Un approccio combinatorio alla poset topology: EL-shellability e algebra booleana

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Relatore:
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Corso di Laurea Triennale in Matematica
Università di Pisa

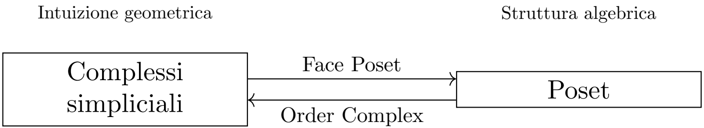
22 settembre 2023

Intuizione geometrica

Complessi simpliciali

Struttura algebrica

Poset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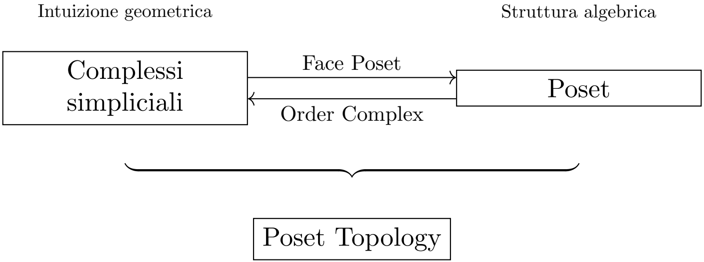
Complessi simpliciali
Face Poset
Order Complex
Poset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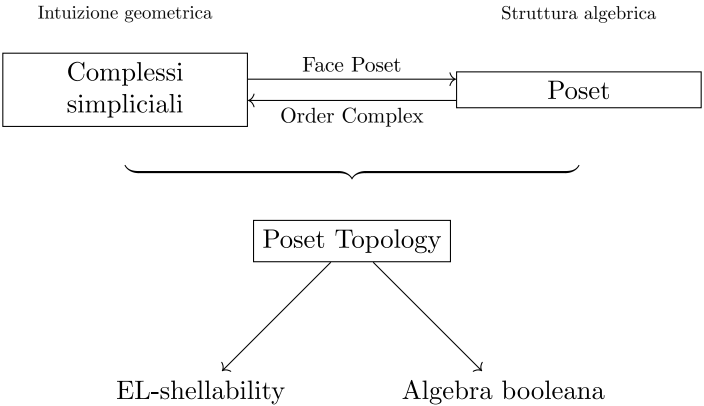
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Poset Topology
EL-shellability
Algebra booleana

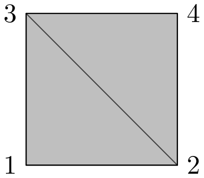
Sezione 1

Complessi simpliciali e poset

Complessi simpliciali

Definizione 1 (Complesso simpliciale astratto)

- **Vertici**: un insieme finito V
- **Complesso simpliciale astratto**: un insieme $\Delta \subseteq \mathcal{P}(V)$ con
  - $\{v\} \in \Delta$ per ogni $v \in V$.
  - se $G \in \Delta$ e $F \subseteq G$ allora $F \in \Delta$.
- **Facce**: $F \in \Delta$
- **Faccette**: facce massimali in $(\Delta, \subseteq)$



$$\Delta = \left\{ \begin{array}{c} \{1\},\{2\},\{3\},\{4\}, \\ \{1,2\},\{1,3\},\{2,3\},\{3,4\}, \\ \{2,4\},\{1,2,3\},\{2,3,4\} \end{array} \right\}$$

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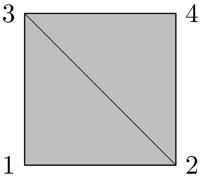

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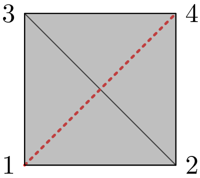

$\{1,4\}$

$\notin$

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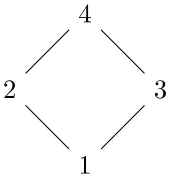
Insiemi parzialmente ordinati

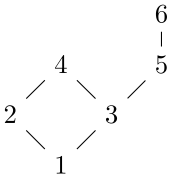
Definizione 2 (Poset)

- **Poset**: una coppia $(P, <_P)$
- Poset **limitato**: esistono $\hat{0}, \hat{1}$ minimo e massimo di $<_P$
- Poset **ridotto**: $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$

