M | + \epsilon \ .$$

Proof. By definition, there exists an (X, μ)-parametrised fundamental cycle $c = \sum_{j=0}^{k} f_j \otimes \sigma_j \in L^\infty(X, \mu, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z})$ such that

$$\sum_{j=0}^{k} \int_X |f_j| d\mu \leq | M |^{(X,\mu)} + \epsilon \ .$$

Set $B(X, \mu, \mathbb{Z}) = \{f : X \to \mathbb{Z} \mid f \text{ is bounded } \mu\text{-almost everywhere}\}$ and $N(X, \mu, \mathbb{Z}) = \{f \in B(X, \mu, \mathbb{Z}) \mid f = 0 \ \mu - \text{almost everywhere}\}$. We have the short exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \to N(X, \mu, \mathbb{Z}) \to B(X, \mu, \mathbb{Z}) \to L^\infty(X, \mu, \mathbb{Z}) \to 0$$

and, since $C_n(\tilde{M}, \mathbb{Z})$ is a free $\mathbb{Z}[\Gamma]$-module, we obtain that

$$L^\infty(X, \mu, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z}) \cong \frac{B(X, \mu, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z})}{N(X, \mu, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z})} \ .$$

As a consequence, we can consider f_j as elements of $B(X, \mu, \mathbb{Z})$ and we can re-write the cycle condition as follows. Let $c_\mathbb{Z}$ be an integral fundamental cycle. By definition, there exist a chain $s \in B(X, \mu, \mathbb{Z}) \otimes_\Gamma C_{n+1}(\tilde{M}, \mathbb{Z})$, a chain $s' \in B(X, \mu, \mathbb{Z}) \otimes_\Gamma C_n(\tilde{M}, \mathbb{Z})$ and a μ-measurable set $A \subset X$, with $\mu(A) = 0$, such that

$$c - c_\mathbb{Z} = d_{n+1}(s) + \chi_A s' \ .$$

The discussion above ensures that, if, after changing the probability measure μ with another probability measure μ', the functions f_j are still bounded μ'-almost everywhere and $\mu'(A) = 0$, then c represents an (X, μ')-parametrised fundamental cycle. We are now looking for a probability measure μ' satisfying these properties such that the action of Γ on (X, μ') is ergodic. By the ergodic decomposition theorem, there exist a probability space (P, ν) and a family of Γ-invariant probability measures $(\mu_p)_{p \in P}$, such that, for every borel set $B \subset X$,

$$\mu(B) = \int_P \mu_p(B) d\nu$$

and for every μ-integrable function f,

$$\int_X f d\mu = \int_P \int_X f d\mu_p d\nu \ .$$

Taking $f = \sum_{j=1}^k |f_j|$ and $B = A$, there exists $p \in P$ such that

- $\mu_p(A) = 0$;
- $\int_X f d\mu_p \leq \int_X f d\mu$.

Consider the chain $c_p = [c] \in \frac{B(X,\mu_p,\mathbb{Z})\otimes_\Gamma C_n(\tilde{M},\mathbb{Z})}{N(X,\mu_p,\mathbb{Z})\otimes_\Gamma C_n(\tilde{M},\mathbb{Z})} \cong L^\infty(X,\mu_p,\mathbb{Z}) \otimes C_n(\tilde{M},\mathbb{Z})$: it is an (X,μ_p)-parametrised fundamental cycle, because the relation $c - c_\mathbb{Z} = d_{n+1}(s) + \chi_A s'$ continues to hold and $\mu_p(A) = 0$. Therefore,

$$\begin{aligned} |M|^{(X,\mu_p)} \leq |c_p|^{(X,\mu_p)} &= \int_X f d\mu_p \leq \int_X f d\mu \\ &= |c|^{(X,\mu)} \leq |M|^{(X,\mu)} + \epsilon \ . \end{aligned}$$

If (X,μ) is a standard Γ-space realizing the integral foliated simplicial volume of M, we obtain the second part of the thesis. □

We conclude this section with the calculation of the simplicial volume of S^1 using ergodic parameter spaces. We first need a general result about ergodic actions:

Theorem 1.53 (Rohlin). *Let X be a standard non-atomic probability space and $T : X \to X$ a measure-preserving automorphism. Given $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a measurable set $B \subset X$ such that*

(1) $B, T(B), \ldots, T^{n-1}(B)$ are pairwise disjoint;

(2) $\mu(\bigcup_{j=0}^{n-1} T^j(B)) \geq 1 - \epsilon$.

Proposition 1.54. *For every ergodic standard $\mathbb{Z}$-space X we have $|S^1|^X = 0$.*

Proof. Let X be an ergodic standard $\mathbb{Z}$-space. Let $n \in \mathbb{N}$ and $\epsilon > 0$. Consider the X-parametrised fundamental cycle $1 \otimes \sigma \in L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} C_1(\mathbb{R},\mathbb{Z})$, where $\sigma : \Delta_1 \to \mathbb{R}$ is defined as $\sigma(t) = t$. For every $j \in \mathbb{Z}$ we indicate with $\sigma_j : \Delta_1 \to \mathbb{R}$ its translation $\sigma_j(t) = t + j$.
By Rohlin's theorem there exists a measurable set $B \subset X$ such that $B, 1\cdot B, \ldots, (n-1)\cdot B$ are pairwise disjoint and $\mu(X \setminus \bigcup_{j=0}^{n-1} j \cdot B) < \epsilon$. Set $A = X \setminus \bigcup_{j=0}^{n-1} j \cdot B$. We can write

$$1 \otimes \sigma = \chi_A \otimes \sigma + \sum_{j=0}^{n-1} \chi_{j\cdot B} \otimes \sigma = \chi_A \otimes \sigma + \sum_{j=0}^{n-1} \chi_B \otimes \sigma_{-j} \ .$$

Notice that $\sum_{j=0}^{n-1} \chi_B \sigma_{-j}$ is homologous to $\chi_B \otimes \bar{\sigma}$, where

$$\begin{aligned} \bar{\sigma} : \Delta_1 &\to \mathbb{R} \\ t &\mapsto -(n-1) + nt \ . \end{aligned}$$

As a consequence, $\chi_A \otimes \sigma + \chi_B \otimes \bar{\sigma}$ is an X-parametrised fundamental cycle and

$$0 \leq \| S^1 \|^X \leq \mu(A) + \mu(B) \leq \epsilon + \frac{1}{n} \xrightarrow{n\to\infty} \epsilon \ .$$

By the arbitrariety of $\epsilon > 0$, we have $\| S^1 \|^X = \| S^1 \| = 0$. □

1.5 Integral foliated simplicial volume of hyperbolic 3-manifolds

In this section we will prove a proportionality principle for the integral foliated simplicial volume of aspherical manifolds. Together with Theorem 1.24, this will imply that integral foliated simplicial volume and simplicial volume of hyperbolic 3-manifolds concide. To this aim we need to introduce the concept of measure equivalence of groups and recall some results about ℓ^1-homology and bounded cohomology of groups.

1.5.1 ℓ^1-homology and bounded cohomology of groups

Homology and cohomology of groups are important tools for the study of a manifold. For instance, the homology of an aspherical manifold is naturally isomorphic to the homology of its fundamental group. In this section, after a brief survey on the classical homology of groups, we will illustrate a slightly more sophisticated version, called ℓ^1-homology, obtained through the introduction of a norm in the classical chain complex. Its dual counterpart is the bounded cohomology introduced by Gromov. The main reference for the missing details and proofs is ([9]).

Let Γ be a discrete group. The bar resolution of Γ with coefficients in $R = \mathbb{R}, \mathbb{Z}$ is the chain complex $C_*(\Gamma)$ defined as follows:

- $C_n(\Gamma) = \{\sum_{\gamma\in\Gamma^{n+1}} a_\gamma \gamma_0[\gamma_1|\cdots|\gamma_n] \,|a_\gamma \in R,\ a_\gamma = 0 \text{ for all but a finite number of } \gamma \in \Gamma^{n+1}\}$ is the free left $R[\Gamma]$-module generated by the n-tuples $[\gamma_1|\ldots|\gamma_n]$ of elements of Γ with the Γ-action given by

$$\begin{aligned} \Gamma \times C_n(\Gamma) &\to C_n(\Gamma) \\ (\gamma', \gamma_0[\gamma_1|\ldots|\gamma_n]) &\mapsto (\gamma' \cdot \gamma_0)[\gamma_1|\cdots|\gamma_n] \ . \end{aligned}$$

- the boundary operator is the G-equivariant homomorphism determined by

$$\partial_n : C_n(\Gamma) \to C_{n-1}(\Gamma)$$
$$\gamma_0[\gamma_1|\cdots|\gamma_n] \mapsto \gamma_0\gamma_1[\gamma_2|\cdots|\gamma_n] + \sum_{j=1}^{n}(-1)^j \gamma_0[\gamma_1|\cdots|\gamma_{j-1}|\gamma_j\gamma_{j+1}|\cdots|\gamma_n]$$
$$+ (-1)^n \gamma_0[\gamma_1|\cdots|\gamma_{n-1}] \ .$$

We can define an ℓ^1-norm on $C_n(\Gamma)$ by setting

$$\Big\| \sum_{\gamma \in \Gamma^{n+1}} a_\gamma \gamma_0[\gamma_1|\cdots|\gamma_n] \Big\|_1 = \sum_{\gamma \in \Gamma^{n+1}} |a_\gamma| \ .$$

Definition 1.55. Let V be a normed $R[\Gamma]$-module.

- The set of the invariants of V is the submodule

$$V^\Gamma := \{v \in V \mid \gamma \cdot v = v \ \forall \gamma \in \Gamma\} \ .$$

- The set of the coinvariants of V is the quotient

$$V_\Gamma := V/\overline{W}$$

  where W is the R-module generated by the elements of the form $(\gamma \cdot v - v)$ for some $v \in V$ and $\gamma \in \Gamma$ and the closure is taken with respect to the topology induced by the norm on V.

Definition 1.56. The homology of Γ with coefficients in R, denoted by $H_*(\Gamma, R)$, is the homology of the chain complex $C_*(\Gamma)_\Gamma$. More in general the homology of Γ with coefficients in a normed right $R[\Gamma]$-module A is the homology of the chain complex $C_*(\Gamma, A) = (A \otimes_\Gamma C_*(\Gamma), id_A \otimes \partial_*)$.

Remark 1.57. We can introduce a norm on $C_*(\Gamma, A)$ by setting

$$\Big\| \sum_{\gamma \in \Gamma^{n+1}} a_\gamma \otimes \gamma_0[\gamma_1|\cdots|\gamma_n] \Big\|_1 = \sum_{\gamma \in \Gamma^{n+1}} \|a_\gamma\|_A \ .$$

It descends to a semi-norm in homology in the following way: if $\alpha \in H_n(\Gamma, A)$

$$\|\alpha\| = \inf\{\|c\| \mid c \in C_n(\Gamma, A) \ \ \partial(c) = 0 \ \ [c] = \alpha\} \ .$$

Let us focus on the case of real coefficients. In this situation $C_n(\Gamma)$ is a normed left $\mathbb{R}[\Gamma]$-module, but it is not necessarily complete. We indicate with $C_n^{\ell^1}(\Gamma)$ the metric completion of $C_n(\Gamma)$. It is easy to verify that the boundary operators of the bar resolution are continuous with respect to the ℓ^1-norm, hence they can be extended uniquely to the completion.

Given V a complete normed right $\mathbb{R}[\Gamma]$-module, we define

$$C_*^{\ell^1}(\Gamma, V) := (V \bar{\otimes}_\Gamma C_*^{\ell^1}(\Gamma), id_V \otimes \partial_*) \ ,$$

the chain complex, which has in each degree the completion of the real vector space $V \otimes_\Gamma C_n^{\ell^1}(\Gamma)$ with respect to the norm

$$\|u\|_1 := \inf\{\sum_{j=1}^{k} \|v_i\|_V \|c_i\|_1 \mid \sum_{j=1}^{k} v_i \otimes c_i \text{ represents } u \in V \otimes_{\mathbb{R}} C_n^{\ell^1}(\Gamma)\} \ .$$

Definition 1.58. Let V be a complete normed $\mathbb{R}[\Gamma]$-module. The ℓ^1-homology of Γ with coefficients in V, denoted by $H_*^{\ell^1}(\Gamma, V)$, is the homology of the chain complex $C_*^{\ell^1}(\Gamma, V)$.

Definition 1.59. Let V be a complete normed $\mathbb{R}[\Gamma]$-module. The bounded cohomology of Γ with coefficients in V, denoted by $H_b^*(\Gamma, V)$, is the cohomology of the cochain complex $C_b^*(\Gamma, V) = B(C_*^{\ell^1}(\Gamma), V)^\Gamma$, where $B(C_*^{\ell^1}(\Gamma), V)$ is the set of all bounded linear functions from $C_*^{\ell^1}(\Gamma)$ to V. The cochain complex $C_b^*(\Gamma, V)$ is endowed with the supremum norm, which induces a semi-norm in cohomology, as usual.

Remark 1.60. If V is the dual space of a complete normed vector space W, then $C_b^*(\Gamma, V) = C_*^{\ell^1}(\Gamma, W)'$.

If V is a complete normed $\mathbb{R}[\Gamma]$-module, both $H_*(\Gamma, V)$ and $H_*^{\ell^1}(\Gamma, V)$ are well-defined. The natural inclusion $\iota : C_*(\Gamma, V) \to C_*^{\ell^1}(\Gamma, V)$ as a dense subcomplex allows us to compare the two homology groups, thanks to this general result:

Proposition 1.61. *Let D be a chain complex endowed with a norm and C be a dense subcomplex. The map induced in homology by the inclusion is isometric.*

Proof. Set $\iota : C \to D$ the inclusion and $H_*(\iota)$ the map induced in homology. Clearly $\|H_*(\iota)\| \leq 1$. We have to prove the other inequality. Let $z \in C_n$ be a cycle and $\bar{z} \in D_n$ a cycle such that

$$[\bar{z}] = H_n(\iota)([z]) \in H_n(D) \ .$$

It is sufficient to prove that for every $\epsilon > 0$ there exists $z' \in C_n$ such that

$$[z'] = [z] \qquad \|z'\| \leq \|\bar{z}\| + \epsilon$$

By definition, there exists a chain $\bar{w} \in D_{n+1}$ such that $\partial_{n+1}(\bar{w}) = \iota(z) - \bar{z}$. Since C_{n+1} is dense in D_{n+1} and the boundary operator is continuous, there exists a chain $w \in C_{n+1}$ such that

$$\|\bar{w} - \iota(w)\| \leq \frac{\epsilon}{\|\partial_{n+1}\|} \ .$$

Set $z' = z - \partial_{n+1}(w)$: it is a cycle homologous to z and

$$\|\bar{z} - \iota(z')\| = \|\partial_{n+1}(\bar{w} - \iota(w))\| \leq \epsilon \ .$$

In particular, we have $\|z'\| \leq \|\bar{z}\| + \epsilon$. □

Since ℓ^1-homology and bounded cohomology are defined from dual complexes, we expect some kind of duality relations between the two theories.
The evaluation $C_b^*(\Gamma, V') \otimes C_*^{\ell^1}(\Gamma, V) \to \mathbb{R}$ descends in (co)-homology, defining a duality pairing

$$\langle \cdot, \cdot \rangle : H_b^*(\Gamma, V') \otimes H_*^{\ell^1}(\Gamma, V) \to \mathbb{R} \ .$$

Definition 1.62. We say that two maps $f : H_*^{\ell^1}(\Gamma, V) \to H_*^{\ell^1}(\Lambda, W)$ and $g : H_b^*(\Lambda, W') \to H_b^*(\Gamma, V')$ are mutually adjoint if for every cohomology class $\phi \in H_b^*(\Lambda, W')$ and for every homology class $\alpha \in H_*^{\ell^1}(\Gamma, V)$ we have

$$\langle \phi, f(\alpha) \rangle = \langle g(\phi), \alpha \rangle \ .$$

Example 1.63. If $f : C_*^{\ell^1}(\Gamma, V) \to C_*^{\ell^1}(\Lambda, W)$ is a bounded morphism of chain complexes and $f' : C_b^*(\Lambda, W') \to C_b^*(\Gamma, V')$ is its dual, then the maps induced in (co)-homology are mutually adjoint.

Proposition 1.64. *([10], Corollary 4.1.) Let $f : C_*^{\ell^1}(\Gamma, V) \to C_*^{\ell^1}(\Lambda, W)$ be a morphism of chain complexes constisting of continuos maps. Let $f' : C_b^*(\Lambda, W') \to C_b^*(\Gamma, V')$ be its dual. Indicate with $H_b^*(f')$ and with $H_*^{\ell^1}(f)$ the homomorphisms induced in (co)-homology.*

1. *$H_b^*(f')$ is an isomorphism of real vector spaces if and only if $H_*^{\ell^1}(f)$ is an isomorphism.*
2. *If $H_b^*(f)$ is an isometric isomorphism, then $H_*^{\ell^1}(f)$ is an isometric isomorphism.*

1.5.2 ME-coupling and proportionality principle

We recall the notion of measure equivalence for groups introduced by Gromov and we use it to prove a proportionality principle for the integral foliated simplicial volume.

Definition 1.65. Two countable groups Γ and Λ are measure equivalent (ME) if there exists a measure space (Ω, m) endowed with commuting and m-preserving actions of Γ and Λ, which admit fundamental domains of finite measure. We say that (Ω, m) is an ME-coupling.

Remark 1.66. Since the actions commute, each group acts on the orbits of the other and consequently on its fundamental domain.

Example 1.67. Consider two lattices $\Gamma, \Lambda \subset \mathrm{Isom}(\mathbb{H}^n) = G$. It is well-known that G is unimodular ([3]), i.e. its Haar measure m_G is both left- and right-invariant. Therefore (G, m_G) endowed with the left translatory action of Γ and with the right translatory action of Λ is an ME-coupling.

Let (Ω, m) be an ME-coupling for two countable groups Γ and Λ. Let X_Λ and X_Γ be two finite-measure fundamental domains for the actions of Λ and Γ respectively. It is possible to associate a measure cocycle to the ME-coupling with the following procedure:

$$\alpha_\Lambda : \Gamma \times X_\Lambda \to \Lambda$$
$$(\gamma, x) \mapsto \alpha_\Lambda(\gamma, x)$$

where $\alpha_\Lambda(\gamma, x)$ is the unique element in Λ such that $\gamma x \in \alpha_\Lambda(\gamma, x)^{-1} X_\Lambda$. The definition for the function α_Γ is similar.
It is quite easy to verify from the definition that the above functions satisfy the following properties:

- for every $\gamma_1, \gamma_2 \in \Gamma$ and for every $x \in X_\Lambda$, we have

$$\alpha_\Lambda(\gamma_1\gamma_2, x) = \alpha_\Lambda(\gamma_1, \gamma_2 \cdot x)\alpha_\Lambda(\gamma_2, x)$$

- if X'_Λ is another fundamental domain for the action of Λ, the new measure cocycle α'_Λ is related to α_Λ by the relation

$$\alpha'_\Lambda(\gamma, \theta(x)) = \zeta(\gamma \cdot x)^{-1}\alpha_\Lambda(\gamma, x)\zeta(x)$$

where $\theta : X_\Lambda \to X'_\Lambda$ is the isomorphism induced by the natural bijections $X_\Lambda \to X/\Lambda \to X'_\Lambda$ and $\zeta(x)$ is defined as the unique element of Λ such that $\zeta(x)x \in X'_\Lambda$.

