Appunti di Analisi Armonica

Appunti dalle Lezioni di J. Bellazini

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Prefazione

Work in progress

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Contents

Chapter 1

The interpolation theory

1.1 The Fourier transform

In this chapter we will introduce the interpolation theory. One of its consequences will be the definition of the Fourier transform in L^p with $p \in [1, 2]$.

Definition 1.1.1 – Fourier transform in L^1

Given $f \in L^1(\mathbb{R}^d)$, we define the *Fourier transform* of f as the function

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

Theorem 1.1.2 – Minkowski's theorem

Let us consider $(X, \mu), (Y, \nu)$ σ-finite measurable spaces. Given $f : X \times Y \to \mathbb{R}$ measurable, if $1 \leq p \leq +\infty$, it holds that

$$\left\| \int_Y f d\nu \right\|_{L^p_X} \leq \int_Y \|f\|_{L^p_X} d\mu.$$

Proof. By duality we have that

$$\|g\|_{L^p} = \sup_{\|f\|_{L^q}=1} |\langle f, g \rangle|.$$

By direct computation we get

$$\begin{aligned}
\left| \int_X \int_Y f g d\nu d\mu \right| &\leq \int_Y \int_X |f||g| d\mu d\nu \\
&\leq \int_Y \|f\|_{L^p_x} \|g\|_{L^q_x} d\nu = \int_Y \|f\|_{L^p_x} d\nu.
\end{aligned}$$

$\square$

Theorem 1.1.3 – Young's inequality

Remembering $f * g(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy$ it holds that, given $f \in L^p$ and $g \in L^1$:

$$\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}.$$

Proposition 1.1.4 – Approximate Identity

Let us consider a family of functions $K_n \in L^1$ $\forall n \in \mathbb{N}$ such that:

- For all $n \in \mathbb{N}$ $\int K_n dx = 1$.
- For all $n \in \mathbb{N}$ $\sup_n \int |K_n| < +\infty$.
- For all $\delta > 0$ $\int_{|x|>\delta} |K_n| dx \overset{n\to+\infty}{\longrightarrow} 0$.

Then for all $1 \le p < +\infty$ we have that

$$K_n * f \longrightarrow f \text{ in } L^p.$$

Proposition 1.1.5

We recall some basic fact about the Fourier transform, where we denote $\tau_h f(x) = f(x-h)$ and $\delta_\lambda f(x) = f\left(\frac{x}{\lambda}\right)$.

1. $\hat{f}$ is linear.
2. $\widehat{\tau_h f}(\xi) = e^{-2\pi i \xi \cdot h} \hat{f}$.
3. $\widehat{\delta_\lambda f}(\xi) = \lambda^d \hat{f}(\lambda \xi)$.
4. If $f, g \in L^1$ then $\widehat{f * g} = \hat{f}\hat{g}$.
5. The function $g_\lambda(x) = e^{-\pi\lambda|x|^2}$ is such that $\widehat{g_\lambda}(\xi) = \lambda^{-\frac{d}{2}} e^{-\frac{\pi|\xi|^2}{\lambda}}$.

Proof. These are consequences of some direct computations. We do explicitly the last one: for the dilatation formula it is sufficient to prove it for the case $\lambda = 1$:

$$\begin{aligned} \hat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-2\pi i \xi \cdot x} dx \\ &= \int_{\mathbb{R}^d} e^{-\pi(x+i\xi)^2} e^{-\pi\xi^2} dx \\ &= e^{-\pi\xi^2} \underbrace{\int_{\mathbb{R}^d} e^{-\pi(x+i\xi)^2} dx}_{h(\xi)} \\ &= e^{-\pi\xi^2} \end{aligned}$$

indeed we have that

$$h'(\xi) = \int_{\mathbb{R}^d} e^{-\pi(x+i\xi)^2}(-2\pi i(x+i\xi))dx = i\int_{\mathbb{R}^d} \frac{d}{dx}[e^{-\pi(x+i\xi)^2}]dx = 0.$$

□

Theorem 1.1.6 – Plancherel

If we have $f \in L^1 \cap L^2$ then $\hat{f} \in L^2$ and $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$. It follows that the map $f \xrightarrow{\mathcal{F}} \hat{f}$ has a unique extension to L^2 and is a surjective isometry.