- **Diagramma di Hasse**:



P



P'

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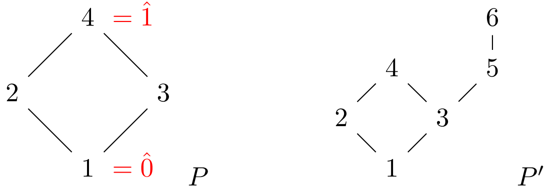

Insiemi parzialmente ordinati

Definizione 3 (Catena)

- **Catena**: sottoinsieme $\mathbf{c} \subseteq P$ tale che $(\mathbf{c}, <_P)$ è ordine totale
- Catena **massimale**: catena che è massimo in $(\mathcal{M}(P), \subseteq)$
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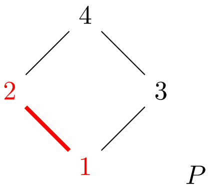

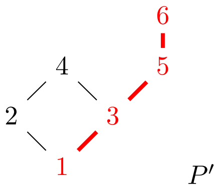

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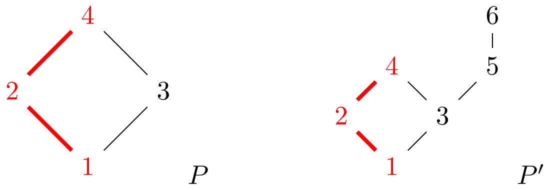
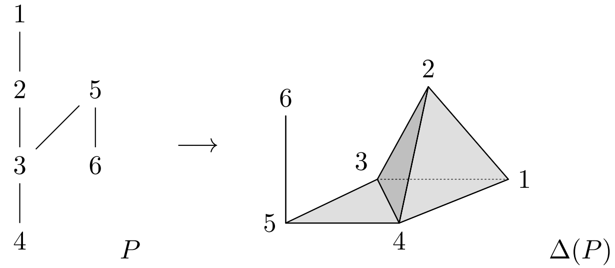


Order complex

Definizione 4 (Order complex)

- **Order complex** di P: un complesso simpliciale astratto $\Delta(P)$
  - **vertici** sono gli elementi di P
  - **facce** sono le catene di P



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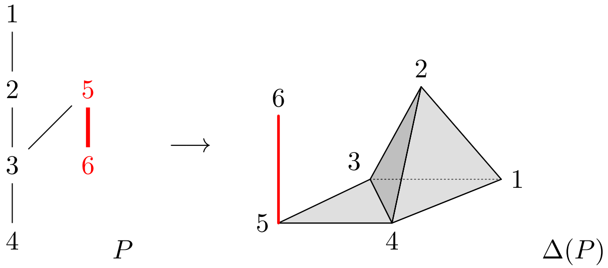

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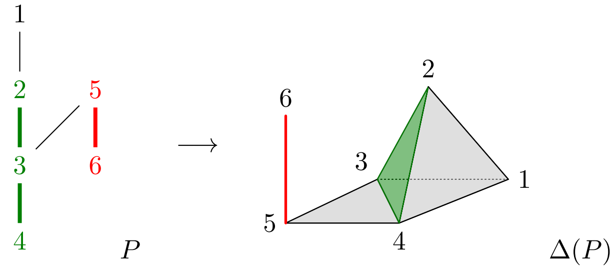

Face poset

Definizione 5

- **Face poset** di Δ: un poset $(P(\Delta), <)$ con
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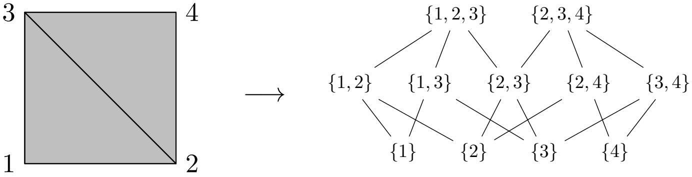
  - relazione $<$ data dall'inclusione $\subset$



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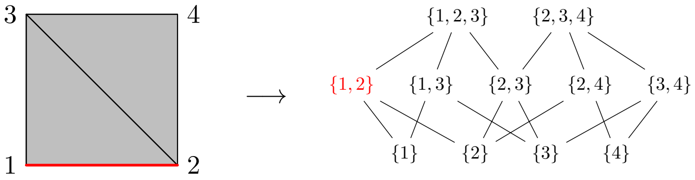