This allows us to describe explicitly the action of each group on the fundamental domain of the other as follows:

$$\begin{aligned} \Gamma \times X_\Lambda &\to X_\Lambda \\ (\gamma, x) &\mapsto \gamma \cdot x = \alpha_\Lambda(\gamma, x)\gamma x \end{aligned} \qquad \begin{aligned} \Lambda \times X_\Gamma &\to X_\Gamma \\ (\lambda, y) &\mapsto \lambda \cdot y = \alpha_\Gamma(\lambda, x)\lambda y \end{aligned}$$

Definition 1.68. Suppose now that Γ and Λ are finitely generated. Let ℓ be some word-norm on the groups. Let (Ω, m) be an ME-coupling and X_Γ, X_Λ two fundamental domains. We say that X_Γ is bounded if for every $\lambda \in \Lambda$ the function

$$\begin{aligned} X_\Gamma &\to \mathbb{R} \\ x &\mapsto \ell(\alpha_\Gamma(\lambda, x)) \end{aligned}$$

is an element of $L^\infty(X_\Gamma, m_\Gamma, \mathbb{R})$. The ME-coupling (Ω, m) is bounded if it is possible to find bounded fundamental domains.

Definition 1.69. An ME-coupling (Ω, m) for Γ and Λ is ergodic (risp. mixing) if the actions of Γ on X_Λ and of Λ on X_Γ are ergodic (risp. mixing).

Definition 1.70. Given an ME-coupling (Ω, m) for the groups Γ and Λ and two finite-measure fundamental domains X_Λ and X_Γ for their actions, the ME-coupling index is the ratio

$$c_\Omega = \frac{m(X_\Lambda)}{m(X_\Gamma)} .$$

Example 1.71. Let $\Gamma, \Lambda \subset \mathrm{Isom}(\mathbb{H}^n)$ be two countable groups acting freely and proper discontinouosly on $\mathbb{H}^n$. We have seen that $\Omega = \mathrm{Isom}(\mathbb{H}^n)$ with its bi-invariant Haar measure is an ME-coupling (Remark 1.67). It can be proved that it is mixing ([2], Theorem III.2.1) and bounded ([18], Corollary 6.12), as well. Let X_Λ and X_Γ be two finite-measure fundamental domains. The ME-coupling index is

$$c_\Omega = \frac{\mathrm{Covol}(\Lambda)}{\mathrm{Covol}(\Gamma)}$$

where $\mathrm{Covol}(\Gamma) = \mathrm{Vol}(\mathbb{H}^n/\Gamma)$.

An ME-coupling enables us to connect the bounded cohomology groups of Γ and Λ with coefficient in a suitable space of functions. Keeping the notations introduced above, we endow the fundamental domains X_Γ and X_Λ with the normalised measures $m_\Gamma = \frac{1}{m(X_\Gamma)} m_{|\Gamma}$ and $m_\Lambda = \frac{1}{m(X_\Lambda)} m_{|\Lambda}$, so that (X_Λ, m_λ) and (X_Γ, m_Γ) become standard probability spaces.

Theorem 1.72. *(Monod-Shalom [15]) Let (Ω, m) be an ME-coupling for two countable groups Λ and Γ. Let X_Γ and X_Λ be two fundamental domains. Let α_Γ be the associated measure cocycle. The map*

$$\begin{aligned} &\alpha_\Gamma^* : C_b^*(\Gamma, L^\infty(X_\Lambda, \mathbb{R})) \to C_b^*(\Lambda, L^\infty(X_\Gamma, \mathbb{R})) \\ &(\alpha_\Gamma^k f)(\lambda_0, \lambda_1, \dots, \lambda_k)(x) = f(\alpha_\Gamma(\lambda_0^{-1}, x), \dots, \alpha_\Gamma(\lambda_k^{-1}, x))(\Lambda x \cap X_\Lambda) \end{aligned}$$

restricts to the invariants and induces an isometric isomorphism $H_*^b(\alpha_\Gamma)$ in cohomology.

Due to Theorem 1.64 and Remark 1.60, we expect a similar result for the ℓ^1-homology with coefficients in the space of integrable functions. We need first an easy lemma:

Lemma 1.73. *Let (X, μ) and (Y, ν) two standard measure spaces. Let $p : X \to Y$ be a measurable map such that*

1. *the fiber $p^{-1}(y)$ is countable for ν-almost every $y \in Y$;*
2. *for every measurable subset $A \subset X$ such that $p_{|A}$ is injective $\mu(A) = \nu(p(A))$.*

Then for every $f \in L^1(X, \mu)$ the function $y \mapsto \sum_{x \in p^{-1}(y)} f(x)$ is integrable and

$$\int_X f d\mu = \int_Y \sum_{x \in p^{-1}(y)} f(x) d\nu$$

Proof. The assumption is true when f is the characteristic function of a set A satisfying 2). Since every function $f \in L^1(X, \mu)$ can be approximated by finite linear combination of such functions, we are done. □

Remark 1.74. We can identify

$$L^1(X_\Lambda, \mathbb{R}) \otimes_{\mathbb{R}} C_k(\Gamma) \leftrightarrow L^1(\Gamma^{k+1} \times X_\Lambda, \mathbb{R})^{fin}$$

where the caption fin indicates that for every function f belonging to the set there exists a finite subset $F \subset \Gamma^{k+1}$ such that the support of f is contained in $F \times X_\Gamma$.
As a consequence we get the following identification for the coinvariants

$$L^1(X_\Lambda, \mathbb{R}) \otimes_\Gamma C_k(\Gamma) \leftrightarrow L^1(\Gamma^{k+1} \times X_\Lambda, \mathbb{R})_\Gamma^{fin}$$

and for the completion

$$L^1(X_\Lambda, \mathbb{R}) \otimes_\Gamma C_k^{\ell^1}(\Gamma) \leftrightarrow L^1(\Gamma^{k+1} \times X_\Lambda, \mathbb{R})_\Gamma \ .$$

Consider the function

$$\begin{aligned} &\phi_k^\alpha : \Lambda^{k+1} \times X_\Gamma \to \Gamma^{k+1} \times X_\Lambda \\ &(\lambda_0, \dots, \lambda_k, x) \mapsto (\alpha_\Gamma(\lambda_0^{-1}, x), \dots, \alpha_\Gamma(\lambda_k^{-1}, x), \Lambda x \cap X_\Lambda) \quad : \end{aligned}$$

it satisfies the hypothesis of the function p in the previous lemma. Hence we obtain the following result:

Theorem 1.75. *Let (Ω, m) be an ME-coupling for two countable groups Λ and Γ. Let X_Γ and X_Λ two fundamental domains. Let α_Γ be the associated measure cocycle. Using the identification of Remark 1.74, the map*

$$\alpha_k^\Gamma : L^1(X_\Gamma, \mathbb{R}) \otimes_\mathbb{R} C_k^{\ell^1}(\Lambda) \to L^1(X_\Lambda, \mathbb{R}) \otimes_\mathbb{R} C_k^{\ell^1}(\Gamma)$$
$$\alpha_k^\Gamma f(\bar{\gamma}, x) = c_\Omega \sum_{(\bar{\lambda}, y) \in \phi_k^{\alpha -1}(\bar{\gamma}, x)} f(\bar{\lambda}, y)$$

is well-defined on the level of coinvariants and the induced map in ℓ^1-homology

$$H_*^{\ell^1}(\alpha_\Gamma) : H_*^{\ell^1}(\Lambda, L^1(X_\Gamma, \mathbb{R})) \to H_*^{\ell^1}(\Gamma, L^1(X_\Lambda, \mathbb{R}))$$

is an isometric isomorphism. Moreover, $H_*^{\ell^1}(\alpha_\Gamma)$ is adjoint to $H_b^*(\alpha_\Gamma)$.

Proof. We first verify that α_k^Γ and α_Γ^k are mutually adjoint on the level of (co)-chains, i.e for every $f \in L^1(X_\Gamma, \mathbb{R}) \otimes_\mathbb{R} C_k^{\ell^1}(\Lambda)$ and for every $g \in C_b^k(\Gamma, L^\infty(X_\Lambda, \mathbb{R}))$ we have

$$\langle \alpha_k^\Gamma(f), g \rangle = \langle f, \alpha_\Gamma^k(g) \rangle \ .$$

This follows from the previous lemma:

$$\begin{aligned}
\langle \alpha_k^\Gamma(f), g \rangle &= \frac{m(X_\Lambda)}{m(X_\Gamma)} \sum_{\gamma \in \Gamma^{n+1}} \int_{X_\Lambda} \sum_{(\bar{\lambda}, y) \in \phi_k^{\alpha -1}(\bar{\gamma}, x)} f(\bar{\lambda}, y) g(\bar{\gamma}, x) \frac{dm}{m(X_\Lambda)} \\
&= \frac{1}{m(X_\Gamma)} \sum_{\gamma \in \Gamma^{n+1}} \int_{X_\Lambda} \sum_{(\bar{\lambda}, y) \in \phi_k^{\alpha -1}(\bar{\gamma}, x)} f(\bar{\lambda}, y)(g \circ \phi_k^\alpha)(\bar{\lambda}, y) dm \\
&= \frac{1}{m(X_\Gamma)} \sum_{\bar{\lambda} \in \Lambda^{k+1}} \int_{X_\Gamma} f(\bar{\lambda}, y)(g \circ \phi_k^\alpha)(\bar{\lambda}, y) dm = \langle f, \alpha_\Gamma^k(g) \rangle
\end{aligned}$$

Therefore, α_k^Γ restricts to the coinvariants and commutes with the boundary operators, because α_Γ^k does. The thesis follows from Theorem 1.64. □

We are now ready to prove the proportionality principle for the integral foliated simplicial volume.

Theorem 1.76 (Proportionality principle for the integral foliated simplicial volume)**.** *Let M and N be closed, connected and oriented aspherical n-manifolds with positive simplicial volume. Suppose that there exists an ergodic and bounded ME-coupling (Ω, m) between their fundamental groups Γ and Λ. Let c_Ω be the ME-coupling index. We have*

1) $\left| M \right|^{\Lambda/\Omega} = c_\Omega \left| N \right|^{\Gamma/\Omega}$ *;*

2) if the coupling is mixing, then $\left| M \right| = c_\Omega \left| N \right|$.

Proof. 1) Let X_Γ and X_Λ be two fundamental domains for the actions of Γ and Λ respectively. Let α_Γ and α_Λ be the measure cocycles associated to them. Like in Remark 1.74, we can identify

$$L^1(X_\Gamma,\mathbb{Z})\otimes_\Lambda C_k(\Lambda,\mathbb{Z}) \leftrightarrow L^1(\Lambda^{k+1}\times X_\Gamma,\mathbb{Z})^{fin}_\Lambda .$$

Consider the function

$$\begin{aligned}\phi_k^{\alpha_\Gamma} &: \Lambda^{k+1}\times X_\Gamma \to \Gamma^{k+1}\times X_\Lambda\\ &(\lambda_0,\dots,\lambda_n,x)\mapsto (\alpha_\Gamma(\lambda_0^{-1},x),\dots,\alpha_\Gamma(\lambda_n^{-1},x),\Lambda x\cap X_\Lambda)\end{aligned}$$

and define

$$\begin{aligned}(\alpha_\Gamma^{\mathbb{Z}})_k &: L^1(\Lambda^{k+1}\times X_\Gamma,\mathbb{Z})^{fin}_\Lambda \to L^1(\Gamma^{k+1}\times X_\Lambda,\mathbb{Z})^{fin}_\Gamma\\ &((\alpha_\Gamma)^{\mathbb{Z}}_k f)(\bar\gamma,x) = \sum_{(\bar\lambda,y)\in(\phi_n^{\alpha_\Gamma})^{-1}(\bar\gamma,x)} f(\bar\lambda,y)\ .\end{aligned}$$

Similarly we define $(\alpha_\Lambda^{\mathbb{Z}})_k$. It can be verified directly that $(\alpha_\Gamma^{\mathbb{Z}})_*$ is well-defined because of the boundedness of the ME-coupling and that it is a chain morphism with norm at most $\frac{1}{c_\Omega}$. As a consequence the induced map in homology $H_n(\alpha_\Gamma^{\mathbb{Z}}): H_n(\Lambda, L^1(X_\Gamma,\mathbb{Z}))\to H_n(\Gamma, L^1(X_\Lambda,\mathbb{Z}))$ has norm at most $\frac{1}{c_\Omega}$, as well.
Consider the following commutative diagram:

$$\begin{array}{ccccc}
H_n^{\ell^1}(\Lambda,L^1(X_\Gamma,\mathbb{R})) & \xrightarrow{H_n^{\ell^1}(\alpha_\Gamma^{\mathbb{R}})} & H_n^{\ell^1}(\Gamma,L^1(X_\Lambda,\mathbb{R})) & \xrightarrow{H_n^{\ell^1}(\alpha_\Lambda^{\mathbb{R}})} & H_n^{\ell^1}(\Lambda,L^1(X_\Gamma,\mathbb{R}))\\
\uparrow{\scriptstyle \iota_\Lambda} & & \uparrow{\scriptstyle \iota_\Gamma} & & \uparrow{\scriptstyle \iota_\Lambda}\\
H_n(\Lambda,L^1(X_\Gamma,\mathbb{R})) & \xrightarrow{H_n(\alpha_\Gamma^{\mathbb{R}})} & H_n(\Gamma,L^1(X_\Lambda,\mathbb{R})) & \xrightarrow{H_n(\alpha_\Lambda^{\mathbb{R}})} & H_n(\Lambda,L^1(X_\Gamma,\mathbb{R}))\\
\uparrow{\scriptstyle j_\Lambda^{\mathbb{R}}} & & \uparrow{\scriptstyle j_\Gamma^{\mathbb{R}}} & & \uparrow{\scriptstyle \iota_\Gamma}\\
H_n(\Lambda,L^1(X_\Gamma,\mathbb{Z})) & \xrightarrow{H_n(\alpha_\Gamma^{\mathbb{Z}})} & H_n(\Gamma,L^1(X_\Lambda,\mathbb{Z})) & \xrightarrow{H_n(\alpha_\Lambda^{\mathbb{Z}})} & H_n(\Lambda,L^1(X_\Gamma,\mathbb{Z}))\\
\uparrow{\scriptstyle j_\Lambda} & & \uparrow{\scriptstyle j_\Gamma} & & \uparrow{\scriptstyle j_\Lambda}\\
H_n(\Lambda,\mathbb{Z}) & & H_n(\Gamma,\mathbb{Z}) & & H_n(\Lambda,\mathbb{Z})\\
\uparrow{\scriptstyle c_\Lambda} & & \uparrow{\scriptstyle c_\Gamma} & & \uparrow{\scriptstyle c_\Lambda}\\
H_n(N,\mathbb{Z}) & & H_n(M,\mathbb{Z}) & & H_n(N,\mathbb{Z})
\end{array}$$

where the vertical maps are induced by the natural inclusions and c_Λ, c_Γ are the isomorphisms between the homology of an aspherical manifold and the homology of its fundamental group. Notice that all vertical maps are norm non-increasing and $c_\Omega H_n^{\ell^1}(\alpha_\Gamma^{\mathbb{R}})$,

$\frac{1}{c_\Omega} H_n^{\ell^1}(\alpha_\Lambda^{\mathbb{R}})$ are exactly the maps involved in the previous theorem, hence they are isometries.
By ergodicity (and Poincaré duality)

$$H_n(\Lambda, L^1(X_\Gamma, \mathbb{Z})) \cong H^0(\Lambda, L^1(X_\Gamma, \mathbb{Z})) \cong L^1(X_\Gamma, \mathbb{Z})^\Gamma \cong \mathbb{Z} \ ,$$

hence there exists $m \in \mathbb{Z}$ such that

$$H_n(\alpha_\Gamma^{\mathbb{R}}) \circ j_\Lambda^{\mathbb{R}} \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}}) = m(j_\Gamma^{\mathbb{R}} \circ j_\Gamma \circ c_\Gamma([M]_{\mathbb{Z}})) \ .$$

Since the simplicial volume of N is positive, by Proposition 1.61 we have that $\|\iota_\Lambda \circ j_\Lambda^{\mathbb{R}} \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}})\| > 0$ and

$$H_n^{\ell^1}(\alpha_\Gamma^{\mathbb{R}}) \circ \iota_\Lambda \circ \iota_\Lambda^{\mathbb{R}} \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}}) \neq 0 \ .$$

From the commutativity of the upper left square we obtain that $m \neq 0$, thus

$$\begin{aligned} \|(j_\Gamma^{\mathbb{R}} \circ j_\Gamma \circ c_\Gamma([M]_{\mathbb{Z}})\| &\leq |m| \|j_\Gamma^{\mathbb{R}} \circ j_\Gamma \circ c_\Gamma([M]_{\mathbb{Z}})\| \\ &\leq \left\| H_n^{\ell^1}(\alpha_\Gamma^{\mathbb{R}}) \circ j_\Lambda^{\mathbb{R}} \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}}) \right\| \\ &\leq \frac{1}{c_\Omega} \|j_\Lambda^{\mathbb{R}} \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}})\| \ . \end{aligned}$$

The same argument applied to the right part of the diagram produces the inequalities

$$\|(j_\Gamma^{\mathbb{R}} \circ j_\Gamma \circ c_\Gamma([M]_{\mathbb{Z}})\| \leq \frac{1}{c_\Omega} \|j_\Lambda^{\mathbb{R}} \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}})\| \leq \|(j_\Gamma^{\mathbb{R}} \circ j_\Gamma \circ c_\Gamma([M]_{\mathbb{Z}})\|,$$

which imply that $m = 1$ and $H_n(\alpha_\Gamma^{\mathbb{Z}}) \circ j_\Lambda \circ c_\Lambda([N]_{\mathbb{Z}})$ is an X_Λ-parametrised fundamental cycle. Therefore,

$$| M |^{X_\Lambda} = \|j_\Gamma \circ c_\Gamma([M]_{\mathbb{Z}})\| = \|H_n(\alpha_\Gamma^{\mathbb{Z}}) \circ j_\Lambda \circ c_\Lambda([M]_{\mathbb{Z}})\| \leq \frac{1}{c_\Omega} | N |^{X_\Gamma} \ .$$