Corollary 1.1.7 – Plancherel-Parseval Identity

For $f, g \in L^2$ we have
$$\int f\bar{g}dx = \int \hat{f}\bar{\hat{g}}dx.$$

Proof. □

Definition 1.1.8 – Fourier transform in L^2

Given[a] $\{f_n\} \in L^1 \cap L^2$ such that $f_n \to f$ in L^2, we define the Fourier transform of f as
$$hatf := \lim_{n\to\infty} \hat{f}_n.$$

[a] a sequence exists because $L^1 \cap L^2$ is dense in L^2.

Osservazione 1.1.1
The limit surely exists because $\{f_n\}$ is a Cauchy sequence in L^2 and, thanks to Plancherel-Parseval identity, also $\{\hat{f}_n\}$ is.

Definition 1.1.9 – Fourier antitrasform

We define the *Fourier antitrasform* as
$$\check{f}(x) = f^{\vee}(x) := \int_{\mathbb{R}^d} f(\xi)e^{2\pi i\xi\cdot x}.$$

Theorem 1.1.10 – Inversion formula

If $f \in L^2$, then $f = (\hat{f})^{\vee}$ almost everywhere.

Proof. □

1.2 Riesz-Thorin theorem

This first result of the interpolation theory will let us define the Fourier[1] transform in L^p, with $1 \leq p \leq 2$.

In order to prove the theorem we first need to develop some results of complex analysis[2]

Theorem 1.2.1 – Hadamard Three lines theorem

Let us consider the strip $\Sigma = \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$ in the complex plane. Given F such that:

- F is analytic on $\mathring{\Sigma}$
- F is bounded and continuous on Σ
- There exists M_0, M_1 such that $\begin{cases} |F(it)| \leq M_0 \\ |F(1+it)| \leq M_1 \end{cases}$ for all $t = \Re(z)$

It follows that:

$$|F(z)| \leq M_0^{1-t} M_1^t -$$

Idea This result allows us to control F inside the strip, knowing only a control on its boundary. ⌟

Proof. □

We are now able to prove the Riesz-Thorin theorem:

Theorem 1.2.2 – Riesz-Thorin Theorem

Let $p_0 \leq p_1$, $q_0 \leq q_1$. Given a linear operator T where

$$T : L^{p_0} \to L^{q_0}$$
$$T : L^{p_1} \to L^{q_1}$$

such that there exist constants M_0 and M_1 which make valid the inequalities

$$\|T_f\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}$$
$$\|T_f\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}.$$

Then, for all p and q of the form

$$\begin{cases} \frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \\ \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1} \end{cases}$$

for $t \in (0,1)$, we have

$$\|Tf\|_{L^q} \leq M_0^{1-t} M_1^t \|f\|_{L^p}.$$

Proof. □

[1]As in L^2 we will have a Cauchy sequence $\{f_n\} \subseteq L^1 \cap L^p$ which approximates f in L^p. Then $\{\hat{f}_n\}$ will be a Cauchy sequence and we will define $\hat{f}$ as its limit.

[2]Historically people proved a lot of results exploiting complex analysis. The breakthrough will arrive with the Calderón-Zygmund theory, which permitted to prove the results without using complex analysis.

Corollary 1.2.3

Considering $\begin{array}{cccc} \mathcal{F}: & L^1 & \longrightarrow & L^\infty \\ & f & \longmapsto & \hat{f} \end{array}$ and $\begin{array}{cccc} \mathcal{F}: & L^2 & \longrightarrow & L^2 \\ & f & \longmapsto & \hat{f} \end{array}$, we can define the Fourier transform from L^p to L^q for all $1 \leq p \leq 2$ (q is the conjugate exponent of p). Furthermore,

$$\|\mathcal{F}f\|_{L^q} \leq \|f\|_{L^p}.$$

Proof. It follows directly from Riesz-Thorin theorem: we know that

$$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$
$$\|\hat{f}\|_{L^2} \leq \|f\|_{L^2}$$