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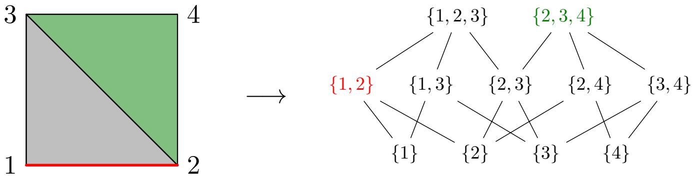
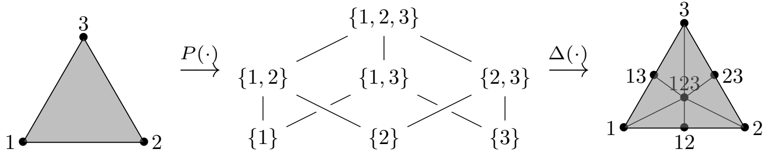


Suddivisione baricentrica

Proposizione 6

*$\Delta(P(\Delta))$ è la **suddivisione baricentrica** del complesso Δ*



Omologia di poset

Siano P poset e $\mathbb{K} = \mathbb{Z}$ o campo.

- **Complesso di catene**: per $j \geq -1$

$$C_j(P;\mathbb{K}) = \mathbb{K}\text{-modulo libero generato dalle } j\text{-catene}$$

- **Mappa di bordo**:

$$\begin{aligned} \partial_j : \quad C_j(P;\mathbb{K}) &\longrightarrow C_{j-1}(P;\mathbb{K}) \\ (x_0 < \cdots < x_j) &\longmapsto \sum_{i=0}^{j} (-1)^i (x_0 < \cdots < \widehat{x_i} < \cdots < x_j) \end{aligned}$$

- **Omologia di poset**:

$$\begin{cases} B_j(P;\mathbb{K}) = \operatorname{Im} \partial_{j+1} \\ Z_j(P;\mathbb{K}) = \ker \partial_j \end{cases} \qquad \tilde{H}_j(P;\mathbb{K}) = {Z_j(P;\mathbb{K})}/{B_j(P;\mathbb{K})}$$

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Coomologia di poset

Siano P poset e $\mathbb{K} = \mathbb{Z}$ o campo.

- **Complesso di cocatene**: per $j \geq -1$

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- **Mappa di cobordo**:

$$\begin{aligned} \delta_j : \quad C_j(P;\mathbb{K}) &\to C_{j+1}(P;\mathbb{K}) \\ (x_0 < \cdots < x_j) &\mapsto \sum_{i=0}^{j+1} (-1)^i \sum_{y \in (x_{i-1}, x_i)} (x_0 < \cdots < x_{i-1} < y < x_i < \cdots < x_j) \end{aligned}$$

- **Coomologia di poset**:

$$\begin{cases} B^j(P;\mathbb{K}) = \operatorname{Im} \delta_{j-1} \\ Z^j(P;\mathbb{K}) = \ker \delta_j \end{cases}$$

$$\tilde{H}^j(P;\mathbb{K}) = {Z^j(P;\mathbb{K})} / {B^j(P;\mathbb{K})}$$

Sezione 2

EL-shellability

Shellability

- **Complesso puro**: faccette di dimensione uguale

Definizione 7 (Complesso simpliciale shellable)

- **Complesso shellable**: esiste un ordine $F_1, F_2, \ldots, F_n$ delle faccette tale che per $j = 1, 2, \ldots, n-1$ il sottocomplesso

$$\left(\bigcup_{i=1}^{j-1} \langle F_i \rangle\right) \cap \langle F_j \rangle$$

  - è puro
  - ha dimensione $\dim(F_j) - 1$

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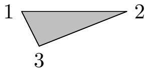

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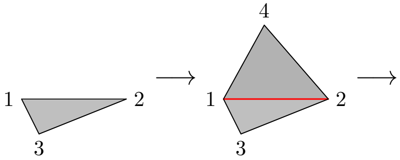