The same argument applied to the right part of the diagram gives the opposite inequality. This implies the thesis as the composition

$$X_\Gamma \to \Omega \to \Gamma/\Omega$$

induces an isometry between $H_n(\Gamma, L^1(X_\Lambda, \mathbb{Z}))$ and $H_n(\Gamma, L^1(\Gamma/\Omega, \mathbb{Z}))$.
2) Assume that the ME-coupling (Ω, m) is mixing. Let (X, m_x) be an ergodic standard Γ-space on which Λ acts trivially. Then $(X \times \Omega, m_x \otimes m)$ is a bounded and ergodic ME-coupling between Γ and Λ with respect to their diagonal actions. Notice that $X \times X_\Gamma$ and $X \times X_\Lambda$ are bounded, finite-measure fundamental domains for the actions of Γ and Λ on $(X \times \Omega)$. By the previous result

$$| N |^{\Gamma/(X\times\Omega)} = c_{X\times\Omega} | M |^{\Lambda/(X\times\Omega)}$$

where $c_{X\times\Omega} = \frac{(m_x\otimes m)(X\times X_\Lambda)}{(m_x\otimes m)(X\times X_\Gamma)} = c_\Omega$. As a consequence,

$$\lvert N \rvert \le \lvert N \rvert^{\Gamma/(X\times\Omega)} = c_\Omega \lvert M \rvert^{X\times\Lambda/\Omega} \le c_\Omega \lvert M \rvert^{X}$$

and taking the infimum over all possible ergodic Γ-spaces X we get $\lvert N \rvert \le c_\Omega \lvert M \rvert$ from Proposition 1.52. Repeating the same argument applied to an ergodic standard Λ-space (Y, m_y) on which Γ acts trivially, we get the other inequality. □

Corollary 1.77. *Let $n \in \mathbb{N}$ and $\Gamma, \Lambda \subset G = Isom(\mathbb{H}^n)$ be uniform lattices. Then*

$$\frac{\lvert \mathbb{H}^n/\Gamma \rvert}{Covol(\Gamma)} = \frac{\lvert \mathbb{H}^n/\Lambda \rvert}{Covol(\Lambda)} .$$

Proof. Consider the hyperbolic manifolds $\mathbb{H}^n/\Gamma$ and $\mathbb{H}^n/\Lambda$. We have seen that the group G endowed with its bi-invariant Haar measure is a mixing and bounded ME-coupling between Γ and Λ (Remark 1.67). Moreover, $c_G = \frac{\mathrm{Covol}(\Lambda)}{\mathrm{Covol}(\Gamma)}$ (Remark 1.71). Therefore, by the proportionality principle

$$\lvert \mathbb{H}^n/\Lambda \rvert = \frac{\mathrm{Covol}(\Lambda)}{\mathrm{Covol}(\Gamma)} \lvert \mathbb{H}^n/\Gamma \rvert .$$

□

Theorem 1.78. *Let $n \in \mathbb{N}$ and let M and N be closed, connected and oriented hyperbolic n-manifolds with fundamental groups Γ and Λ, respectively. Let $G = Isom^+(\mathbb{H}^n)$. Let S be a set of uniform lattices of G containing one representative for each isometry class of hyperbolic n-manifolds. The product $X = \prod_{\Lambda'\in S} G/\Lambda'$ endowed with the product probability measure and the translatory action of Γ is a standard Γ-space and*

$$\lvert M \rvert \le \lvert M \rvert^{X} \le \frac{Vol(M)}{Vol(N)} \|N\|_{\mathbb{Z}}^{\infty} .$$

Proof. X is a standard Γ-space because, once the dimension n is fixed, there is only a countable quantity of distinct isometry classes of hyperbolic n-manifolds. Hence S is countable and Proposition 1.31 holds.
We know that $M = \mathbb{H}^n/\Gamma$, $N = \mathbb{H}^n/\Lambda$ and $\mathrm{Covol}(\Gamma) = \mathrm{Vol}(M)$, $\mathrm{Covol}(\Lambda) = \mathrm{Vol}(N)$. Let $N' \to N$ be a finite covering of N. Let $\pi_1(N') = \Lambda' < \Lambda$ with index d. Then

$$\begin{aligned} \lvert M \rvert &= \lvert \mathbb{H}^n/\Gamma \rvert \le \lvert \mathbb{H}^n/\Gamma \rvert^{\prod_{\Lambda''\in S} G/\Lambda''} \le \lvert \mathbb{H}^n/\Gamma \rvert^{G/\Lambda'} = \frac{\mathrm{Covol}(\Gamma)}{\mathrm{Covol}(\Lambda')} \lvert \mathbb{H}^n/\Lambda' \rvert^{G/\Gamma} \\ &= \frac{\mathrm{Covol}(\Gamma)}{\mathrm{Covol}(\Lambda)} \frac{1}{[\Lambda' : \Lambda]} \lvert N' \rvert^{G/\Gamma} \le \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(N)} \frac{1}{d} \|N'\|_{\mathbb{Z}} \end{aligned}$$

Taking the infimum over all possible finite coverings of N we get the thesis. □

Theorem 1.79 (Integral foliated simplicial volume for hyperbolic 3-manifolds)**.** *Let M be a closed, connected and oriented hyperbolic 3-manifold. Then*

$$\lvert M \rvert = \|M\|$$

Proof. Let S be a set of uniform lattices of $G = \mathrm{Isom}^+(\mathbb{H}^n)$ containing one representative for each isometry class of hyperbolic 3-manifolds. By Theorem 1.24 there exists a sequence of closed, connected and oriented hyperbolic 3-manifolds $(M_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{\|M_n\|_{\mathbb{Z}}^{\infty}}{\|M_n\|} = 1 \ .$$

By the previous theorem

$$\|M\| \leq |\!|\!| M |\!|\!| \leq |\!|\!| M |\!|\!|^{\prod_{\Lambda\in S} G/\Lambda} \leq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(M_n)} \|M_n\|_{\mathbb{Z}}^{\infty} \ .$$

Taking the limit for n tending to infinity we get the thesis. □

Chapter 2

ℓ^2-Betti numbers

A classical invariant of a finite CW-complex X is its p-th Betti number $b_p(X)$, which is the dimension of the real vector space $H_p(X,\mathbb{R})$, where $H_p(X,\mathbb{R})$ denotes the p-th singular homology module of X with real coefficients. Consider a G-covering $p : \overline{X} \to X$: if G is infinite the p-th Betti number of $\overline{X}$ may be infinite and hence useless. We can overcome this problem by considering the reduced homology of $\overline{X}$ with coefficients in the Hilbert space of square summable functions from G to $\mathbb{R}$. In this way, the p-th homology group will have a structure of a finite Hilbert G-module and the p-th ℓ^2-Betti number will be defined as its Von Neumann dimension. In this chapter we will introduce this invariant and illustrate its basic properties.

2.1 Von Neumann dimension

In this section we define the Von Neumann dimension, which provides the required concept of dimension in a non-commutative setting.

Let G be a group and R be a ring (usually $R = \mathbb{R}, \mathbb{Z}$). We denote with $R[G]$ the set whose elements are $\sum_{x\in G} r(x)x$, where $r(x) \in R$ and $r(x) \neq 0$ only for a finite number of $x \in G$. This is a ring with the operations:

$$\left(\sum_{x\in G} r(x)x\right) + \left(\sum_{x\in G} s(x)x\right) = \sum_{x\in G} (r(x) + s(x))x \ ;$$

$$\left(\sum_{x\in G} r(x)x\right) \cdot \left(\sum_{y\in G} s(y)y\right) = \sum_{x\in G}\sum_{y\in G} r(x)s(y)xy \ .$$

We denote with $\ell^2(G)$ the real Hilbert space of square summable functions $f : G \to \mathbb{R}$, with the scalar product

$$\langle \cdot,\cdot\rangle : \ell^2(G) \times \ell^2(G) \to \mathbb{R}$$

$$\langle f, g\rangle = \sum_{x\in G} f(x)g(x) \ .$$

There is a canonical immersion of $\mathbb{R}$-vector spaces $\mathbb{R}[G] \hookrightarrow \ell^2(G)$, by considering the elements of $\mathbb{R}[G]$ as square summable functions with finite support.
With this in mind, we indicate a function $f \in \ell^2(G)$ as a formal series $f = \sum_{x\in G} f(x)x$, with $f(x) \in \mathbb{R}$ and $\sum_{x\in G} f(x)^2 < \infty$.

Definition 2.1. We say that M is a left $R[G]$-module if it is a left module over the ring $R[G]$ or, equivalently, if it is an R-module endowed with a left action of G.

We can define on $\ell^2(G)$ both a left and a right isometric action of G by setting

$$\left(\sum_{x\in G} f(x)x\right) \cdot y = \sum_{x\in G} f(xy^{-1})x \ .$$

$$y \cdot \left(\sum_{x\in G} f(x)x\right) = \sum_{x\in G} f(y^{-1}x)x \ .$$

These actions can be extended to $\ell^2(G)^n$ by letting G act diagonally. Therefore, $\ell^2(G)^n$ has a structure of $\mathbb{R}[G]$-bimodule.

Remark 2.2. These actions are bounded, i.e. for every $c = \sum_{x\in G} c(x)x \in \mathbb{R}[G]$ and for every $f \in \ell^2(G)$ we have

$$\|f \cdot c\| \leq |c|\|f\| \qquad\qquad \|c \cdot f\| \leq |c|\|f\|$$

where $|c| = \sum_{x\in G} |c(x)|$.

Definition 2.3. A Hilbert G-module is a left $\mathbb{R}[G]$-module M endowed with a Hilbert structure with respect to which G acts via isometries. In addition, we require that M is G-equivariantly isometric to a G-invariant Hilbert subspace of $\ell^2(G)^n$ for some $n \in \mathbb{N}$.

Example 2.4. The left translation of G described above endows $\ell^2(G)$ with a structure of Hilbert G-module.

Remark 2.5. If M is a Hilbert G-module and $H < G$ is of finite index d, then M has a structure of Hilbert H-module, as well. Fix a set of representatives $\{x_1, \ldots, x_d\}$ such that $G/H = \{Hx_1, \ldots, Hx_d\}$. Then $G = \bigcup_{i=1}^d Hx_i$ and $\ell^2(G) \cong \perp_{i=1}^d \ell^2(H)\cdot x_i \cong \ell^2(H)^d$ as left H-module. Therefore, if M is G-equivariantly isometric to a G-invariant Hilbert subspace of $\ell^2(G)^n$, then it is H-equivariant isometric to an H-invariant Hilbert subspace of $\ell^2(H)^{nd}$.

We want to define a function $\dim_G$: {Hilbert G-modules} $\to \mathbb{R}^+$ satisfying the following properties:

- $\dim_G M \geq 0$ and $\dim_G M = 0$ iff $M = 0$;
- if $M \cong N$ then $\dim_G M = \dim_G N$;
- if $N \subseteq M$ then $\dim_G N \leq \dim_G M$;

- $\dim_G(M \oplus N) = \dim_G M + \dim_G N$;
- $\dim_G \ell^2(G) = 1$;
- if G is finite, then $\dim_G M = \frac{1}{|G|} \dim_{\mathbb{R}} M$;
- if $H < G$ has finite index, then $\dim_G M = \frac{1}{[G:H]} \dim_H M$.

Definition 2.6. The Von Neumann algebra $\mathcal{N}(G)$ is the algebra of the bounded G-equivariant operators from $\ell^2(G)$ into $\ell^2(G)$, where $\ell^2(G)$ is considered as a left G-module.

Remark 2.7. Since the right action of $\mathbb{R}[G]$ on $\ell^2(G)$ defines a bounded G-equivariant operator from $\ell^2(G)$ to $\ell^2(G)$, we can consider $\mathbb{R}[G]$ as a subalgebra of the Von Neumann algebra.

Given $\phi \in \mathcal{N}(G)$, we denote with $\phi^* \in \mathcal{N}(G)$ its adjoint operator, i.e. the unique operator such that for every $f, g \in \ell^2(G)$ the relation $\langle \phi(f), g\rangle = \langle f, \phi^*(g)\rangle$ holds. The adjoint of an element $\phi \in \mathcal{N}(G)$ always exists because it is a bounded operator between Hilbert spaces. Moreover, $\phi^* \in \mathcal{N}(G)$ because if ϕ is G-equivariant, then ϕ^* is G-equivariant, as well.

Example 2.8. The adjoint of the right translation by an element $x \in G$ is the right translation by x^{-1}, which will be indicated as $\bar{x}$ thereafter. More in general, given $f = \sum_{x\in G} f(x)x \in \ell^2(G)$ we denote with $\bar{f}$ the element $\sum_{x\in G} f(x)x^{-1}$.

The Kaplansky trace is the map

$$\begin{aligned} \rho : \mathbb{R}[G] &\to \mathbb{R} \\ \sum_{x\in G} r(x)x &\mapsto r(1) \end{aligned}$$

where $1 \in G$ is the identity.
We want to extend the Kaplansky trace to all elements of $\mathcal{N}(G)$.

Definition 2.9. Let $\phi \in \mathcal{N}(G)$ and $1 \in \mathbb{R}[G] \subset \ell^2(G)$ be the identity. We define

$$\begin{aligned} \mathrm{trace}_G : \mathcal{N}(G) &\to \mathbb{R} \\ \phi &\mapsto \langle \phi(1), 1\rangle \ . \end{aligned}$$

This is an extention of the Kaplansky trace, because, considering $w = \sum_{x\in G} r(x)x \in \mathbb{R}[G]$ as an element of $\mathcal{N}(G)$, we have

$$\mathrm{trace}_G(w) = \langle 1 \cdot \sum_{x\in G} r(x)x, 1\rangle = \langle \sum_{x\in G} r(x)x, 1\rangle = r(1) = \rho(w) \ .$$

Remark 2.10. $\mathrm{trace}_G(\phi) = \mathrm{trace}_G(\phi^*)$ since $\mathrm{trace}_G(\phi) = \langle \phi(1), 1\rangle = \langle 1, \phi^*(1)\rangle = \mathrm{trace}_G(\phi^*)$.

Lemma 2.11. *Let $\theta : \mathcal{N}(G) \to \ell^2(G)$ be the $\mathbb{R}$-linear map defined by $\phi \mapsto \phi(1)$. Then, θ is injective and satisfies $\theta(\phi^*) = \overline{\phi(1)}$.*

Proof. Let $\phi \in \mathcal{N}(G)$ belong to $\ker \theta$. Then $\phi(1) = 0$ and for every $x \in G$ we have $\phi(x) = x\phi(1) = 0$. Therefore, ϕ vanishes on $\mathbb{R}[G]$, which is dense in $\ell^2(G)$. Being bounded, ϕ is the null map.
Moreover, for every $x \in G$ we have

$$\langle \phi^*(1), x\rangle = \langle 1, \phi(x)\rangle = \langle 1, x\phi(1)\rangle = \langle \bar{x}, \phi(1)\rangle = \langle \phi(1), \bar{x}\rangle = \langle \overline{\phi(1)}, x\rangle \ ,$$

thus, by the same density argument used before, $\theta(\phi^*) = \phi^*(1) = \overline{\phi(1)}$. □

Corollary 2.12. *We can see $\mathcal{N}(G)$ as a G-submodule of $\ell^2(G)$.*

Lemma 2.13 (Properties of the Von Neumann trace)**.** *The Von Neumann trace satisfies the following properties:*

1) the Von Neumann trace is linear;

2) let $\{\phi_i\}_{i\in I}$ be a directed system of positive operators in $\mathcal{N}(G)$ (i.e. $\langle \phi(x), x\rangle > 0$ for every $x \in \ell^2(G)$, $x \neq 0$ or, equivalently, $\phi = \psi^*\psi$ for some $\psi \in \mathcal{N}(G)$) such that if $i \leq j$ then $\phi_i \leq \phi_j$. If ϕ_i converges weakly to ϕ, then

$$trace_G(\phi) = \sup_{i\in I} trace_G(\phi_i) \ ;$$

3) If $\phi \in \mathcal{N}(G)$ is a positive operator, then

$$trace_G(\phi) = 0 \Leftrightarrow \phi = 0 \ ;$$

4) if $\phi, \psi \in \mathcal{N}(G)$ are positive and $\phi \leq \psi$, then $trace_G(\phi) \leq trace_G(\psi)$;

5) for every $\phi, \psi \in \mathcal{N}(G)$ we have

$$trace_G(\phi\psi) = trace_G(\psi\phi) \ .$$

Proof. 1) It is clear by definition.
2) We recall that ϕ_i converges weakly to ϕ if and only if for every $x, y \in \ell^2(G)$ we have $|\langle \phi_i(x), y\rangle| \to |\langle \phi(x), y\rangle|$. We start proving that $\phi_i \leq \phi$ for every $i \in I$. Let $x \in \ell^2(G)$ be a fixed element and $i \in I$ be a fixed index. Given $\epsilon > 0$, there exists an index $i \leq i(\epsilon)$ such that

$$\langle \phi_i(x), x\rangle - \epsilon \leq \langle \phi_{i(\epsilon)}(x), x\rangle - \epsilon \leq \langle \phi(x), x\rangle \ .$$

It follows that $\mathrm{trace}_G(\phi) \geq \sup_{i\in I} \mathrm{trace}_G(\phi_i)$. By weakly convergence, for every $\epsilon > 0$ there exists an index $i(\epsilon) \in I$ such that $\langle \phi(1), 1\rangle \leq \langle \phi_{i(\epsilon)}(1), 1\rangle + \epsilon$. Therefore,

$$\mathrm{trace}_G(\phi) \leq \mathrm{trace}_G(\phi_{i(\epsilon)}) + \epsilon \leq \sup_{i\in I} \mathrm{trace}_G(\phi_i) + \epsilon \ .$$

3) By polar decomposition, there exists a self-adjoint operator $h \in \mathcal{N}(G)$ such that $\phi = h^* \circ h$. Therefore,

$$\text{trace}_G(\phi) = \langle \phi(1), 1 \rangle = \langle h(1), h(1) \rangle = \|h(1)\|^2$$

is null if and only if $h(1) = 0$. Since h is G-equivariant and linear ϕ vanishes on $\mathbb{R}[G]$, which is dense in $\ell^2(G)$. Being bounded, it is the null map, whence $\phi = 0$.
4) By definition, $\psi - \phi$ is a positive operator, hence its trace is positive. The result follows by linearity.
5) Using the previous lemma, we have

$$\begin{aligned}\text{trace}_G(\phi\psi) &= \langle \phi(\psi(1)), 1 \rangle = \langle \psi(1), \phi^*(1) \rangle = \langle \psi(1), \overline{\phi(1)} \rangle \\ &= \langle \overline{\psi(1)}, \phi(1) \rangle = \langle \psi^*(1), \phi(1) \rangle = \langle 1, \psi(\phi(1)) \rangle = \text{trace}_G(\psi\phi).\end{aligned}$$

□

We denote with $M_n(\mathcal{N}(G))$ the $\mathbb{R}$-algebra of G-equivariant bounded operators from $\ell^2(G)^n$ into $\ell^2(G)^n$. An element $F \in M_n(\mathcal{N}(G))$ is uniquely determined by a matrix $(F_{i,j})_{i,j=1,\dots,n}$ where $F_{i,j} \in \mathcal{N}(G)$. We extend the trace operator to an element $F \in M_n(\mathcal{N}(G))$, by defining

$$\text{trace}_G(F) = \sum_{i=1}^{n} \text{trace}_G(F_{i,i}) \ .$$

The following properties are straightforward consequences of the previous lemmas:

- $\text{trace}_G(F) = \text{trace}_G(F^*)$;
- $\text{trace}_G(F_1 \circ F_2) = \text{trace}_G(F_2 \circ F_1)$

Lemma 2.14. *Suppose $F \in M_n(\mathcal{N}(G))$ is self-adjoint and idempotent, then*

$$\textit{trace}_G(F) = \sum_{i,j=1}^{n} \|F_{i,j}(1)\|^2 \ .$$

In particular, $\textit{trace}_G(F) \geq 0$ and $\textit{trace}_G(F) = 0$ if and only if F is identically null.