The estimate follows because

$$\begin{cases} \frac{1-t}{\infty} + \frac{t}{2} = \frac{1}{q} \\ \frac{1-t}{1} + \frac{t}{2} = \frac{1}{p} \end{cases} \Rightarrow \begin{cases} \frac{1}{q} = \frac{t}{2} \\ 1 - \frac{t}{2} = \frac{1}{p} \end{cases} \Rightarrow 1 = \frac{1}{p} + \frac{1}{q}$$

□

Osservazione 1.2.1
The estimate given by the corollary is called *Hausdorff-Young* inequality[3]: for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\hat{u}\|_{L^q} \leq \|u\|_{L^p}.$$

Corollary 1.2.4 – Young's inequality

Let us consider $p_1, p_2, r \in [1, +\infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{r}$. It follows that $\forall f \in L^{p_1}(\mathbb{R}^d)$ and $\forall g \in L^{p_2}(\mathbb{R}^d)$

$$\|f * g(x)\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}(\mathbb{R}^d)} < +\infty.$$

Proof. Let us consider p and its conjugate exponent p'. It is known that

$$\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^1} \quad \text{and} \quad \|f * g\|_{L^\infty} \leq \|f\|_{L^p}\|g\|_{L^{p'}}.$$

Given $f \in L^p$ let us consider the operator $T : g \to f * g$. We have that

$$\|Tg\|_{L^\infty} \leq \underbrace{\|f\|_{L^p}}_{=M_0} \|g\|_{L^{p'}} \quad \text{and} \quad \|Tg\|_{L^1} \leq \underbrace{\|f\|_{L^p}}_{=M_1} \|g\|_{L^1}.$$

For Riesz-Thorin we have that

$$\|Tg\|_{L^R} \leq \underbrace{\|f\|_{L^p}}_{=M_0^{1-t}M_1^t} \|g\|_{L^s}$$

where

$$\begin{cases} \frac{1}{R} = \frac{1-t}{\infty} + \frac{t}{p} \\ \frac{1}{s} = \frac{1-t}{p'} + \frac{t}{1} = 1 + \frac{t-1}{p} \end{cases}.$$

[3]One can see that the optimal constant in not 1!

This means

$$\frac{1}{s}+\frac{1}{p}=1+\frac{1}{R}$$

hence the Young inequality is valid. □

1.2.1 A generalisation of Riesz-Thorin theorem

We can generalise the Riesz-Thorin theorem to particular families of operators. First let us give the following definition:

Definition 1.2.5 – Admissible family of operators

Let $\Sigma = \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$ be the strip in the complex plane and T_z a family of operators parameterised on Σ. We say the family $\{T_z\}$ is admissible if we have the following conditions:

- The function
$$\begin{array}{rccc} R: & \Sigma & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \int_Y T_z(f)g\,dx \end{array}$$
is analytic in $\mathring{\Sigma}$ and continuous in Σ, where f, g are finitely simple functions considered on two σ-finite measure space.

- There exists $a < \pi$ such that
$$\log\left(\left|\int_Y T_z(f)g\,dx\right|\right) \leq C(f,g)e^{a|y|}.$$

Lemma 1.2.6 – Hirshman lemma

If we have a function F which is analytic on $\mathring{\Sigma}$, continuous in Σ and a constant $a < \pi$ such that

$$\sup_{\substack{0\leq x\leq 1 \\ -\infty<y<+\infty}} e^{-a|y|}\log(|F(x+iy)|) < +\infty.$$

Then

$$\log_{x\in[0,1]}|F(x)| \leq \frac{1}{2}\sin(\pi x)\int_{-\infty}^{+\infty}\frac{\log(|F(iy)|)}{\cosh(\pi y)-\cos(\pi x)}+\frac{\log(F|(1+iy)|)}{\cosh(\pi y)+\cos(\pi x)}dy.$$

Osservazione 1.2.2
We recall that

$$\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\sin(\pi x)}{\cosh(\pi y)-\cos(\pi x)}dy = 1-x \quad \text{and} \quad \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\sin(\pi x)}{\cosh(\pi y)+\cos(\pi x)} = x.$$

Osservazione 1.2.3
If we suppose that both $\log(|F(iy)|)$ and $\log(|F(1+iy)|)$ are constant or bounded we get back $|F(x)| \leq M_0^{1-x}M_1^x$, which recalls Riesz-Thorin.