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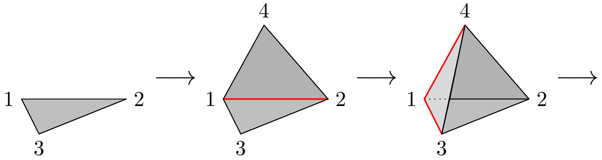

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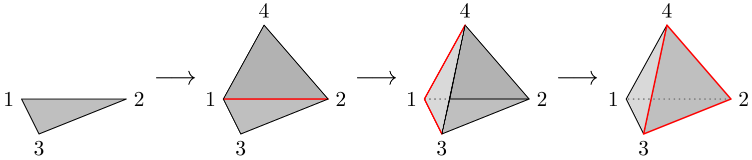

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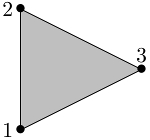

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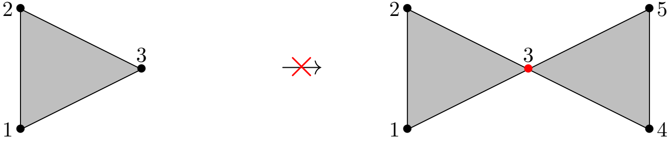
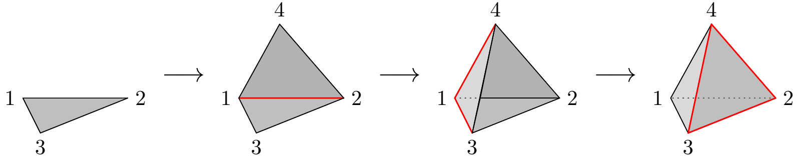


Shellability

Teorema 8

*Ogni complesso shellable ha omotopia di tipo **wedge di sfere**:*

$$\#\{i\text{-}sfere\ del\ wedge\} = \#\{\textbf{\textit{faccette omologiche}}\ di\ dimensione\ i\}$$



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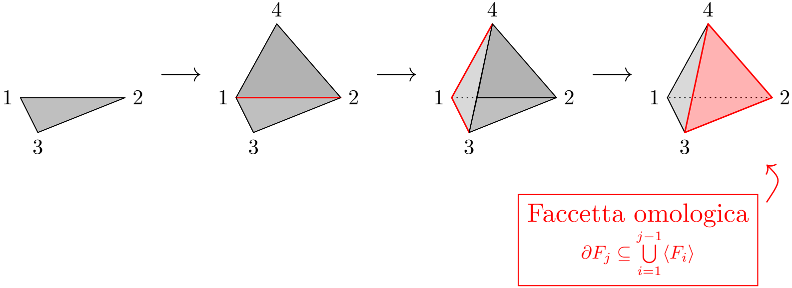


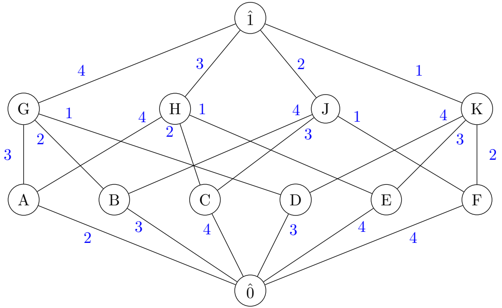
EL-shellability

- **Edge labeling** per un poset P:

$$\lambda : \mathcal{E}(P) \longrightarrow \mathbb{Z}$$

- **Word** associata a $\mathbf{c} : (x_0 < x_1 < \cdots < x_n)$ massimale:

$$\lambda(\mathbf{c}) = \lambda(x_0, x_1)\lambda(x_1, x_2)\dots\lambda(x_{n-1}, x_n)$$



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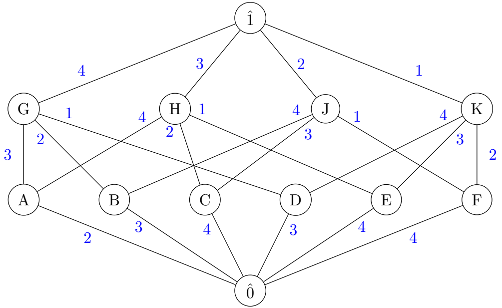