Proof. Since F is self-adjoint, $F_{i,j} = F^*_{j,i}$, hence

$$\begin{aligned}\text{trace}_G(F) &= \sum_{j=1}^{n} \langle F_{j,j}(1), 1 \rangle = \sum_{j=1}^{n} \langle F^2_{j,j}(1), 1 \rangle \\ &= \sum_{i,j=1}^{n} \langle F_{j,i} F_{i,j}(1), 1 \rangle = \sum_{i,j=1}^{n} \langle F_{i,j}(1), F_{i,j}(1) \rangle = \sum_{i,j=1}^{n} \|F_{i,j}(1)\|^2 \ .\end{aligned}$$

□

Let $V \subset \ell^2(G)^n$ be a G-invariant Hilbert subspace of $\ell^2(G)^n$ and let π_V be the orthogonal projection onto V. Since for every $T \in \ell^2(G)^n$ and for every $x \in G$ we have

$$xT = \underbrace{\pi_V(xT)}_{\in V} + \underbrace{(xT - \pi_V(xT))}_{\in V^\perp} = \underbrace{x\pi_V(T)}_{\in V} + \underbrace{(xT - x\pi_V(T))}_{\in V^\perp} \ ,$$

π_V is G-equivariant and hence is a element of $M_n(\mathcal{N}(G))$.

Definition 2.15. We define the Von Neumann dimension of a G-invariant Hilbert subspace $V \subset \ell^2(G)^n$ as

$$\dim_G V = \operatorname{trace}_G(\pi_V) \ .$$

Remark 2.16. Since π_V is self-adjoint and idempotent, $\dim_G V \geq 0$ and $\dim_G V = 0$ if and only if $V = 0$.

Example 2.17. If $V = \ell^2(G)$, the orthogonal projection onto V is just the identity. Therefore $\dim_G \ell^2(G) = \operatorname{trace}_G(id) = \langle 1, 1\rangle = 1$.

In the general case, let M be a Hilbert G-module. By definition there exists a G-equivariant isometry $\alpha : M \to V \subset \ell^2(G)^n$. We define

$$\dim_G M = \dim_G V \ .$$

We must verify that this definition is independent of the choice of the isometry α. Suppose that $\beta : M \to W \subset \ell^2(G)^m$ is another G-equivariant isometry. Without loss of generality we can suppose that $m = n + k \geq n$. Let V' be the image of the inclusion of V into $\ell^2(G)^m = \ell^2(G)^n \oplus \ell^2(G)^k$. It is obvious that $\dim_G V = \dim_G V'$, hence we can suppose $m = n$. Define $h = \beta \circ \alpha^{-1} : V \to W$ and extend it to an element $H \in M_n(\mathcal{N}(G))$ by putting $H_{|_V} = h$ and $H_{|_{V^\perp}} = 0$. By construction $H^*H : \ell^2(G) \to V$ is the orthogonal projection onto V and $HH^* : \ell^2(G) \to W$ is the orthogonal projection onto W. Therefore, we have

$$\dim_G V = \operatorname{trace}_G(H^*H) = \operatorname{trace}_G(HH^*) = \dim_G W \ .$$

Remark 2.18. Let $G = \{1\}$ be the trivial group and M be a Hilbert G-module. We want to calculate the Von Neumann dimension of M in this simple situation. Since G acts trivially, M is just an $\mathbb{R}$-vector space. We can suppose $M = \mathbb{R}^n = \ell^2(G)^n$. Then the orthogonal projection from $\ell^2(G)^n$ to M is the identity. Therefore

$$\dim_G M = \operatorname{trace}_G(id_{\ell^2(G)^n}) = \sum_{k=1}^{n} \langle id_{\ell^2(G)^n}(1), 1\rangle = n = \dim_{\mathbb{R}} M$$

It is easy to verify from the very definition that $\dim_G$ satisfies the following properties:

- $\dim_G(M \oplus N) = \dim_G M + \dim_G N$;
- if $M \cong N$, then $\dim_G M = \dim_G N$;
- if $N \subseteq M$, then $\dim_G N \leq \dim_G M$;

Let us verify the other properties we pointed out at the beginning of the section.

Lemma 2.19. *Let $H < G$ be a subgroup of G of finite index and M a Hilbert G-module. Then $\dim_G M = \frac{1}{[G:H]} \dim_H M$.*

Proof. Let $d = [G : H]$. By Remark 2.5, there exist elements $x_1, \dots, x_d$ such that $G = \bigcup_{i=1}^d Hx_i$ and $\ell^2(G) = \perp_{i=1}^n \ell^2(H) \cdot x_i$. Therefore, for every element $F \in \mathcal{N}(G)$ we have

$$\text{trace}_H(F) = \sum_{i=1}^d \langle F(x_i), x_i \rangle = \sum_{i=1}^d \langle F(1), 1 \rangle = d \cdot \text{trace}_G(F) \ .$$

The same result holds for $F \in M_n(\mathcal{N}(G))$, thus $\dim_H M = d \dim_G M$. □

Corollary 2.20. *If G is finite and M is a Hilbert G-module, then* $\dim_G M = \frac{1}{|G|} \dim_{\mathbb{R}} M$.

Proof. Since G is finite, we can take $H = \{1\}$ as subgroup of G with finite index. By Remark 2.18 and Lemma 2.19 we have

$$\dim_{\mathbb{R}} M = \dim_H M = [G : H] \dim_G M = |G| \dim_G M \ .$$

□

Let V_1 and V_2 be Hilbert spaces. Let $V_1 \otimes_{\mathbb{R}} V_2$ be the tensor product of V_1 and V_2 in the category of vector spaces, i.e the set of finite linear combinations of the elementary tensors $v_1 \otimes v_2$, where $v_1 \in V_1$ and $v_2 \in V_2$. This vector space is naturally endowed with the scalar product determined by the relation

$$\langle\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle\rangle = \langle v_1, w_1 \rangle_{V_1} \langle v_2, w_2 \rangle_{V_2} \ .$$

The tensor product of the Hilbert spaces V_1 and V_2, denoted by $V_1 \bar{\otimes} V_2$, is the completion of $V_1 \otimes V_2$ with respect to the metric induced by the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$.

Proposition 2.21. *Let M_1 be a Hilbert G-module and M_2 be a Hilbert H-module. Then the tensor product $M_1 \bar{\otimes}_{\mathbb{R}} M_2$ is a Hilbert $(G \times H)$-module and*

$$\dim_{(G \times H)}(M_1 \bar{\otimes}_{\mathbb{R}} M_2) = \dim_G M_1 \cdot \dim_H M_2 \ .$$

Proof. Obviously $\ell^2(G) \bar{\otimes}_{\mathbb{R}} \ell^2(H)$ is isometrically $(G \times H)$-equivariantly isomorphic to $\ell^2(G \times H)$. Hence $M_1 \bar{\otimes}_{\mathbb{R}} M_2$ is a Hilbert $(G \times H)$-module. Let $f_1 \in \mathcal{N}(G)$ and $f_2 \in \mathcal{N}(H)$, then $f_1 \otimes f_2 \in \mathcal{N}(G \times H)$ and

$$\begin{aligned} \text{trace}_{G \times H}(f_1 \otimes f_2) &= \langle\langle (f_1 \otimes f_2)(1_G \otimes 1_H), 1_G \otimes 1_H \rangle\rangle \\ &= \langle\langle f_1(1_G) \otimes f_2(1_H), 1_G \otimes 1_H \rangle\rangle \\ &= \langle f_1(1_G), 1_G \rangle \cdot \langle f_2(1_H), 1_H \rangle \\ &= \text{trace}_G(f_1) \cdot \text{trace}_H(f_2) \ . \end{aligned}$$

A similar formula holds when $f_1 \in M_n(\mathcal{N}(G))$ and $f_2 \in M_n(\mathcal{N}(H))$ and the claim follows by applying it to the orthogonal projection onto M_1 and M_2. □

2.2 ℓ^2-chain complexes and ℓ^2-Betti numbers

We are going to apply the theory of Von Neumann dimension to the Hilbert G-modules resulting from a G-equivariant homology theory of CW-complexes. This will lead to the definition of ℓ^2-Betti numbers as the Von Neumann dimension of the reduced ℓ^2-homology groups.

Definition 2.22. A chain complex of Hilbert G-modules

$$\cdots \to V_{i+1} \xrightarrow{d_{i+1}} V_i \xrightarrow{d_i} V_{i-1} \to \dots$$

is said to be an ℓ^2-chain complex if every homomorphism d_i is bounded and G-equivariant. The reduced homology of the ℓ^2-chain complex (V_*, d_*) is defined as

$$^{red}H_i(V_*) = \mathrm{Ker}(d_i)/\overline{\mathrm{Im}(d_{i+1})} \ .$$

The chain complex (V_*, d_*) is weak-exact if $^{red}H_i(V_*) = 0$.

Definition 2.23. Let (V_*, d_*) and $(W_*, \tilde{d}_*)$ be two ℓ^2-chain complexes. A morphism $\phi_* : V_* \to W_*$ is a family of bounded and G-equivariant homomorphisms $\phi_i : V_i \to W_i$ such that $\phi_i \circ d_{i+1} = \tilde{d}_{i+1} \circ \phi_{i+1}$ for every i.
Two morphisms $\phi_*, \psi_* : V_* \to W_*$ are ℓ^2-homotopic if there exists a family of G-equivariant and bounded operators $K_i : V_i \to W_{i+1}$ such that $K_i \circ d_{i+1} \pm \tilde{d}_{i+1} \circ K_i = \phi_{i+1} - \psi_{i+1}$ for every i.

Remark 2.24. A morphism between ℓ^2-chain complexes $\phi_* : V_* \to W_*$ induces a bounded and G-equivariant map $\phi_* :^{red} H_*(V_*) \to^{red} H_*(W_*)$, which depends only on the ℓ^2-homotopy class of ϕ_*.

Definition 2.25. A map $f : M_1 \to M_2$ between Hilbert G-modules is

- a weak isomorphism, if it is injective, bounded, G-equivariant and with dense image;
- a strong isomorphism, if it is a G-equivariant bijective isometry.

Lemma 2.26. *If there exists a weak isomorphism between two G-modules M_1 and M_2, then there exists a strong one, as well.*

Proof. Let $f : M_1 \to M_2$ be a weak isomorphism. The operator $f^* \circ f : M_1 \to M_1$ is positive, as for every $v \in M_1$, $v \neq 0$, we have $\langle f^* f(v), v\rangle = \langle f(v), f(v)\rangle > 0$, and with dense image, as $w \in M_1$ is orthogonal to $\mathrm{Im}(f^* \circ f)$ if and only if $f(w)$ is orthogonal to $\mathrm{Im}(f)$, which is dense. For the polar decomposition there exists a self-adjoint operator $g : M_1 \to M_2$ such that $g^2 = f^* \circ f$ and $\mathrm{Im}(g^2) \subset \mathrm{Im}(g)$ is dense. Define $h = f \circ g^{-1} : \mathrm{Im}(g) \to M_2$. Notice that $\mathrm{Im}(h) = \mathrm{Im}(f)$ is dense. In addition, $\forall x, y \in \mathrm{Im}(g)$ we have

$$\begin{aligned}\langle h(x), h(y)\rangle &= \langle f(g^{-1}(x)), f(g^{-1}(y))\rangle = \langle f^*(f(g^{-1}(x))), g^{-1}(y)\rangle \\ &= \langle g^2(g^{-1}(x), g^{-1}(y)\rangle = \langle g(g^{-1}(x)), g(g^{-1}(y))\rangle = \langle x, y\rangle \ .\end{aligned}$$

Therefore, $h : \mathrm{Im}(g) \to \mathrm{Im}(f)$ is an isometry. Since $\mathrm{Im}(g) \subset M_1$ and $\mathrm{Im}(f) \subset M_2$ are dense, h can be extended in a unique way to a bijective isometry $\tilde{h} : M_1 \to M_2$. Since f and f^* are G-equivariant, g is G-equivariant too and so is h. It follows that $\tilde{h}$ is a strong isomorphism of Hilbert G-modules. □

If V is a Hilbert G-module and $W \subset V$ is a G-equivariant closed subspace, then the quotient V/W has a natural structure of Hilbert G-module, where the norm is given by

$$\|[v]\| = \inf\{\|z\| \mid [z] = [v]\}.$$

Corollary 2.27. *Let $\phi : M_1 \to M_2$ be a bounded G-equivariant operator between Hilbert G-modules. Then we have the following strong isomorphisms:*

$$(Ker(\phi))^\perp \cong M_1/Ker(\phi) \cong \overline{Im(\phi)} \ .$$

Lemma 2.28. *Let $0 \to U \to V \xrightarrow{\alpha} W \to 0$ be a weak-exact ℓ^2-chain complex. Then*

$$\dim_G V = \dim_G U + \dim_G W \ .$$

Proof. It is easy to verify that $V \cong \mathrm{Ker}(\alpha) \oplus \mathrm{Ker}(\alpha)^\perp$ as Hilbert G-modules and, by the previous corollary, $\mathrm{Ker}(\alpha)^\perp \cong \overline{\mathrm{Im}(\alpha)}$. Therefore,

$$\begin{aligned} \dim_G V &= \dim_G \mathrm{Ker}(\alpha) + \dim_G \overline{\mathrm{Im}(\alpha)} = \dim_G U + \dim_G \overline{\mathrm{Im}(\alpha)} \\ &= \dim_G U + \dim_G W \ . \end{aligned}$$

□

Corollary 2.29. *Let $0 \to V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \cdots \to \ldots \xrightarrow{d_1} V_0 \to 0$ be an $\ell^2(G)$-chain complex. Then $\sum_{i=0}^n (-1)^i \dim_G V_i = \sum_{i=0}^n (-1)^i \dim_G^{red} H_i(V_*)$.*

Proof. Let $K_i = \mathrm{Ker}(d_i)$ and $J_i = \overline{\mathrm{Im}(d_{i+1})}$. We have the following weak-exact complexes of Hilbert G-modules:

$$0 \to K_i \to V_i \to J_{i-1} \to 0 \qquad 0 \to J_i \to K_i \to^{red} H_i(V_*) \to 0$$

Using repeatedly the relations of the previous lemma, we are done. □

2.2.1 ℓ^2-homology of CW-complexes

We are going to produce the fundamental example of $\ell^2(G)$-chain complex. Let Y be a connected CW-complex and G be a group acting on Y via permutations of the cells. Suppose $X = Y/G$ is a compact CW-complex. We denote with $K_i(Y)$ the free $\mathbb{Z}$ module generated by the i-dimensional cells of Y: since the action of G is cellular, G acts on $K_i(Y)$ giving it a structure of a left $\mathbb{Z}[G]$-module, generated by the i-dimensional cells of X. We define $C_i(Y, \ell^2(G)) = \ell^2(G) \otimes_{\mathbb{Z}[G]} K_i(Y)$, where $\ell^2(G)$ is considered as a right $\mathbb{Z}[G]$-module (i.e. G acts via right translations) and $K_i(Y)$ has the structure of left $\mathbb{Z}[G]$-module described above. The resulting tensor product is a left $\mathbb{R}[G]$-module: let $\bar{\tau}_i^\mu$ be a representative i-cell of each G-orbit for $\mu = 1, \dots, \alpha_i$, where $\alpha_i = \mathrm{rank}_{\mathbb{Z}} K_i(X)$, then the action is determined by:

$$\begin{aligned} \mathbb{R}[G] \times C_i(Y, \ell^2(G)) &\to C_i(Y, \ell^2(G)) \\ \Big(a_g g, \sum_{\mu=1}^{k} f_\mu \otimes \bar{\tau}_i^\mu\Big) &\mapsto \sum_{\mu=1}^{k} a_g f_\mu \otimes g\bar{\tau}_i^\mu \ . \end{aligned}$$

We can endow $C_i(Y, \ell^2(G))$ with a Hilbert structure: we declare that

$$\{x \otimes \bar{\tau}_i^\mu \mid x \in G,\ \mu \in \{1, \dots \alpha_i\}\}$$

is a complete orthonormal basis for $C_i(Y, \ell^2(G))$.

Remark 2.30. It follows immediately that the map

$$\begin{aligned} (\ell^2(G))^{\alpha_i} &\to C_i(Y) \\ (f_1, \dots, f_{\alpha_i}) &\mapsto \sum_{\mu=1}^{\alpha_i} f_\mu \otimes \bar{\tau}_i^\mu \end{aligned}$$

is a G-equivariant isometry, i.e. $C_i(Y, \ell^2(G))$ is a Hilbert G-module.

Denoting with d_i the usual boundary operator of the cellular chain complex, we define the ℓ^2-boundary operator

$$\begin{aligned} id_{\ell^2} \otimes d_i : C_i(Y, \ell^2(G)) &\to C_{i-1}(Y, \ell^2(G)) \\ f \otimes \sigma &\mapsto f \otimes d_i(\sigma) \ . \end{aligned}$$

It is evidently G-equivariant and the following lemma ensures that it is bounded.

Lemma 2.31. *Let $\phi : (\mathbb{R}[G])^n \to (\mathbb{R}[G])^m$ be a G-equivariant morphism between $\mathbb{R}[G]$-modules. Then the unique operator $\tilde{\phi} : \ell^2(G)^n \to \ell^2(G)^m$ which concides with ϕ on $(\mathbb{R}[G])^n$ is bounded.*

Proof. Let $(\phi_{i,j})$ be the elements of a matrix representing ϕ. Notice that

$$\mathrm{Hom}_G(\mathbb{R}[G], \mathbb{R}[G]) \cong \mathbb{R}[G]$$

by the map that associates to a G-morphism the value it takes on the identity. Therefore, each $\phi_{i,j}$ can be thought of as an element of $\mathbb{R}[G]$ acting on $\mathbb{R}[G]$ by right translation and it can be written as $\phi_{i,j} = \sum_{x \in G} c_{i,j}(x)x$. Set $|\phi_{i,j}| = \sum_{x \in G} |c_{i,j}(x)|$. It is easy to verifiy that $\|f \cdot \phi_{i,j}\| \le |\phi_{i,j}| \|f\|$. Therefore

$$\|\tilde{\phi}(f_1, \dots, f_n)\|^2 = \sum_j \Big\| \sum_i f_i \phi_{i,j} \Big\|^2 \le \sum_{i,j} |\phi_{i,j}|^2 \|f_i\|^2 \le \Big(\sum_{i,j} |\phi_{i,j}|^2\Big) \|(f_1, \dots, f_n)\|^2 \ .$$

□

The reduced homology of the ℓ^2-chain complex $(C_i(Y,\ell^2(G)), id_{\ell^2}\otimes d_i)$ defines the ℓ^2-homology groups of Y:

$$^{red}H_i(Y,\ell^2(G)) := \mathrm{Ker}(id_{\ell^2}\otimes d_i)/\overline{\mathrm{Im}(id_{\ell^2}\otimes d_{i+1})}$$

The particular case in which we are interested in is when X is a compact connected CW-complex, G is its fundamental group and Y is its universal covering.