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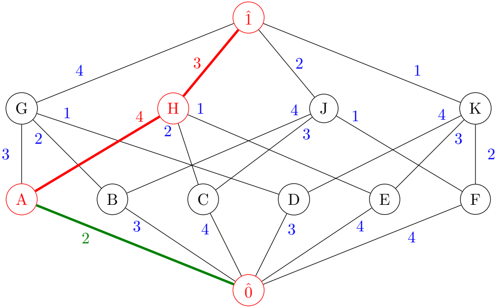
$$(\hat{0} < A < H < \hat{1}) \quad \leftrightarrow \quad 2$$



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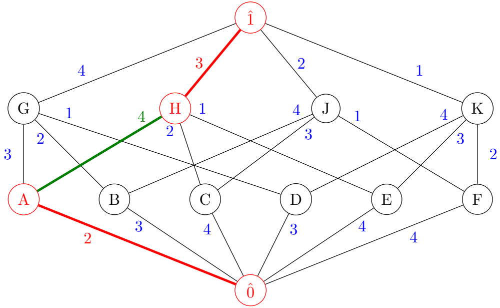

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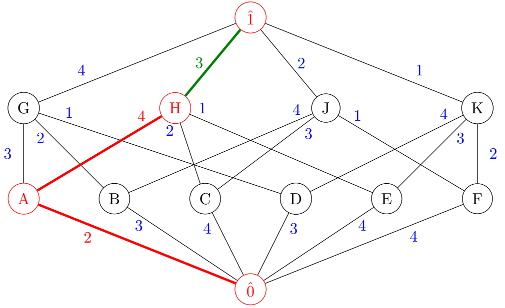
$$\lambda(\mathbf{c}) = \lambda(x_0, x_1)\lambda(x_1, x_2)\ldots\lambda(x_{n-1}, x_n)$$

$$(\hat{0} < A < H < \hat{1}) \quad \leftrightarrow \quad 243$$



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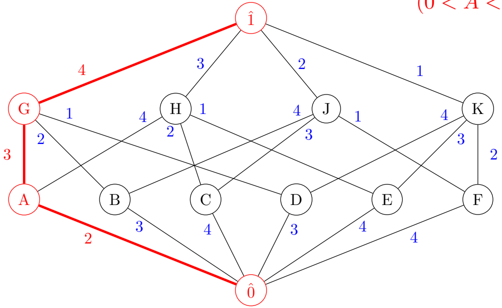

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$(\hat{0} < A < H < \hat{1}) \quad \leftrightarrow \quad 243$

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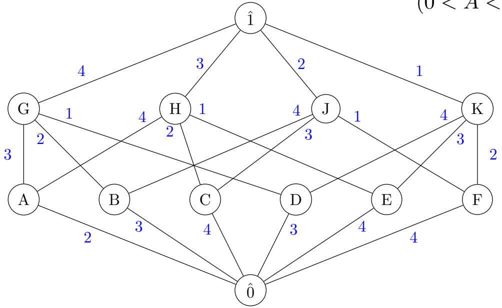
increasing



EL-shellability

Definizione 9 (Edge-lexicographic labeling)

- **EL-labeling**: $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ tale che per P limitato
  - esiste un'unica $\mathbf{c} \in \mathcal{M}(P)$ increasing
  - $\lambda(\mathbf{c}) = \min_{<_{\text{lex}}}\{\lambda(\mathbf{d}) \mid \mathbf{d} \in \mathcal{M}(P)\}$

P

EL-shellability

Definizione 9 (Edge-lexicographic labeling)

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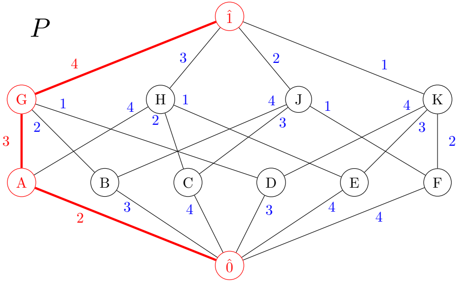

EL-shellability

Definizione 9 (Edge-lexicographic labeling)

- **EL-labeling**: $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ tale che per ogni $[x, y] \subseteq P$
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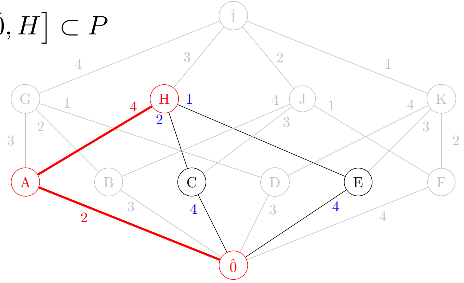
$[\hat{0}, H] \subset P$



$(\hat{0} < A < H) \quad \leftrightarrow \quad 24$

$(\hat{0} < E < H) \quad \leftrightarrow \quad 41$

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EL-shellability

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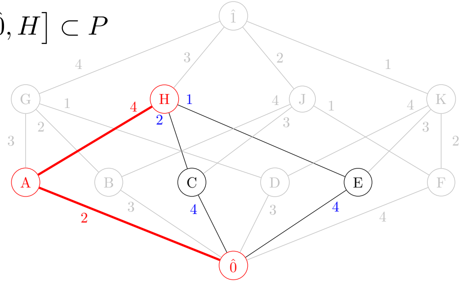
$\Rightarrow$ **EL-shellability**

$[\hat{0}, H] \subset P$



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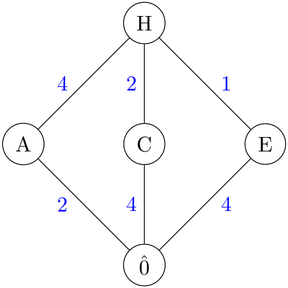
Risultati

Teorema 10 (Shelling order)

Sia P poset EL-shellable.

- *l'ordine $<_{lex}$ su $\mathcal{M}(P) \rightsquigarrow$ shelling order di $\Delta(P)$*
- *l'ordine indotto su $\mathcal{M}(\overline{P}) \rightsquigarrow$ shelling order di $\Delta(\overline{P})$*



$$(\hat{0} < A < H) <_{\text{lex}} (\hat{0} < E < H) <_{\text{lex}} (\hat{0} < C < H)$$

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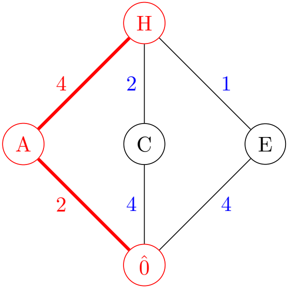

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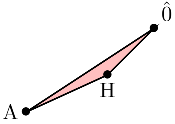

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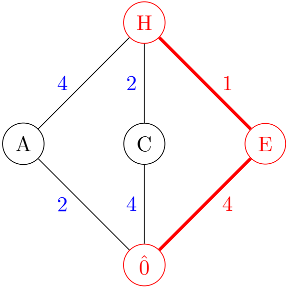

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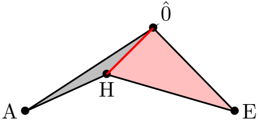

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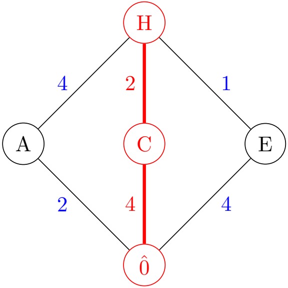

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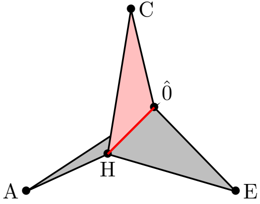

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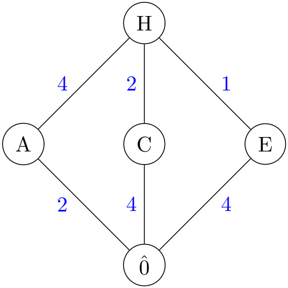

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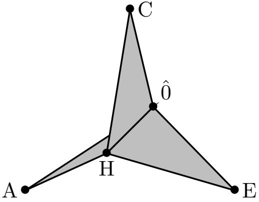
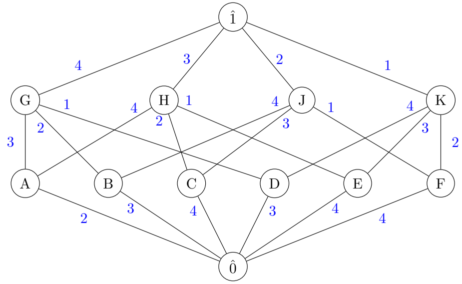