There is another possible definition of the reduced ℓ^2-homology, which will be useful later. Let $\delta_i : C_i(Y,\ell^2(G)) \to C_{i+1}(Y,\ell^2(G))$ be the adjoint operator of d_{i+1}. We define the following G-equivariant closed subspaces of $C_i(Y,\ell^2(G))$:

$$Z_i(Y,\ell^2(G)) = \mathrm{Ker}(d_i) \qquad Z^i(Y,\ell^2(G)) = \mathrm{Ker}(\delta_i)$$

$$\mathcal{H}_i(Y,\ell^2(G)) = Z_i(Y,\ell^2(G)) \cap Z^i(Y,\ell^2(G)) \ .$$

We can interpret $\mathcal{H}_i(Y,\ell^2(G))$ as the kernel of a bounded self-adjoint G-equivariant map: if we define the laplacian $\Delta_i = d_{i+1}\circ\delta_i + \delta_{i-1}\circ d_i : C_i(Y,\ell^2(G)) \to C_i(Y,\ell^2(G))$, then $\mathcal{H}_i(Y,\ell^2(G)) = \mathrm{Ker}(\Delta_i)$. Namely, if $c \in \mathrm{Ker}(\Delta_i)$, then $\langle \delta_{i-1}\circ d_i(c), d_{i+1}\circ\delta_i(c)\rangle = \langle \delta_i\circ\delta_{i-1}\circ d_i(c), \delta_i(c)\rangle = 0$, so

$$0 = \|\Delta_i(c)\|^2 = \|\delta_{i-1}\circ d_i(c)\|^2 + \|d_{i+1}\circ\delta_i(c)\|^2$$

which implies that $d_i(c)\in\mathrm{Ker}(\delta_{i-1})$ and $\delta_i(c)\in\mathrm{Ker}(d_{i+1})$. Therefore,

$$0 = \langle \delta_{i-1}\circ d_i(c), c\rangle + \langle d_{i+1}\circ\delta_i(c), c\rangle = \|d_i(c)\|^2 + \|\delta_i(c)\|^2$$

which gives the thesis.
We denote by $B_i(Y,\ell^2(G))$ and $B^i(Y,\ell^2(G))$ the images of d_{i+1} and δ_{i-1} respectively: they are not necessarily closed.
Recall this general result of functional analysis:

Lemma 2.32. *Let $T : H \to H$ be a bounded operator between Hilbert spaces and T^* be its adjoint. Then $H = Ker(T) \perp \overline{Im(T^*)} = Ker(T^*) \perp \overline{Im(T)}$.*

Therefore, we have the following orthogonal decompositions:

$$\begin{aligned} C_i(Y,\ell^2(G)) &\cong \overline{B^i(Y,\ell^2(G))} \perp Z_i(Y,\ell^2(G)) \\ &\cong \overline{B_i(Y,\ell^2(G))} \perp Z^i(Y,\ell^2(G)) \\ &\cong \overline{B^i(Y,\ell^2(G))} \perp \overline{B_i(Y,\ell^2(G))} \perp \mathcal{H}_i(Y,\ell^2(G)) \end{aligned}$$

The latter isomorphism follows from the fact that $\mathcal{H}_i(Y,\ell^2(G)) = \mathrm{Ker}(\Delta_i) = Z_i(Y,\ell^2(G))\cap Z^i(Y,\ell^2(G))$ and, by the previous lemma, $\overline{\mathrm{Im}(\Delta_i)} \cong (\mathrm{Ker}(\Delta_i))^\perp = (\mathrm{Ker}(d_i)\cap\mathrm{Ker}(\delta_i))^\perp = \mathrm{Ker}(d_i)^\perp \oplus \mathrm{Ker}(\delta_i)^\perp \cong \overline{\mathrm{Im}(\delta_{i-1})} \perp \overline{\mathrm{Im}(d_{i+1})}$.
In particular $Z_i(Y,\ell^2(G)) \cong \overline{B_i(Y,\ell^2(G))} \perp \mathcal{H}_i(Y,\ell^2(G))$ and the orthogonal projection $Z_i(Y,\ell^2(G)) \to \mathcal{H}_i(Y,\ell^2(G))$ induces an isomorphism between $\mathcal{H}_i(Y,\ell^2(G))$ and $^{red}H_i(Y,\ell^2(G))$. Similarly, if we define

$$^{red}H^i(Y,\ell^2(G)) = Z^i(Y,\ell^2(G))/\overline{B^i(Y,\ell^2(G))} \ ,$$

the orthogonal projection $Z^i(Y,\ell^2(G)) \to \mathcal{H}_i(Y,\ell^2(G))$ induces an isomorphism between $\mathcal{H}_i(Y,\ell^2(G))$ and ${}^{red}H^i(Y,\ell^2(G))$.
Moreover, the map $\delta_i|_{(Z_i)^\perp} : (Z^i(Y,\ell^2(G)))^\perp \to B^{i+1}(Y,\ell^2(G))$ induces an isomorphism between $B_i(Y,\ell^2(G))$ and $B^{i+1}(Y,\ell^2(G))$.

Proposition 2.33. *${}^{red}H_*$ defines a covariant functor from the category of CW-complexes with a cocompact action of a group G and G-homotopy classes of cellular maps to the category of Hilbert G-modules with bounded G-equivariant maps.*

Proof. Let $f : Y \to Z$ be a cellular G-equivariant map between CW-complexes. It induces a morphism $f_* : K_*(Y) \to K_*(Z)$, whose extension $\tilde{f}_* : C_*(Y,\ell^2(G)) \to C_*(Z,\ell^2(G))$ is bounded and G-equivariant. By continuity, $f_i(\overline{B_{i+1}(Y,\ell^2(G))}) = \overline{B_{i+1}(Z,\ell^2(G))}$, so f induces a well-defined map ${}^{red}H_i(\tilde{f}) :{}^{red} H_i(Y,\ell^2(G)) \to{}^{red} H_i(Z,\ell^2(G))$.
Moreover, if $g : Y \to Z$ is G-homotopic to f via a cellular homotopy, we know that f_* and g_* are algebraically homotopic, i.e. there exists a family of G-equivariant maps $K_i : K_i(Y) \to K_{i+1}(Z)$ such that $K_i \circ d^Y_{i+1} \pm d^Z_{i+1} \circ K_i = f_{i+1} - g_{i+1}$ for every i. Their extensions define a bounded G-equivariant homotopy between $\tilde{f}_*$ and $\tilde{g}_*$. Therefore ${}^{red}H_i(\tilde{f}) ={}^{red} H_i(\tilde{g})$. □

Remark 2.34. We will see (Theorem 3.22) that ℓ^2-Betti numbers can be computed via singular homology, as well. This will imply that they are independent of the cellular structure. Moreover, they are homotopy invariants, where the homotopy needn't be cellular.

Let us now calculate explicitly ${}^{red}H_0(Y,\ell^2(G))$. We need the following lemma:

Lemma 2.35. *If G is an infinite group, then the only G-invariant element of $\ell^2(G)^n$ is the trivial one.*

Proof. Set $F = (f_1,\dots,f_n) \in \ell^2(G)^n$. Since F is G-invariant iff every f_i is G-invariant, it is sufficient to prove the thesis when $n = 1$. If $\sum_{x\in G} f(x)x \in \ell^2(G)$ is G-invariant, then $f(x)$ does not depend on x. On the other hand, by definition, $\sum_{x\in G} f(x)^2 < \infty$. Therefore, $f(x) = 0$ for every $x \in G$. □

Proposition 2.36. *Let Y be a connected CW-complex endowed with a cellular cocompact action of an infinite group G. Then*

$${}^{red}H_0(Y,\ell^2(G)) = 0 \ .$$

Proof. By definition ${}^{red}H_0(Y,G) = C_0(Y,\ell^2(G))/\overline{\mathrm{Im}(d_1)} \cong \mathrm{Im}(d_1)^\perp$. We want to show that $\mathrm{Im}(d_1)^\perp \subseteq C_0(Y,\ell^2(G))^G$, the submodule of G-invariant ℓ^2-chains.

If $\sum_{\mu=1}^{\alpha_0} f_\mu \otimes \bar{\tau}_0^\mu \in \mathrm{Im}(d_1)^\perp$, then $\forall\ g, x \in G$ and $\forall\ \beta \in \{1, \ldots, \alpha_0\}$ we have

$$\begin{aligned}
0 &= \sum_{\mu=1}^{\alpha_0} \langle f_\mu \otimes \bar{\tau}_0^\mu, x \otimes (1 - g^{-1})\bar{\tau}_0^\beta \rangle \\
&= \langle \sum_{\mu=1}^{\alpha_0} f_\mu \otimes \bar{\tau}_0^\mu, x \cdot (1 - g^{-1}) \otimes \bar{\tau}_0^\beta \rangle \\
&= \langle \sum_{\mu=1}^{\alpha_0} f_\mu \cdot (1 - g) \otimes \bar{\tau}_0^\mu, x \otimes \bar{\tau}_0^\beta \rangle \\
&= \langle f_\beta \cdot (1 - g), x \rangle = f_\beta \cdot (1 - g)(x) = f_\beta(x) - (f_\beta \cdot g)(x) \ ,
\end{aligned}$$

which implies that $\sum_{\mu=1}^{\alpha_0} f_\mu \otimes \bar{\tau}_0^\mu$ is G-invariant.
As a consequence of the previous lemma, ${}^{red}H_0(Y, \ell^2(G)) \subseteq C_0(Y, \ell^2(G))^G = \{0\}$. □

Example 2.37. Let $Y = \mathbb{R}$ be endowed with the action of $\mathbb{Z}$ by translations. It is well known that $X = Y/G = S^1$ has a cellular structure with only one 0-cell e_0 and one 1-cell e_1. We obtain the ℓ^2-chain complex

$$0 \to \ell^2(\mathbb{Z}) \xrightarrow{d_1} \ell^2(\mathbb{Z}) \to 0 \ .$$

If $x \in \mathbb{Z}$ is a generator, then we can write an element $f \in \ell^2(\mathbb{Z})$ as $f = \sum_{n \in \mathbb{Z}} a_n x^n$, where a_n are real numbers. With this identification

$$d_1(f) = (1 - x) \sum_{n \in \mathbb{Z}} a_n x^n \ .$$

It is easy to verify that d_1 is injective and $\mathrm{Im}(d_1)$ is dense. Therefore,

$${}^{red}H_1(\mathbb{R}, \ell^2(\mathbb{Z})) = 0 \quad \text{and} \quad {}^{red}H_0(\mathbb{R}, \ell^2(\mathbb{Z})) = 0 \ .$$

2.2.2 ℓ^2-Betti numbers

Let Y be a connected CW-complex endowed with a free, cellular and cocompact action of a group G.

Definition 2.38. The i-th ℓ^2-Betti number of Y with respect to G is

$$b_i^{(2)}(Y, G) = \dim_G^{red} H_i(Y, \ell^2(G)) \ .$$

We are particularly interested in the case where Y is the universal covering of a finite CW-complex X and $G = \pi_1(X)$. In this situation we set $b_i^{(2)}(Y, G) := b_i^{(2)}(X)$.
The properties of the Von Neumann dimension give us some information on the ℓ^2-Betti numbers of X, which are summarised in the following propositions:

Proposition 2.39. *Let X be a finite n-dimensional CW-complex with fundamental group G. Let α_i be the number of its i-cells and Y be its universal covering.*

(1) $b_i^{(2)}(X) \leq \alpha_i$;

(2) if $\tilde{X}$ *is an* m*-sheet covering of* X*, then* $b_i^{(2)}(\tilde{X}) = m \cdot b_i^{(2)}(X)$;

(3) $\chi(X) = \sum_{i=0}^{n}(-1)^i b_i^{(2)}(X)$*, where* $\chi(X)$ *is the Euler characteristic of* X;

(4) $\sum_{i=0}^{k}(-1)^{k-i}\alpha_i \geq \sum_{i=0}^{k}(-1)^{k-i} b_i^{(2)}(X)$;

(5) if G *is infinite, then* $b_0^{(2)}(X) = 0$;

(6) if G *is finite, then* $b_i^{(2)}(X) = \frac{1}{|G|} b_i(Y)$.

Proof. (1) Since $\mathrm{Ker}(\tilde{d}_i) \subset C_i(Y, \ell^2(G))$, we have $\dim_G(\mathrm{Ker}(\tilde{d}_i)) \leq \dim_G C_i(Y, \ell^2(G)) = \alpha_i$, because $C_i(Y, \ell^2(G)) \cong (\ell^2(G))^{\alpha_i}$. The additivity of the Von Neumann dimension on the weak-exact complex (Lemma 2.28) applied to

$$0 \to \mathrm{Im}(\tilde{d}_{i+1}) \to \mathrm{Ker}(\tilde{d}_i) \to^{red} H_i(Y, \ell^2(G)) \to 0$$

implies that $b_i^{(2)}(X) \leq \dim_G \mathrm{Ker}(\tilde{d}_i) \leq \alpha_i$.

(2) Let H be a subgroup of G associated to the covering $\tilde{X}$, so that $\tilde{X} = Y/H$. H has index m in G. Therefore,

$$b_i^{(2)}(\tilde{X}) = \dim_H^{red} H_i(Y, \ell^2(G)) = [G : H] \dim_G^{red} H_i(Y, \ell^2(G)) = m \cdot b_i^{(2)}(X) \ .$$

(3) It follows from Corollary 2.29 applied to the cellular ℓ^2-chain complex of Y.

(4) Recalling the orthogonal decomposition $C_i(Y, \ell^2(G)) = B^i(Y, \ell^2(G)) \perp B_i(Y, \ell^2(G)) \perp \mathcal{H}_i(Y, \ell^2(G))$ and the isomorphism $B_i(Y, \ell^2(G)) \cong B^{i+1}(Y, \ell^2(G))$, we have

$$\begin{aligned}
\sum_{i=0}^{k}(-1)^{k-i}\alpha_i - \sum_{i=0}^{k}(-1)^{k-i}\beta_i(X) &= \sum_{i=0}^{k}(-1)^{k-i} \dim_G[B_i(Y, \ell^2(G)) \perp B^i(Y, \ell^2(G))] \\
&= \sum_{i=0}^{k}(-1)^{k-i} \dim_G[B_i(Y, \ell^2(G)) \oplus B_{i-1}(Y, \ell^2(G))] \\
&= \sum_{i=0}^{k}(-1)^{k-i}[\dim_G B_i(Y, \ell^2(G)) + \dim_G B_{i-1}(Y, \ell^2(G))] \\
&= \dim_G B_k(Y, \ell^2(G)) \geq 0 \ .
\end{aligned}$$

(5) It follows directly from Proposition 2.36.

(6) From the properties of the Von Neumann dimension

$$b_i^{(2)}(X) = \dim_G^{red} H_i(Y, \ell^2(G)) = \frac{1}{|G|} \dim_{\mathbb{R}} H_i(Y) = \frac{1}{|G|} b_i(Y) \ .$$

□

Proposition 2.40 (Künneth formula). *Let X and Y be finite CW-complexes endowed with free cellular actions of the groups G and H respectively. Then $X \times Y$ is a finite CW-complex with a $G \times H$ free cellular action and*

$$b_i^{(2)}(X \times Y, G \times H) = \sum_{p+q=i} b_p^{(2)}(X, G) b_q^{(2)}(Y, H) \ .$$

Proof. Using the cross product, there is a $\mathbb{Z}[G \times H]$-isomorphism between the cellular complexes $K_*(X) \otimes_{\mathbb{Z}} K_*(Y)$ and $K_*(X \times Y)$. It induces an isomorphism between the ℓ^2-chain complexes $C_*(X, \ell^2(G)) \bar{\otimes}_{\mathbb{R}} C_*(Y, \ell^2(H))$ and $C_*(X \times Y, \ell^2(G \times H))$. The assertion follows from Proposition 2.21. □

We are going to prove a Poincaré Duality formula for ℓ^2-Betti numbers: in Chapter 3 we will give an alternative definition of ℓ^2-Betti numbers, which will imply that they are indipendent of the cellular structure (Theorem 3.22). Therefore, we will not describe any particular cellular structure on the manifold involved in the next proposition.

Proposition 2.41 (Poincaré Duality). *Let M be an n-manifold without boundary endowed with a free, proper and cocompact action of a group G, such that M and M/G are orientable. Then*

$$b_i^{(2)}(M, G) = b_{n-i}^{(2)}(M, G) \ .$$

Proof. It is possible to define a suitable cap product on the cellular chain complex ([20], Theorem 2.1)

$$\cap[M/G] : K^{n-*}(M) \to K_*(M) \ ,$$

which induces a homotopy equivalence between the ℓ^2-chain complexes $C^{n-*}(M, \ell^2(G))$ and $C_*(M, \ell^2(G))$. The assertion follows because

$$\dim_G H_i(C^{n-*}(M, \ell^2(G))) = \dim_G \mathcal{H}_{n-i}(M, \ell^2(G)) = b_{n-i}^{(2)}(M, G) \ .$$

□

Example 2.42. We give the values of the ℓ^2-Betti numbers for all compact connected orientable 1- and 2-manifolds. Due to Theorem 3.22, ℓ^2-Betti numbers are independent of the cellular structure and they are homotopy invariants. Thus, we will not introduce an explicit cellular structure on 1- and 2-manifolds and we will use homotopy arguments to calculate ℓ^2-Betti numbers.
In dimension 1 there are only S^1 and the unit interval I. Since the fundamental group of S^1 is infinite and the euler characteristic is zero, we have $b_i^{(2)}(S^1) = 0$ for all $i \geq 0$. As I is contractible, the ℓ^2-Betti numbers of I concide with the classical ones.
Let F_g^d be the orientable closed surface of genus g with d discs removed. Obviously, $b_i^{(2)}(F_g^d) = 0$ for $i \geq 3$. If $g = 0$ and $d = 0, 1$, F_g^d is simply connected, hence the ℓ^2-Betti numbers coincide with the classical ones. If $d > 0$, F_g^d is homotopy equivalent to a bouquet of $2g + d - 1$ circles, hence $b_2^{(2)}(F_g^d) = 0$. Moreover, $b_0^{(2)}(F_g^d) = 0$ because the fundamental group is infinite and $b_1^{(2)}(F_g^d) = d + 2(g - 1)$ is the opposite of the Euler characteristic (Proposition 2.39). In the closed case with $g > 0$, the 0-th ℓ^2 Betti number

is zero, because the fundamental group is infinite. By Poincaré duality, the second ℓ^2-Betti-number vanishes, as well. By Proposition 2.39, the first ℓ^2-Betti number must coincide with the opposite of the Euler characteristic. To sum up,

$$b_0^{(2)}(F_g^d) = \begin{cases} 1 & if\ g = 0\ d = 0,1 \\ 0 & otherwise \end{cases}$$

$$b_1^{(2)}(F_g^d) = \begin{cases} 0 & if\ g = 0\ d = 0,1 \\ d + 2(g-1) & otherwise \end{cases}$$

$$b_2^{(2)}(F_g^d) = \begin{cases} 1 & if\ g = 0\ d = 0 \\ 0 & otherwise \end{cases}$$

Chapter 3

A vanishing result for the ℓ^2-Betti numbers

A conjecture by Gromov suggests a connection between the vanishing of the simplicial volume of a closed, connected, oriented and aspherical n-manifold M and the vanishing of its ℓ^2-Betti numbers. More precisely, if M is a closed, connected, oriented and aspherical manifold with $\|M\| = 0$, then $b_k^{(2)}(M) = 0$ for every $k \geq 0$. The conjecture is trivial in dimension 2 and it has been proved in dimension 3. On the other hand, little is known for $n \geq 4$. An important step in this field has been made by Schmidt, who proved (in [17]) that if M is a closed, connected and oriented manifold, then $b_k^{(2)}(M) \leq \binom{n+1}{k} \mid M \mid$. In order to explain this result, we will introduce a particular Von Neumann algebra related to the orbit equivalence relation of the fundamental group acting on the universal covering and study some algebraic properties of modules over Von Neumann algebras.