Risultati

Teorema 11 (Base coomologia di poset)

Sia $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ EL-shellable.

$$\{\overline{\boldsymbol{c}} \mid \boldsymbol{c} \text{ è } (i+2)\text{-catena decreasing di } \hat{P}\} \text{ è base di } \tilde{H}^i(P;\mathbb{C})$$



Catene decreasing

$\mathbf{c}_1 : (\varnothing < \{3,4\} < \{2,3,4\} < \{1,2,3,4\})$

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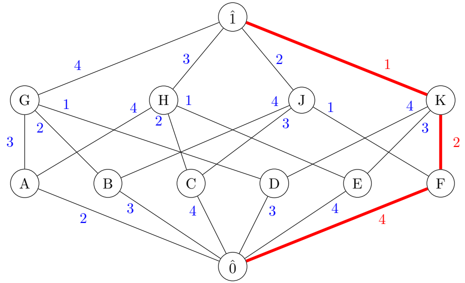

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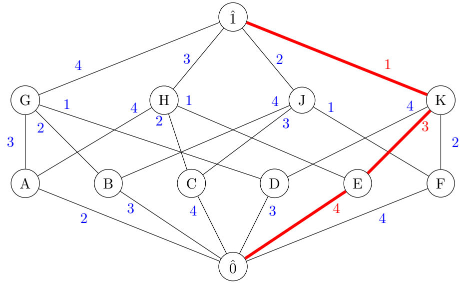

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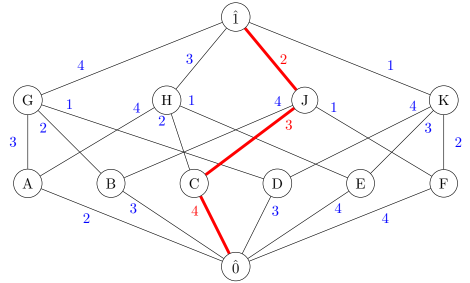

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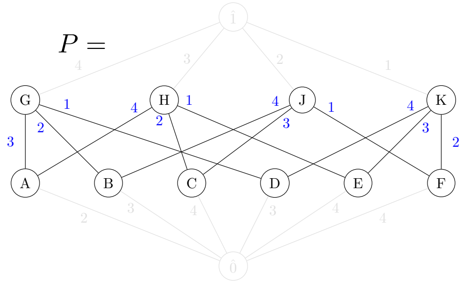

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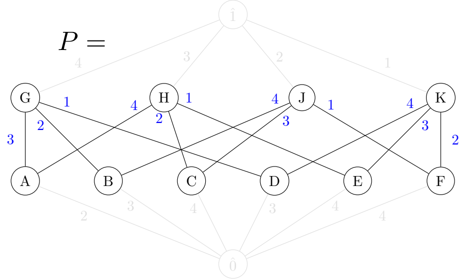

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$$\tilde{H}^1(P) = \langle \overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2, \overline{\mathbf{c}}_3 \rangle_{\mathbb{C}}$$

Risultati

Teorema 11 (Base coomologia di poset)

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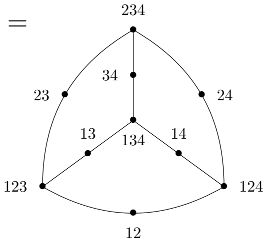
$$\{\overline{\boldsymbol{c}} \mid \boldsymbol{c} \text{ è } (i+2)\text{-catena decreasing di } \hat{P}\} \text{ è base di } \tilde{H}^i(P;\mathbb{C})$$

$\Delta(P) =$



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Sezione 3

Algebra booleana

Definizione

Definizione 12 (Algebra booleana)

- **Algebra booleana**: poset dato da

$$\left(B_n = \{F \subseteq [n]\},\ \subset\right)$$

- **Algebra booleana troncata**: per $0 \leq k \leq n$

$$\left(B_n^k = \{F \subseteq [n] \;\middle|\; |F| \geq k\} \cup \{\varnothing\},\ \subset\right)$$

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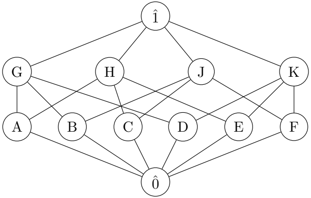