3.1 Dimension theory for modules over a Von Neumann algebra

In Section 2.1 we defined the notion of dimension for Hilbert G-modules, which are particular examples of finitely generated modules over a Von Neumann algebra (in that case $\mathcal{N}(G)$). In this section we generalise the theory so that we can deal with not necessarily finitely generated modules over an arbitrary Von Neumann algebra. The construction is purely algebraic and can be applied to every ring.

Definition 3.1. Given a (real) Hilbert space H, we endow the space of linear and bounded operators $B(H, H)$ of H into itself with the topology generated by the semi-norms $\eta_{x,y}(f) = \langle f(x), y\rangle_H$. A Von Neumann algebra $\mathcal{N}$ is a closed $*$-subalgebra of $B(H, H)$ containing the identity. The involution * assigns to an element $f \in \mathcal{N}$ its adjoint $f^* \in \mathcal{N}$.

Definition 3.2. Let $\mathcal{N}$ be a Von Neumann algebra. A finite trace on $\mathcal{N}$ is a linear map $\mathrm{trace}_{\mathcal{N}} : \mathcal{N} \to \mathbb{R}$ satisfying the following properties:

i) $\text{trace}_{\mathcal{N}}(ab) = \text{trace}_{\mathcal{N}}(ba)$;

ii) if $a \in \mathcal{N}$ is positive (i.e. $\langle a(x), x\rangle_H \geq 0$ for every $x \in H$) then $\text{trace}(a) = 0$ if and only if $a = 0$;

iii) let $\{a_i\}_{i\in I} \in \mathcal{N}$ be a directed family of positive operators such that if $i < j$, then $a_i < a_j$. If $\{a_i\}_{i\in I}$ converges to $a \in \mathcal{N}$, then

$$\text{trace}_{\mathcal{N}}(a) = \sup_{i\in I}\{\text{trace}_{\mathcal{N}}(a_i)\} \ .$$

A Von Neumann algebra is finite if it admits a finite trace.

Remark 3.3. The same theory can be developed for complex Hilbert spaces. In that case a finite trace is required to be $\mathbb{C}$-linear. The extension to complex numbers is useful to prove that $\text{trace}_{\mathcal{N}}(x^*y) = \text{trace}_{\mathcal{N}}(y^*x)$ for every $x, y \in H$: namely, by linearity and the relation

$$4x^*y = (y+x)^*(y+x) - (y-x)^*(y-x) + i(y+ix)^*(y+ix) - i(y-ix)^*(y-ix) \ ,$$

it is sufficient to show that $\text{trace}_{\mathcal{N}}(x^*x) = \text{trace}_{\mathcal{N}}(xx^*)$ for every $x \in H$. By polar decomposition, $x = sq$ where s is self-adjoint and $q^*q = Id$. Therefore,

$$\text{trace}_{\mathcal{N}}(x^*x) = \text{trace}_{\mathcal{N}}(q^*s^2q) = \text{trace}_{\mathcal{N}}(s^2) = \text{trace}_{\mathcal{N}}(xx^*) \ .$$

The trace defined in Section 2.1 satisfies these properties (Proposition 2.13), so the Von Neumann algebra $\mathcal{N}(G)$ is finite. Actually, it can be proved that every finite Von Neumann algebra is of this form:

Theorem 3.4. *Let $\mathcal{N}$ be a finite Von Neumann algebra. Let $\ell^2(\mathcal{N})$ be the completion of $\mathcal{N}$ with respect to the metric induced by the scalar product $\langle x, y\rangle = \text{trace}_{\mathcal{N}}(x^*y)$. Then $\ell^2(\mathcal{N})$ is a left $\mathcal{N}$-module on which $\mathcal{N}$ acts by left translations. Moreover, the map*

$$\mathcal{N} \to B(\ell^2(\mathcal{N}), \ell^2(\mathcal{N}))^{\mathcal{N}}$$

assigning to an element $a \in \mathcal{N}$ the right moltiplication by a is an isometric homomorphism.

The first step to generalise the dimension theory introduced in Section 2.1 consists in replacing the Hilbert structure of a G-module with a purely algebraic property. We will show that the Hilbert G-modules introduced in Definition 2.3 correspond to the finitely generated projective $\mathcal{N}(G)$-modules.
Let us begin with free (hence projective) finitely generated $\mathcal{N}(G)$-modules. We define a correspondence τ as follows:

- $\tau(\mathcal{N}(G)^n) = \ell^2(G)^n$;

- given an $\mathcal{N}(G)$-homomorphism, i.e. a linear $\mathcal{N}(G)$-equivariant map, $f : \mathcal{N}(G)^m \to \mathcal{N}(G)^n$, choose a matrix $A \in M(m, n, \mathcal{N}(G))$ such that $f(x) = xA$ for every $x \in \mathcal{N}(G)^m$. We define
$$\tau(f) : \ell^2(G)^m \to \ell^2(G)^n$$
$$y \mapsto ((A^t)^* y^t)^t \ .$$

We sketch briefly how this correspondence can be extended to every finitely generated projective $\mathcal{N}(G)$-module.

Definition 3.5. Let P be a finitely generated projective $\mathcal{N}(G)$-module. An inner product on P is a map $\mu : P \times P \to \mathcal{N}(G)$ satisfying the following properties

i μ is $\mathcal{N}(G)$-linear in the first variable;

ii $\mu(x, y) = \mu(y, x)^*$ for every $x, y \in P$;

iii $\mu(x, x)$ is positive and $\mu(x, x) = 0$ if and only if $x = 0$;

iv the map $\bar{\mu} : P \to \mathrm{Hom}_{\mathcal{N}(G)}(P, \mathcal{N}(G))$ given by $\bar{\mu}(x)(y) = \mu(x, y)$ is bijective.

The following result holds:

Proposition 3.6. *Every finitely generated projective $\mathcal{N}(G)$-module admits an inner product. Moreover, two finitely generated projective $\mathcal{N}(G)$-modules endowed with an inner product are isometrically isomorphic if and only if they are $\mathcal{N}(G)$-isomorphic.*

Example 3.7. The standard inner product μ_{st} on the free $\mathcal{N}(G)$-module $\mathcal{N}(G)^n$ is defined as
$$\mu_{st} : \mathcal{N}(G)^n \times \mathcal{N}(G)^n \to \mathcal{N}(G)^n$$
$$(x, y) \mapsto \sum_{i=1}^{n} x_i y_i^*$$

Given two finitely generated projective $\mathcal{N}(G)$-modules endowed with inner products (P_0, μ_0) and (P_1, μ_1), we can define an involution

$$\mathrm{Hom}_{\mathcal{N}(G)}(P_0, P_1) \to \mathrm{Hom}_{\mathcal{N}(G)}(P_1, P_0)$$
$$f \mapsto f^*$$

where f^* is the unique homomorphism such that $\mu_1(f(x), y) = \mu_0(x, f^*(y))$ for every $x \in P_0$ and $y \in P_1$.

Given a finitely generated projective $\mathcal{N}(G)$-module P endowed with an inner product μ, we can define on P a pre-Hilbert structure by setting

$$P \times P \to \mathbb{R}$$
$$(x, y) \mapsto \mathrm{trace}_{\mathcal{N}(G)}(\mu(x, y))$$

We denote with $\tau(P, \mu)$ the associated Hilbert space.

Example 3.8. The Hilbert space corresponding to $(\mathcal{N}(G)^n, \mu_{st})$ is exactly $\ell^2(G)^n$ endowed with the scalar product introduced in Section 2.1.

The left G-action on P, given by

$$\begin{aligned} G \times P &\to P \\ (g, x) &\mapsto g \cdot x = R_{g^{-1}} x \end{aligned}$$

where $R_{g^{-1}} \in \mathcal{N}(G)$ is the right translation on $\ell^2(G)$ induced by the element g^{-1}, induces a G-action on $\tau(P, \mu)$. Then $\tau(P, \mu)$ is a Hilbert G-module: namely, by projectivity and Proposition 3.6 there exists another finitely generated projective $\mathcal{N}(G)$-module P' endowed with an inner product μ' such that $(P, \mu) \oplus (P', \mu') \cong (\mathcal{N}(G)^n, \mu_{st})$.
In addition, every $\mathcal{N}(G)$-homomorphism $f : (P, \mu) \to (P', \mu')$ between finitely generated projective $\mathcal{N}(G)$-modules with inner product extends to a morphism of Hilbert G-modules $\tau(f) : \tau(P, \mu) \to \tau(P', \mu')$.

Theorem 3.9. *([12], Theorem 6.24) The construction above defines an equivalence τ from the category of finitely generated projective $\mathcal{N}(G)$-modules to the category of Hilbert G-modules. Moreover, denoting with τ^{-1} its inverse, both τ and τ^{-1} are exact functors and commute with the adjoint.*

Definition 3.10. The Von Neumann dimension of a finitely generated projective $\mathcal{N}(G)$-module P is

$$\dim_{\mathcal{N}(G)} P = \dim_G(\tau(P))$$

Actually, there is a more practical definition for the dimension of a finitely generated projective $\mathcal{N}(G)$-module P. By projectivity there exists an $\mathcal{N}(G)$-homomorphism $q : \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ such that $q \circ q = q$ and $\mathrm{Im}(q) = P$. Let $p : \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ be such that the corresponding map $\tau(p)$ is the orthogonal projection onto $\tau(P)$. Then $p \circ p = p$, $p^* = p$ and $\mathrm{Im}(p) = \mathrm{Im}(q)$. We denote with $A = (a_{i,j})_{i,j}$ a matrix with coefficients in $\mathcal{N}(G)$ such that $p(x) = xA$ for every $x \in \mathcal{N}(G)^n$. By definition, $\tau(p)$ is represented by the same matrix A, thus

$$\dim_G(\tau(P)) = \mathrm{trace}_G(A) \ .$$

Therefore, we could define $\dim_{\mathcal{N}(G)} P = \mathrm{trace}_G(A)$, where $A \in M_n(\mathcal{N}(G))$ is a matrix such that $A^2 = A$, $A^* = A$ and $P \cong \mathcal{N}(G)^n A$. This definition is independent of the choice of the matrix A: suppose $B \in M_m(\mathcal{N}(G))$ is another matrix satisfying the same properties. By possibly taking the direct sum with a zero-square matrix we can achieve that $n = m$ without changing their traces and their images. We denote with r_A the right moltiplication by the matrix A so that $P = \mathrm{Im}(r_A)$. Let $C \in M_n(\mathcal{N}(G))$ be an invertible matrix such that r_C maps $\mathrm{Im}(r_A)$ to $\mathrm{Im}(r_B)$ and $\mathrm{Im}(r_{1-A})$ to $\mathrm{Im}(r_{1-B})$. Then $r_B \circ r_C = r_B \circ r_C \circ r_A = r_C \circ r_A$ and hence $CBC^{-1} = A$. This implies

$$\mathrm{trace}_G(B) = \mathrm{trace}_G(C^{-1}CB) = \mathrm{trace}_G(CBC^{-1}) = \mathrm{trace}_G(A) \ .$$

This suggests how to define the dimension of a finitely generated projective $\mathcal{N}$-module P over an arbitrary finite Von Neumann algebra: choose a matrix $A \in M_k(\mathcal{N})$ such that $A^2 = A$, $A^* = A$ and $P \cong \mathcal{N}^k A$. Then

$$\dim_{\mathcal{N}} P = \mathrm{trace}_{\mathcal{N}}(A) \ .$$

Definition 3.11. Let $M \subset N$ be a submodule of the R-module N. The closure of M in N is the R-submodule

$$\overline{M} = \{x \in N \mid f(x) = 0 \ \forall \ f \in \mathrm{Hom}_R(N, R) \ s.t. \ M \subset \mathrm{Ker}(f)\} \ .$$

We define

$$\mathbb{T}M := \{x \in M \ | f(x) = 0 \ \forall \ f \in \mathrm{Hom}_R(N, R)\} = \overline{\{0\}}$$

$$\mathbb{P}M := \frac{M}{\mathbb{T}M} \ .$$

The dimension defined above satisfies the following key properties, which are essential to extend the notion of dimension to arbitrary modules:

Theorem 3.12. *([12], Theorem 6.5) Let Q be a finitely generated projective $\mathcal{N}$-module. Then*

1) if P is another finitely generated projective $\mathcal{N}$-module such that $P \cong Q$ then $\dim_{\mathcal{N}} P = \dim_{\mathcal{N}} Q$;

2) if P is another finitely generated projective $\mathcal{N}$-module then $\dim_{\mathcal{N}} P \oplus Q = \dim_{\mathcal{N}} P + \dim_{\mathcal{N}} Q$;

3) if $K \subset Q$ is a submodule, then the closure $\overline{K}$ is a direct summand in Q and

$$\dim_{\mathcal{N}} \bar{K} = \sup\{\dim_{\mathcal{N}} P \mid P \subset K \text{ finitely generated and projective}\} \ .$$

We can now extend the dimension theory to every $\mathcal{N}$-module. This procedure can be applied to modules over an arbitrary ring R, provided one can define a dimension over finitely generated projective R-modules satisfying Theorem 3.12.

Definition 3.13. Let M be an $\mathcal{N}$-module. The extended dimension of M is

$$\dim'_{\mathcal{N}} M = \sup\{\dim_{\mathcal{N}} P \ |P \subset M, \ P \text{ is finitely generated and projective}\} \ .$$

The following proposition describes the main properties of the extended dimension.

Theorem 3.14. *1) Every finitely generated submodule of a projective $\mathcal{N}$-module is projective;*

2) if M is a finitely generated $\mathcal{N}$-module and $K \subset M$ is a submodule, then $\overline{K}$ is a direct summand of M and $M/\overline{K}$ is finitely generated and projective;

3) if M is a finitely generated $\mathcal{N}$-module, then $\mathbb{P}M$ is projective finitely generated and $M \cong \mathbb{P}M \oplus \mathbb{T}M$;

4) the extended dimension satisfies the following properties:

a) if M is a finitely generated projective $\mathcal{N}$-module, then $\dim_{\mathcal{N}} M = \dim'_{\mathcal{N}} M$;

b) (additivity) if $0 \to M_0 \xrightarrow{\iota} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of $\mathcal{N}$-modules, then $\dim'_{\mathcal{N}} M_1 = \dim'_{\mathcal{N}} M_0 + \dim'_{\mathcal{N}} M_2$;

c) (cofinality) if $M = \bigcup_{i \in I} M_i$ and for every $i, j \in I$ there exists $k \in I$ such that $M_i, M_j \subset M_k$, then $\dim'_{\mathcal{N}} M = \sup\{\dim'_{\mathcal{N}} M_i \mid i \in I\}$;

d) (continuity) if M is a finitely generated $\mathcal{N}$-module and $K \subset M$ is a submodule, then $\dim'_{\mathcal{N}} K = \dim'_{\mathcal{N}} \overline{K}$;

e) if M is a finitely generated $\mathcal{N}$-module, then

$$\dim'_{\mathcal{N}} M = \dim_{\mathcal{N}}(\mathbb{P}M) \qquad \dim'_{\mathcal{N}}(\mathbb{T}M) = 0$$

Proof. 1) Let $M \subset P$ be a finitely generated submodule of a projective $\mathcal{N}$-module P. Let $q : \mathcal{N}^n \to P$ be a homomorphism such that $\mathrm{Im}(q) = M$. It can be verified that $\mathrm{Ker}(q) = \overline{\mathrm{Ker}(q)}$, so, by Theorem 3.12, $\mathrm{Ker}(q)$ is a direct summand of $\mathcal{N}^n$. Therefore, $M \cong \mathcal{N}^n/\mathrm{Ker}(q)$ and $\mathcal{N}^n \cong M \oplus \mathrm{Ker}(q)$, hence M is projective.
2) Let $q : \mathcal{N}^n \to M$ be a surjective homomorphism. A direct computation shows that $q^{-1}(\overline{K}) = \overline{q^{-1}(K)}$ and $\mathcal{N}^n/q^{-1}(\overline{K}) \cong M/\overline{K}$. By Theorem 3.12, $\mathcal{N}^n/\overline{q^{-1}(K)}$ is a direct summand of $\mathcal{N}^n$, hence $M/\overline{K}$ is projective.
3) It follows from the previous point applied to $K = \{0\}$.
4) Let us verify the properties of the extended dimension.
a) Let $P \subset M$ be a finitely generated projective $\mathcal{N}$-submodule. By Theorem 3.12, $\overline{P}$ is a direct summand of M and

$$\dim_{\mathcal{N}} P \leq \dim_{\mathcal{N}} \overline{P} = \dim_{\mathcal{N}} M - \dim_{\mathcal{N}}(M/\overline{P}) \leq \dim_{\mathcal{N}} M \ .$$

Since M is finitely generated and projective we have

$$\dim_{\mathcal{N}} M \leq \dim'_{\mathcal{N}} M = \sup_{P \subset M} \dim_{\mathcal{N}} P \leq \dim_{\mathcal{N}} M \ .$$

b) Let $P \subset M_2$ be a finitely generated projective submodule. We have a short exact sequence

$$0 \to M_0 \to p^{-1}(P) \to P \to 0$$

and by projectivity $p^{-1}(P) \cong P \oplus M_0$. Therefore,

$$\dim'_{\mathcal{N}} M_0 + \dim_{\mathcal{N}} P \leq \dim'_{\mathcal{N}} p^{-1}(P) \leq \dim'_{\mathcal{N}} M_1 \ .$$

Since this relation holds for every finitely generated and projective submodule $P \subset M_2$, we have

$$\dim'_{\mathcal{N}} M_0 + \dim'_{\mathcal{N}} M_2 \leq \dim'_{\mathcal{N}} M_1 \ .$$

Let $Q \subset M_1$ be a finitely generated projective submodule. Let $\overline{\iota(M_0) \cap Q}$ be the closure of $\iota(M_0) \cap Q$ in Q. We get the following short exact sequences

$$0 \to \iota(M_0) \cap Q \to Q \to p(Q) \to 0$$

$$0 \to \overline{\iota(M_0) \cap Q} \to Q \to Q/\overline{\iota(M_0) \cap Q} \to 0 \ .$$

By Theorem 3.12, the submodule $\overline{\iota(M_0) \cap Q}$ is a direct summand of Q, so

$$\begin{aligned}\dim_{\mathcal{N}} Q &= \dim_{\mathcal{N}}(\overline{\iota(M_0) \cap Q}) + \dim_{\mathcal{N}}(Q/\overline{\iota(M_0) \cap Q}) \\ &\leq \dim'_{\mathcal{N}}(\iota(M_0) \cap Q) + \dim'_{\mathcal{N}}(p(Q)) \\ &\leq \dim'_{\mathcal{N}}(M_0) + \dim'_{\mathcal{N}}(M_2) \ .\end{aligned}$$

c) Let $P \subset M$ be a finitely generated projective submodule. By cofinality, there exists $i \in I$ such that $P \subset M_i$. Therefore,

$$\dim'_{\mathcal{N}} M = \sup_{P \subset M} \dim_{\mathcal{N}} P \leq \sup_{i \in I} \dim'_{\mathcal{N}} M_i \leq \dim'_{\mathcal{N}} M \ .$$

d) Let $0 \to L \to \mathcal{N}^n \xrightarrow{q} M \to 0$ be a short exact sequence. Since $q^{-1}(\overline{K}) = \overline{q^{-1}(K)}$ we have

$$\begin{aligned}\dim'_{\mathcal{N}}(\overline{K}) &= \dim'_{\mathcal{N}}(q^{-1}(\overline{K})) - \dim'_{\mathcal{N}}(L) \\ &= \dim'_{\mathcal{N}}(\overline{q^{-1}(K)}) - \dim'_{\mathcal{N}}(L) \\ &= \dim'_{\mathcal{N}}(q^{-1}(K)) - \dim'_{\mathcal{N}}(L) \\ &= \dim'_{\mathcal{N}}(K) \ .\end{aligned}$$

e) If follows by the previous point applied to $K = \{0\}$ and the decomposition $M = \mathbb{P}M \oplus \mathbb{T}M$. □

Remark 3.15. In view of Theorem 3.14, we will not distinguish between $\dim_{\mathcal{N}}$ and $\dim'_{\mathcal{N}}$, when dealing with finitely generated projective modules.

Proposition 3.16 (Dimension and colimit)**.** *Let $\{M_i\}_{i \in I}$ be a directed system of $\mathcal{N}(G)$-modules over a directed set I. For $i \leq j$ let $\phi_{i,j} : M_i \to M_j$ be the associated morphism. For $i \in I$ let $\psi_i : M_i \to colim_{i \in I} M_i$ be the canonical morphism. Then*

$$\dim_{\mathcal{N}(G)}(colim_{i \in I} M_i) = \sup_{i \in I} \{\dim_{\mathcal{N}(G)}(Im(\psi_i))\} \ .$$

Proof. Recall that the colimit $\operatorname{colim}_{i \in I} M_i$ is $\bigcup_{i \in I} M_i / \sim$ where the equivalence relation is the following: $x \in M_i$ is equivalent to $y \in M_j$ if there exists $k \in I$ with $i \leq k$ and $j \leq k$ such that $\phi_{i,k}(x) = \phi_{j,k}(x)$. Therefore,

$$\operatorname{colim}_{i \in I} M_i = \bigcup_{i \in I} \operatorname{Im}(\psi_i)$$

and the thesis follows by cofinality. □

We will now prove some results about flat modules over Von Neumann algebras.

Definition 3.17. A $*$-homomorphism $f : \mathcal{N} \to \mathcal{M}$ between finite Von Neumann algebras is a homomorphism of algebras such that $f(a^*) = f(a)^*$ for every $a \in \mathcal{N}$.

Definition 3.18. A $*$-homomorphism $f : \mathcal{N} \to \mathcal{M}$ between finite Von Neumann algebras is trace preserving if

$$\mathrm{trace}_{\mathcal{M}}(f(x)) = \mathrm{trace}_{\mathcal{N}}(x) \qquad \forall\, x \in \mathcal{N}$$

Lemma 3.19. *([16], Theorem 1.48) A trace preserving $*$-homomorphism $f : \mathcal{N} \to \mathcal{M}$ between finite Von Neumann algebras is flat, i.e $\mathcal{M}$ is a flat $\mathcal{N}$-module via f.*

Theorem 3.20. *Let $\phi : \mathcal{N} \to \mathcal{M}$ be a trace preserving $*$-homomorphism between finite Von Neumann algebras. Then, for every $\mathcal{N}$-module N we have*

$$\dim_{\mathcal{N}} N = \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} N)\ .$$

Proof. We start checking that the result holds when N is a projective finitely generated $\mathcal{N}$-module. Let $A \in M_n(\mathcal{N})$ be such that $A^2 = A$, $A^* = A$ and $N \cong \mathcal{N}^n A$. Then $\mathcal{M} \otimes_{\mathcal{N}} N \cong \mathcal{M}^n \phi(A)$ as $\mathcal{N}$-modules. Therefore,

$$\dim_{\mathcal{N}} N = \sum_{i=1}^{n} \mathrm{trace}_{\mathcal{N}}(a_{i,i}) = \mathrm{trace}_{\mathcal{M}}(\phi(a_{i,i})) = \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} N)\ .$$

We will now deduce that the thesis holds for finitely presented $\mathcal{N}(G)$-modules. By Theorem 3.14, we can decompose $N = \mathbb{P}N \oplus \mathbb{T}N$ and there exists a short exact sequence

$$0 \to P \to \mathcal{N}^n \to \mathbb{T}N \to 0\ ,$$

where P is finitely generated and projective. By the previous lemma $\mathcal{M}$ is a flat $\mathcal{N}$-module, hence when tensoring with $\mathcal{M}$ the sequence remains exact and by additivity $\dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} \mathbb{T}N) = n - \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} P) = n - \dim_{\mathcal{N}}(P) = 0$. Therefore,

$$\begin{aligned} \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} N) &= \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} \mathbb{P}N) + \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} \mathbb{T}N) \\ &= \dim_{\mathcal{N}}(\mathbb{P}N) \end{aligned}$$

where the last equality holds because $\mathbb{P}N$ is a finitely generated projective $\mathcal{N}$-module. Suppose now that N is finitely generated. There exists a short exact sequence

$$0 \to K \to P \to N \to 0$$

where P is a finitely generated projective $\mathcal{N}$-module. We write $K = \mathrm{colim}_{i \in I} K_i$ as colimit of its finitely generated submodules. Then

$$\begin{aligned} \dim_{\mathcal{N}} N &= \dim_{\mathcal{N}} P - \dim_{\mathcal{N}} K = \dim_{\mathcal{N}} P - \sup_{i \in I} \dim_{\mathcal{N}} K_i \\ &= \inf_{i \in I}(\dim_{\mathcal{N}} P - \dim_{\mathcal{N}} K_i) = \inf_{i \in I} \dim_{\mathcal{N}}(P/K_i)\ . \end{aligned}$$

Since the tensor product preserves colimits, we have $\mathcal{M} \otimes_{\mathcal{N}} K = \mathrm{colim}_{i \in I}(\mathcal{M} \otimes_{\mathcal{N}} K_i)$. As a consequence, the same computation shows that

$$\begin{aligned}\dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} N) &= \inf_{i \in I} \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} P/K_i) = \inf_{i \in I}(\dim_{\mathcal{N}}(P/K_i)) \\ &= \dim_{\mathcal{N}} N\end{aligned}$$

where the last inequality holds because P/K_i is finitely presented.
By writing an $\mathcal{N}$-module N as the colimit of its finitely generated submodules, a similar computation implies the thesis. □

The extended dimension enables us to define ℓ^2-Betti numbers in terms of singular homology. Therefore, one could deal with CW-complexes endowed with the action of a group G, which is not necessarily cocompact. Actually, we are only interested in the case of the universal covering of a closed, connected and oriented manifold M with the action of the fundamental group $G = \pi_1(M)$, but, in order to obtain an upper bound for the ℓ^2-Betti numbers, we will use singular homology. We recall here the main notions we will use in the next section.

Let X be a CW-complex endowed with a free cellular action of a group G. The singular chain complex $(C_*^{sing}(X), \partial_*)$ consists of $\mathbb{Z}[G]$-modules, as G acts on a singular simplex by left translation, and the boundary operators are G-equivariant.

Definition 3.21. The p-th (singular) ℓ^2-Betti number of X is

$$b_p^{(2)}(X, \mathcal{N}(G)) := \dim_{\mathcal{N}(G)} H_p((\mathcal{N}(G) \otimes_G C_*^{sing}(X), id_{\mathcal{N}(G)} \otimes \partial_*))$$

The following result ensures that this definition is equivalent to Definition 2.38 in case the action of G is free and cocompact:

Theorem 3.22. *Let X be a free G-CW-complex of finite type. Then*

$$\dim_{\mathcal{N}(G)} H_p(X, \mathcal{N}(G)) = \dim_G^{red} H_p(X, \ell^2(G))$$

Proof. It is well-known that there exists a chain map

$$f : C_*^{sing}(X) \to C_*^{cell}(X)$$

which induces isomorphisms in homology. Defining

$$\mathrm{Cone}(f)_n = C_{n-1}^{sing}(X) \oplus C_n^{cell}(X)$$

with the boundary operator $d_n : \mathrm{Cone}(f)_n \to \mathrm{Cone}(f)_{n-1}$ given by

$$d_n(c, c') = (-\partial_{n-1}^{sing}(c), f(c) + \partial_n^{cell}(c')) ,$$

we obtain a chain complex of free left $\mathbb{Z}[G]$-modules. It fits in the exact sequence

$$0 \to C_*^{cell}(X) \to \mathrm{Cone}(f)_* \to C_{*-1}^{sing}(X) \to 0 .$$

Studying the corresponding long exact sequence in homology, we obtain that $\mathrm{Cone}(f)_*$ has trivial homology. Since it consists of projective modules, it is also contractible. This implies that f is a chain homotopy equivalence. By tensoring the involved maps, the same holds for

$$id_{\mathcal{N}(G)} \otimes f : \mathcal{N}(G) \otimes_G C_*^{sing}(X) \to \mathcal{N}(G) \otimes_G C_*^{cell}(X) \ ,$$

thus we obtain an isomorphism $H_p(\mathcal{N}(G) \otimes_G C_*^{cell}) \xrightarrow{\cong} H_p(\mathcal{N}(G) \otimes_G C^{sing}(X))$. In addition, there exists an isomorphism

$$h_p : \tau^{-1}({}^{red}H_p(X, \ell^2(G))) \to \mathbb{P}H_p(X, \mathcal{N}(G))$$

defined by the following diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\tau^{-1}({}^{red}H_p(X,\ell^2(G))) & \xrightarrow{h_p} & \mathbb{P}H_p(X,\mathcal{N}(G)) \\
\uparrow{\scriptstyle \tau^{-1}(q)} & & \uparrow{\scriptstyle q} \\
\tau^{-1}(\mathrm{Ker}(id_{\ell^2} \otimes \partial_p)) & \to & \mathrm{Ker}(id_{\mathcal{N}(G)} \otimes \partial_p) \\
\uparrow{\scriptstyle \tau^{-1}(j)} & & \uparrow{\scriptstyle j} \\
\tau^{-1}(\overline{\mathrm{Im}(id_{\ell^2} \otimes \partial_{p+1})}) & \to & \overline{\mathrm{Im}(id_{\mathcal{N}(G)} \otimes \partial_{p+1})} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

The columns are exact, the middle and lower arrows are isomorphisms and h_p is an isomorphism by the five lemma. Therefore,

$$\begin{aligned}
\dim_{\mathcal{N}(G)} H_p(X, \mathcal{N}(G)) &= \dim_{\mathcal{N}(G)} \mathbb{P}H_p(X, \mathcal{N}(G)) \\
&= \dim_{\mathcal{N}(G)} \tau^{-1}({}^{red}H_p(X, \ell^2(G))) \\
&= \dim_G^{red} H_p(X, \ell^2(G)) \ .
\end{aligned}$$

□

3.2 An upper bound for the ℓ^2-Betti numbers

In this section we will establish a connection between the integral foliated simplicial volume of a closed, connected and oriented manifold M and its ℓ^2-Betti numbers. It is based on a generalised cap product which induces a sort of Poincaré isomorphism in (co)-homology with coefficients in the Von Neumann algebra $\mathcal{NR}_{G\curvearrowright X}$.

3.2.1 The Von Neumann algebra $\mathcal{NR}_{G\curvearrowright X}$

We are going to define the Von Neumann algebra related to the orbit equivalence relation of a group G acting on a standard G-space. The construction is analogue to that of $\mathcal{N}(G)$.

We recall this general result which will be useful later:

Theorem 3.23. *([16], Theorem 1.3) Let $f : X \to Y$ be a measurable map between standard Borel spaces which is countable-to-one, i.e. for each $y \in Y$ the preimage $f^{-1}(y)$ is countable. Then the image $f(X)$ is measurable and there is a countable partition $(X_n)_{n\in\mathbb{N}}$ of X by measurable subsets X_n, such that $f_{|X_n}$ is injective and a Borel isomorphism onto $f(X_n)$ for every $n \in \mathbb{N}$.*

Definition 3.24. Let (X, μ) be a standard probability space. A standard equivalence relation $\mathcal{R}$ on X is an equivalence relation $\mathcal{R} \subset X \times X$ such that

i) $\mathcal{R}$ is a measurable subset of $X \times X$;

ii) the equivalence classes of $\mathcal{R}$ are countable;

iii) For each Borel isomorphism $\phi : A \to B$ between measurable subsets $A, B \subset X$ such that $(a, \phi(a)) \in \mathcal{R}$ for every $a \in A$ one has $\mu(A) = \mu(B)$.

The main example is given by the orbit equivalence relation of a countable group G acting on a standard G-space.

Lemma 3.25. *Let (X, μ) be a standard G-space. The orbit equivalence relation*

$$\mathcal{R}_{G\curvearrowright X} = \{(x, gx) \mid x \in X \ , \ g \in G\}$$

is a standard equivalence relation.

Proof. For a fixed $g_0 \in G$ the subset $\{(x, g_0x) \mid x \in X\}$ is a Borel subset, hence $\mathcal{R}_{G\curvearrowright X}$ is measurable because it is the countable union of Borel sets. The equivalence classes are countable because G is. Let $\phi : A \to B$ be a Borel isomorphism between measurable subsets $A, B \subset X$ such that $(a, \phi(a)) \in \mathcal{R}_{G\curvearrowright X}$ for every $a \in A$. Fix an enumeration of $G = \{g_1, g_2, \dots\}$. Then A is the disjoint countable union of A_n where

$$A_n = \{a \in A \mid \phi(a) = g_n a \text{ and } \phi(a) \neq g_j a \ \forall j < n\} \ .$$

Since the G-action is measure-preserving, we have $\mu(g_n A_n) = \mu(A_n)$. Therefore

$$\mu(\phi(A)) = \sum_{i=1}^{\infty} \mu(\phi(A_n)) = \sum_{i=1}^{\infty} \mu(g_n A_n) = \sum_{i=1}^{\infty} \mu(A_n) = \mu(A) \ .$$

□

The following fact is important for the definition of a somehow canonical measure on a standard equivalence relation $\mathcal{R}$.

Lemma 3.26. *Let $\mathcal{R} \subset X \times X$ be a standard equivalence relation and $A \subset \mathcal{R}$ be a measurable subset. Then there exists a partition $A = \bigcup_{n\in\mathbb{N}} A_n$ into measurable subsets A_n such that both coordinate projections are injective on each A_n.*

Proof. Since the equivalence classes are countable, the coordinate projections are countable-to-one. Hence we can apply Theorem 3.23 twice. □

We define a measure on a standard equivalence relation $\mathcal{R}$, which is induced by the measure on X in a natural way. By the previous lemma, there exists a countable partition $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ such that both coordinate projections p_1 and p_2 are injective on each $\mathcal{R}_n$. If follows that for a measurable subset $B \subset \mathcal{R}$ the function

$$\begin{aligned} X &\to \mathbb{N} \cup \{\infty\} \\ x &\mapsto \sum_{n\in\mathbb{N}} \chi_{p_1(\mathcal{R}_n \cap B)}(x) = |B \cap p_1^{-1}(x)| \end{aligned}$$

is measurable.

Definition 3.27. The measure ν on $\mathcal{R}$ is given by

$$\nu(B) = \int_X |B \cap p_1^{-1}(x)| d\mu \ .$$

Remark 3.28. We could define the measure using the coordinate projection p_2. The resulting measure would be the same: namely, the map

$$p_2 \circ p_1^{-1} : p_1(B \cap \mathcal{R}_n) \to p_2(B \cap \mathcal{R}_n)$$

is a Borel isomorphism because both coordinate projections are injective on each $\mathcal{R}_n$. Hence, by condition (iii) in the definition of a standard equivalence relation, we have $\mu(p_1(B \cap \mathcal{R}_n)) = \mu(p_2(B \cap \mathcal{R}_n))$. As a consequence,

$$\begin{aligned} \nu(B) &= \int_X |B \cap p_1^{-1}(x)| d\mu = \sum_{n\in\mathbb{N}} \int_X \chi_{p_1(\mathcal{R}_n \cap B)}(x) d\mu \\ &= \sum_{n\in\mathbb{N}} \mu(p_1(B \cap \mathcal{R}_n)) = \sum_{n\in\mathbb{N}} \mu(p_2(B \cap \mathcal{R}_n)) \\ &= \int_X |B \cap p_2^{-1}(x)| d\mu \ . \end{aligned}$$

We will now define the Von Neumann algebra of a standard equivalence relation $\mathcal{R}$.

Definition 3.29. Let $\mathcal{R}$ be a standard equivalence relation and $S = \mathbb{Z}, \mathbb{R}$ a ring. The equivalence relation ring $S[\mathcal{R}]$ is defined by

$$\begin{aligned} S[\mathcal{R}] = \{ f \in L^\infty(\mathcal{R}, S) \mid \exists\, n \in \mathbb{N} \; s.t. \\ \forall x \in X \; |\{y \in X \mid f(x,y) \neq 0\}| \leq n \; and \; |\{y \in X \mid f(y,x) \neq 0\}| \leq n \} \end{aligned}$$

where the addition is pointwise and the multiplication is given by

$$(f \cdot g)(x,y) = \sum_{z \sim x} f(x,z) g(z,y) \; .$$

Remark 3.30. There is an embedding of rings

$$\begin{aligned} j : L^\infty(X,S) \to S[\mathcal{R}] \\ f \mapsto \left((x,y) \mapsto \begin{cases} f(x) & if \; x = y \\ 0 & otherwise \end{cases} \right) \end{aligned}$$

Thereafter, we will never distinguish between f and $j(f)$, if the ring we are considering is clear from the context.

Remark 3.31. In the special case $\mathcal{R}$ is the orbit equivalence relation of a group G on a standard probability space X, the map

$$\begin{aligned} S[G] \to S[\mathcal{R}_{G \curvearrowright X}] \\ \sum_{g \in G} a_g g \mapsto \Big((gx, x) \mapsto a_g \Big) \end{aligned}$$

is an injective homomorphism of rings.