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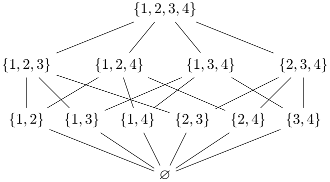
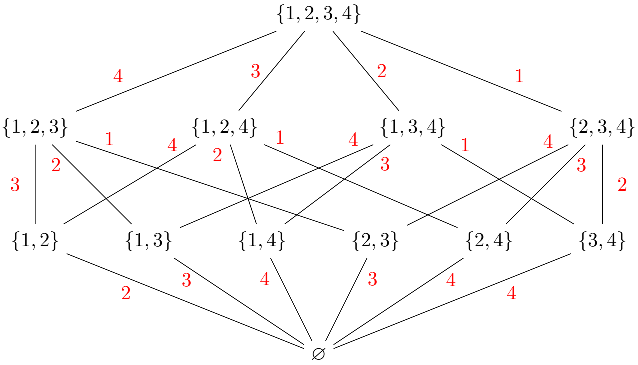
$B_4^2 =$ 

EL-labeling

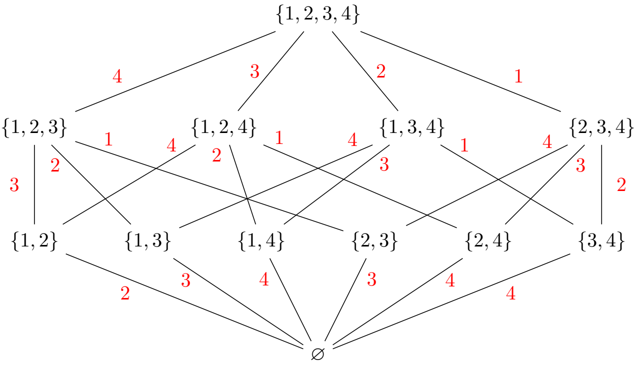
$B_4^2 =$



EL-labeling

$$\lambda(F,G) = \begin{cases} \max G & \text{se } F = \varnothing \text{ e } |G| = k \\ a & \text{se } G \setminus F = \{a\} \end{cases}$$

$B_4^2 =$



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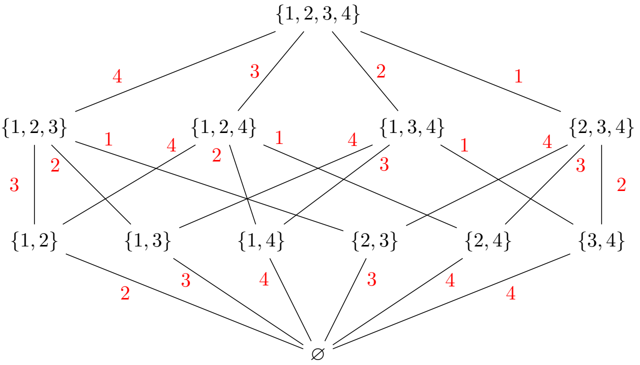

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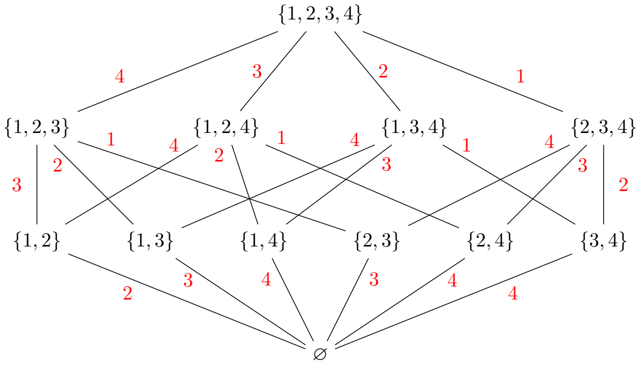

Rappresentazione della coomologia

Teorema 13 (Solomon)

Dati $0 \leq k \leq n$, *sia* $\lambda = (k, 1^{n-k})$.

$$\tilde{H}^{n-k-1}(\overline{B}_n^k; \mathbb{C}) \cong_{\mathfrak{S}_n} S^\lambda$$

Grazie per l'attenzione!