We indicate with $\ell^2(\mathcal{R})$ the Hilbert space of real square summable functions on $\mathcal{R}$ with respect to the measure ν introduced in Definition 3.27. The scalar product on $\ell^2(\mathcal{R})$ is given by

$$\begin{aligned} \langle \cdot, \cdot \rangle : \ell^2(\mathcal{R}) \times \ell^2(\mathcal{R}) \to \mathbb{R} \\ \langle \phi, \psi \rangle = \int_{\mathcal{R}} \phi(x,y) \psi(x,y) d\nu \; . \end{aligned}$$

It is a left $\mathbb{R}[\mathcal{R}]$-module, where the action is defined by

$$\begin{aligned} \rho_l : \mathbb{R}[\mathcal{R}] \to B(\ell^2(\mathcal{R}), \ell^2(\mathcal{R})) \\ \rho_l(f)(\phi)(x,y) = \sum_{z \sim x} f(x,z) \phi(z,y) \; . \end{aligned}$$

The ring $\mathbb{R}[\mathcal{R}]$ can act on $\ell^2(\mathcal{R})$ via right translations, as well

$$\begin{aligned} \rho_r : \mathbb{R}[\mathcal{R}] \to B(\ell^2(\mathcal{R}), \ell^2(\mathcal{R})) \\ \rho_r(f)(\phi)(x,y) = \sum_{z \sim x} \phi(x,z) f(z,y) \; . \end{aligned}$$

Definition 3.32. The Von Neumann algebra $\mathcal{NR}$ is the weak closure of $\rho_r(\mathbb{R}[\mathcal{R}])$ in $B(\ell^2(\mathcal{R}), \ell^2(\mathcal{R}))$. It is finite and the trace is given by

$$\begin{aligned} \mathrm{trace}_{\mathcal{NR}} : \mathcal{NR} &\to \mathbb{R} \\ T &\mapsto \langle T(\chi_{\Delta_X}), \chi_{\Delta_X} \rangle_{\ell^2(\mathcal{R})} \end{aligned}$$

where $\Delta_X \subset \mathcal{R}$ is the diagonal in $X \times X$.

Remark 3.33. In Section 2.1 we presented the definition of the Von Neumann algebra $\mathcal{N}(G)$ using a different procedure. Actually, one could verify that $\mathcal{N}(G)$ is exactly the weak closure of $\mathbb{R}[G]$ in $B(\ell^2(G), \ell^2(G))$, where $\mathbb{R}[G]$ acts on $\ell^2(G)$ via right translation. Moreover, the trace on $\mathcal{NR}$ is based on the same idea of Definition 2.9, as χ_{Δ_X} is the identity in the ring $\mathbb{R}[\mathcal{R}]$.

In the special case $\mathcal{R}$ is the orbit equivalence relation of a group G, Sauer ([16]) proved the following result:

Proposition 3.34. *There is a trace-preserving $*$-homomorphism $\mathcal{N}(G) \to \mathcal{NR}_{G \curvearrowright X}$. So $\mathcal{NR}_{G \curvearrowright X}$ is a flat $\mathcal{N}(G)$-module.*

This result implies that we can calculate the ℓ^2-Betti numbers of a G-CW-complex Z using the Von Neumann algebra $\mathcal{NR}_{G \cap X}$, where X is a standard G-space. We denote with

$$C_n(Z, \mathcal{NR}_{G \curvearrowright X}) := \mathcal{NR}_{G \curvearrowright X} \otimes_G C_n(Z)$$

the set of n-singular chains of Z with coefficients in $\mathcal{N}R_{G \curvearrowright X}$. In the definition, $C_n(Z)$ stands for the left $\mathbb{Z}[G]$-module of the integral n-singular chains and $\mathcal{NR}_{G \curvearrowright X}$ is considered as a right $\mathbb{Z}[G]$-module. Notice that $C_n(Z, \mathcal{NR}_{G \curvearrowright X})$ has a natural structure of left $\mathcal{NR}_{G \curvearrowright X}$-module given by

$$\begin{aligned} \mathcal{NR}_{G \curvearrowright X} \times \mathcal{NR}_{G \curvearrowright X} \otimes_G C_n(Z) &\to \mathcal{NR}_{G \curvearrowright X} \otimes_G C_n(Z) \\ (g, f \otimes \sigma) &\mapsto f g^* \otimes \sigma \ . \end{aligned}$$

where g^* denotes the adjoint operator.

Definition 3.35. The homology of Z with coefficients in $\mathcal{NR}_{G \curvearrowright X}$ is

$$H_*(Z, \mathcal{NR}_{G \curvearrowright X}) := H_*((C_n(Z, \mathcal{NR}_{G \curvearrowright X}), id_{\mathcal{NR}_{G \curvearrowright X}} \otimes \partial_*))$$

Proposition 3.36. *Let G be a countable group and X be a standard G-space. Let Z be a G-CW-complex. Then*

$$b_k^{(2)}(X, \mathcal{N}(G)) = \dim_{\mathcal{NR}_{G \curvearrowright X}} H_k(Z, \mathcal{NR}_{G \curvearrowright X})$$

Proof. Since $\mathcal{N}(G)$ is a subring of $\mathcal{NR}_{G \curvearrowright X}$, we have

$$\mathcal{NR}_{G \curvearrowright X} \otimes_G C_n(Z) = \mathcal{NR}_{G \curvearrowright X} \otimes_{\mathcal{N}(G)} \mathcal{N}(G) \otimes_G C_n(Z) \ .$$

By flatness, we deduce that

$$H_n(Z, \mathcal{NR}_{G \curvearrowright X}) \cong \mathcal{NR}_{G \curvearrowright X} \otimes_{\mathcal{N}(G)} H_n(Z, \mathcal{N}(G)) \ .$$

The thesis follows from Theorem 3.20. □

3.2.2 A generalised cap product

Let M be a closed, connected and oriented manifold with fundamental group G and universal covering $\tilde{M}$. Let X be a standard G-space. Let A be a ring with involution, denoted like the complex conjugation, containing $\mathbb{Z}[G]$ as a subring. Then A has a natural $\mathbb{Z}[G]$-bimodule structure. Let $B \subset A$ be a subring closed under involution such that for every $b \in B$ and $g \in G$ the product gbg^{-1} belongs to B. We explain the other module stuctures we will use:

- B has a left $\mathbb{Z}[G]$-module structure given by
$$g \star b = gbg^{-1} \ .$$
- In Chapter 2 we defined a left action of G on the set of singular n-chains $C_n(\tilde{M})$ via left translations. We can introduce a right action by setting
$$\sigma \cdot g = g^{-1} \cdot \sigma \ .$$
We will denote with $C_n^r(\tilde{M})$ the corresponding right $\mathbb{Z}[G]$-module.
- On $\mathrm{Hom}_G(C_j(\tilde{M}), A) \otimes_{\mathbb{Z}} C_n^r(\tilde{M})$ a right $\mathbb{Z}[G]$-module structure is given by
$$(\phi \otimes \sigma) \cdot g = \phi \otimes g^{-1} \cdot \sigma \ .$$
In addition, it is a left A-module with the action defined by
$$a \cdot (\phi \otimes \sigma) = \phi \cdot \bar{a} \otimes \sigma \ .$$
- On $A \otimes_{\mathbb{Z}} C_{n-j}^r(\tilde{M})$, a right $\mathbb{Z}[G]$-module structure is given by
$$(a \otimes \sigma) \cdot g = a \cdot g \otimes g^{-1} \cdot \sigma \ \ .$$

Definition 3.37. Given a cochain $\phi \in \mathrm{Hom}_G(C_j(\tilde{M}), A)$ and a chain $c \in B \otimes_G C_n(\tilde{M})$, the generalised cap product $\_ \cap \_$ is the linear map determined by the formula

$$\begin{aligned} \mathrm{Hom}_G(C_j(\tilde{M}), A) \otimes_{\mathbb{Z}} B \otimes_G C_n(\tilde{M}) &\to A \otimes_G C_{n-j}(\tilde{M}) \\ \phi \otimes b \otimes \sigma &\mapsto \overline{\phi(\sigma_{|_j})} \cdot \bar{b} \otimes_{|_{n-j}} \sigma \end{aligned}$$

where $\sigma_{|_j}$ indicates the singular j-simplex obtained by restricting σ to the front j-dimensional face of Δ^n and $_{|_{n-j}}\sigma$ denotes the singular $(n-j)$-simplex obtained by restricting σ to the back $(n-j)$-dimensional face of Δ^n.

A direct computation shows that

$$\partial_{n-j}(\phi \cap c) = (-1)^j (\delta_{j-1} \cap c - \phi \cap \partial_n(c))$$

hence the generalised cap product descends to (co)-homology.

We will apply this construction in the case $A = \mathcal{NR}_{G\curvearrowright X}$ and $B = L^\infty(X,\mathbb{Z})$. Recall that the involution in A is the map which associates to an operator $T \in \mathcal{NR}_{G\curvearrowright X}$ its adjoint $T^* \in \mathcal{NR}_{G\curvearrowright X}$. Moreover, $L^\infty(X,\mathbb{Z})$ can be seen as a subring of $\mathcal{NR}_{G\curvearrowright X}$, because, given a function $f \in L^\infty(X,\mathbb{Z})$, we defined in Remark 3.30 the element $j(f) \in \mathbb{Z}[\mathcal{R}_{G\curvearrowright X}]$, which acts on $\ell^2(\mathcal{R}_{G\curvearrowright X})$ by right translation. In addition, $L^\infty(X,\mathbb{Z})$ is closed under involution, as every element $f \in L^\infty(X,\mathbb{Z})$ is self-adjoint.

With the aid of the change of coefficient morphism $\iota_M^X : \mathbb{Z} \otimes_G C_n(\tilde{M}) \to L^\infty(X,\mathbb{Z}) \otimes_G C_n(\tilde{M})$ we can establish a relationship between the extended cap product and the classical one.

Lemma 3.38. *The diagram*

$$\begin{array}{ccc} Hom_G(C_j(\tilde{M}), \mathcal{NR}_{G\curvearrowright X}) \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_G C_n(\tilde{M}) & \xrightarrow{\_\cap\_} & \mathcal{NR}_{G\curvearrowright X} \otimes_G C_{n-j}(\tilde{M}) \\ \Big\downarrow {\scriptstyle id_{Hom}\otimes \iota_M^X} & & \Big\downarrow {\scriptstyle id} \\ Hom_G(C_j(\tilde{M}), \mathcal{NR}_{G\curvearrowright X}) \otimes_{\mathbb{Z}} L^\infty(X,\mathbb{Z}) \otimes_G C_n(\tilde{M}) & \xrightarrow{\_\cap\_} & \mathcal{NR}_{G\curvearrowright X} \otimes_G C_{n-j}(\tilde{M}) \end{array}$$

commutes, i.e for a cochain $\phi \in Hom_G(C_j(\tilde{M}), \mathcal{NR}_{G\curvearrowright X})$ and a chain $c \in \mathbb{Z} \otimes_G C_n(\tilde{M})$ one gets

$$\phi \cap c = \phi \cap \iota_M^X(c) \ .$$

Proof. Since cap products are linear, it is sufficient to prove commutativity when $c = 1 \otimes \sigma$, where $\sigma : \Delta^n \to \tilde{M}$ is a singular simplex. We get

$$\begin{aligned} \phi \cap (1 \otimes \sigma) &= \overline{\phi(\sigma_{|j})} \otimes \ {}_{|n-j}\sigma = \overline{\phi(\sigma_{|j})} \cdot \chi_{\Delta_X} \otimes \ {}_{|n-j}\sigma \\ &= \overline{\chi_{\Delta_X} \cdot \phi(\sigma_{|j})} \otimes \ {}_{|n-j}\sigma = \overline{\mathrm{const}_1 \cdot \phi(\sigma_{|j})} \otimes \ {}_{|n-j}\sigma \\ &= \phi \cap (\mathrm{const}_1 \otimes \sigma) = \phi \cap \iota_M^X(1 \otimes \sigma) \end{aligned}$$

because the characteristic function of the diagonal in $X \times X$ is the identity in $\mathcal{NR}_{G\curvearrowright X}$ and it is the image of the constant function $\mathrm{const}_1 \in L^\infty(X,\mathbb{Z})$ under the inclusion $j : L^\infty(X,\mathbb{Z}) \to \mathbb{Z}[\mathcal{R}_{G\curvearrowright X}]$ (Remark 3.30) . ☐

Corollary 3.39. *Let $[M]_{\mathbb{Z}} \in H_n(M,\mathbb{Z})$ be the integral fundamental class. Then the map*

$$\_ \cap H_n(\iota_M^X)([M]) : H^j(\tilde{M}, \mathcal{NR}_{G\curvearrowright X}) \to H_{n-j}(\tilde{M}, \mathcal{NR}_{G\curvearrowright X})$$

is an $\mathcal{NR}_{G\curvearrowright X}$-isomorphism.

3.2.3 An upper bound for the ℓ^2-Betti numbers

We now use the above construction to prove an upper bound for the ℓ^2-Betti numbers.

Theorem 3.40. *Let M be a closed, connected and oriented n-manifold with fundamental group G and universal covering $\tilde{M}$. Let (X,μ) be a standard G-space. Let*

$$z = \sum_{i=1}^{k} f_i \otimes \sigma_i \in L^\infty(X,\mathbb{Z}) \otimes_G C_n(\tilde{M},\mathbb{Z})$$

be an X-parametrised fundamental cycle. Then for every $j \geq 0$ we have

$$b_j^{(2)}(M) \leq \binom{n+1}{j} \sum_{i=1}^{k} \mu(supp(f_i)) \ .$$

Proof. Consider the evaluation homomorphism

$$\begin{aligned} \mathrm{ev}_{n-j} : \mathrm{Hom}_G(C_{n-j}(\tilde{M}), \mathcal{NR}_{G\curvearrowright X}) &\to \bigoplus_{i=1}^{k} \bigoplus_{(n-j)-faces} \mathcal{NR}_{G\curvearrowright X} \cdot f_i \\ \phi &\mapsto \overline{\phi(\sigma_i^l)} \cdot f_i \end{aligned}$$

where σ_i^l denotes the l-th $(n-j)$-face of σ_i.
Let $\phi \in H^{n-j}(\tilde{M}, \mathcal{NR}_{G\curvearrowright X})$ be a cohomology class which can be represented by a cocycle in $\mathrm{Ker}(\mathrm{ev}_{n-j})$. By definition of the cap product, we obtain that $\phi \cap z = 0$. Since the cap product $\_ \cap \iota_M^X(z)$ induces an isomorphism in homology, it follows that $\phi = 0$. Therefore, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}(\delta_{n-j}) & \longrightarrow & H^{n-j}(\tilde{M}, \mathcal{NR}_{G\curvearrowright X}) \\ & \searrow \qquad \nearrow & \\ & \dfrac{\mathrm{Ker}(\delta_{n-j})}{\mathrm{Ker}(\mathrm{ev}_{n-j}) \cap \mathrm{Ker}(\delta_{n-j})} & \end{array}$$

Due to the additivity of the Von Neumann dimension, we conclude that

$$\dim_{\mathcal{NR}_{G\curvearrowright X}}(H^{n-j}(\tilde{M}, \mathcal{NR}_{G\curvearrowright X})) \leq \dim_{\mathcal{NR}_{G\curvearrowright X}} \Big(\frac{\mathrm{Ker}(\delta_{n-j})}{\mathrm{Ker}(\mathrm{ev}_{n-j}) \cap \mathrm{Ker}(\delta_{n-j})} \Big) \ .$$

Consider the composition of injective $\mathcal{NR}_{G\curvearrowright X}$-homomorphisms

$$\frac{\mathrm{Ker}(\delta_{n-j})}{\mathrm{Ker}(\mathrm{ev}_{n-j}) \cap \mathrm{Ker}(\delta_{n-j})} \to \frac{\mathrm{Hom}_G(C_{n-j}(\tilde{M}, \mathcal{NR}_{G\curvearrowright X}))}{\mathrm{Ker}(\mathrm{ev}_{n-j})} \xrightarrow{\mathrm{ev}_{n-j}} \bigoplus_{i=1}^{k} \bigoplus_{(n-j)-faces} \mathcal{NR}_{G\curvearrowright X} \cdot f_i \ :$$

noticing that $\mathcal{NR}_{G\curvearrowright X} \cdot f_i \subset \mathcal{NR}_{G\curvearrowright X} \cdot \chi_{\mathrm{supp}(f_i)}$, the additivity of the Von Neumann dimension implies that

$$\dim_{\mathcal{NR}_{G\curvearrowright X}} \Big(\frac{\mathrm{Ker}(\delta_{n-j})}{\mathrm{Ker}(\mathrm{ev}_{n-j}) \cap \mathrm{Ker}(\delta_{n-j})} \Big) \leq \dim_{\mathcal{NR}_{G\curvearrowright X}} \Big(\bigoplus_{i=1}^{k} \bigoplus_{(n-j)-faces} \mathcal{NR}_{G\curvearrowright X} \cdot \chi_{\mathrm{supp}(f_i)} \Big) \ .$$

From the definition of the Von Neumann dimension, it follows that

$$\begin{aligned}\dim_{\mathcal{NR}_{G\curvearrowright X}}(\mathcal{NR}_{G\curvearrowright X}\cdot\chi_{\mathrm{supp}(f_i)}) &= \mathrm{trace}_{\mathcal{NR}_{G\curvearrowright X}}(\chi_{\mathrm{supp}(f_i)})\\ &= \langle \chi_{\Delta_X}\cdot\chi_{\mathrm{supp}(f_i)}, \chi_{\Delta_X}\rangle_{\ell^2(\mathcal{NR}_{G\curvearrowright X})}\\ &= \int_{\Delta_X}\chi_{\mathrm{supp}(f_i)}d\nu = \nu(\Delta_X\cap\mathrm{supp}(f_i))\\ &= \int_X \chi_{\mathrm{supp}(f_i)}d\mu = \mu(\mathrm{supp}(f_i))\ .\end{aligned}$$

Hence we obtain

$$\dim_{\mathcal{NR}_{G\curvearrowright X}}(H_j(\tilde{M},\mathcal{NR}_{G\curvearrowright X})) \le \binom{n+1}{j}\cdot\sum_{i=1}^{k}\mu(\mathrm{supp}(f_i))$$

and Proposition 3.36 completes the proof. □

Corollary 3.41. *Let M be a closed, connected and oriented n-manifold. Then*

$$b_j^{(2)}(M) \le \binom{n+1}{j}\cdot \| M \|\ .$$

Proof. Fix $\epsilon > 0$. Let (X,μ) be a standard G-space such that $\| M \| = \| M \|^X$. Let

$$z = \sum_{i=1}^{k} f_i\otimes\sigma_i$$

be an X-parametrized fundamental cycle such that $\| z \|^X \le \| M \|^X + \epsilon$. By the previous theorem we know that

$$b_j^{(2)}(M) \le \binom{n+1}{j}\sum_{i=1}^{k}\mu(\mathrm{supp}(f_i))\ .$$

Since the functions f_j are integer valued, for almost every $x\in\mathrm{supp}(f_i)$ we have $|f_i(x)|\ge 1$, so

$$\sum_{i=1}^{k}\mu(\mathrm{supp}(f_i)) \le \sum_{i=1}^{k}\int_X |f_i(x)|d\mu = \| z \|^X \le \| M \| + \epsilon\ .$$

The arbitrariety of $\epsilon > 0$ gives the thesis. □

Corollary 3.42. *Let M be a closed, connected and oriented n-manifold. If the integral foliated simplicial volume vanishes, then the Euler characteristic does, as well.*

Proof. By Proposition 2.39 the Euler characteristic can be computed via the ℓ^2-Betti numbers. Therefore, by the previous corollary,

$$|\chi(M)| \le \sum_{j=0}^{n} b_j^{(2)}(M) \le \sum_{j=0}^{n}\binom{n+1}{j}\| M \| \le 2^{n+1}\| M \|$$

which implies the assertion. □

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