Theorem 1.2.7 – Interpolation of analytic family of operators

Let T_z be a family of operators such that

$$\begin{cases} \|T_{iy}f\|_{L^{q_0}} \leq M_0(y)\|f\|_{L^{p_0}} \\ \|T_{1+iy}f\|_{L^{q_1}} \leq M_1(y)\|f\|_{L^{p_1}} \end{cases}$$

where there exists $b < \pi$ such that

$$\sup_{-\infty<y<+\infty} e^{-b|y|}\log(M_j(y)) < +\infty.$$

Then for all q_t, p_t of the form

$$\begin{cases} \frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \\ \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \end{cases}$$

where $t \in (0, 1)$, we have that

$$\|T_t f\|_{L^{q_t}} \leq M(t)\|f\|_{L^{p_t}}$$

where

$$M(t) \leq \exp\left(\frac{\sin(\pi t)}{2}\int_{-\infty}^{+\infty} \frac{\log(M_0(y))}{\cosh(\pi y) - \cos(\pi t)} + \frac{\log(M_1(y))}{\cosh(\pi y) + \cos(\pi t)}dt\right).$$

Corollary 1.2.8 – Stein Theorem

Take[a] K_0, K_1, u_0, u_1 measurable functions. Suppose that

$$\begin{cases} \|K_0T(f)\|_{L^{q_0}} \leq M_0\|fu_0\|_{L^{p_0}} \\ \|K_1T(f)\|_{L^{q_1}} \leq M_1\|fu_1\|_{L^{p_1}} \end{cases}$$

Then for all q_t, p_t of the form

$$\begin{cases} \frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \\ \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \end{cases}$$

where $t \in (0, 1)$, it holds

$$\|K_tT(f)\|_{L^{q_t}} \leq M_0^{1-t}M_1^t\|u_tf\|_{L^{p_t}}$$

where $K_t = K_0^{1-t}K_1^t$ and $u_t = u_0^{1-t}u_1^t$.

[a] With this theorem we can have an interpolation of operators with weighted condition.

1.3 Marcinkiewitz theorem

Definition 1.3.1 – Distribution function

Given a measurable function f we define the *distribution function* of f as

$$d_f(\alpha) = \mu\{x \in \mathbb{R}^d : |f(x)| > \alpha\}.$$

Proposition 1.3.2 – Layer-Cake decomposition

For every $p \in (0, +\infty)$ it holds that

$$\|f\|_{L^p}^p = p \int_0^{+\infty} \alpha^{p-1} d_f(\alpha) d\alpha.$$

Proof. By applying Fubini we get

$$\begin{aligned} p \int_0^{+\infty} \alpha^{p-1} d_f(\alpha) d\alpha &= p \int_0^{+\infty} \alpha^{p-1} \mu\{|f| > \alpha\} d\alpha \\ &= p \int_0^{+\infty} \alpha^{p-1} \left(\int_{\mathbb{R}^d} \chi_{\{|f|>\alpha\}}(x) dx \right) d\alpha \\ &= \int_{\mathbb{R}^d} \left(\int_0^{|f|} p\alpha^{p-1} d\alpha \right) dx \\ &= \int_{\mathbb{R}^d} |f|^p dx. \end{aligned}$$

□

Definition 1.3.3 – Weak L^p spaces

For all $p \in (0, +\infty)$ we define the *weak* L^p space, also denoted by $L^{p,\infty}$, as the set of measurable function such that

$$\|f\|_{L^{p,\infty}} = \inf \left\{ c : \forall \alpha > 0 d_f(\alpha) \leq \frac{c^p}{\alpha^p} \right\} < +\infty.$$

We have that $L^{p,\infty}$ are quasi-normed spaces:

$$\|f + g\|_{L^{p,\infty}} \leq C_p(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$$

where $C_p = \max(2, 2^{\frac{1}{p}})$.

Proposition 1.3.4

For all $p \in (0, +\infty)$

$$\|f\|_{L^{p,\infty}} \leq \|f\|_{L^p}$$

hence $L^p \subset L^{p,+\infty}$.

Proof. □

Corollary 1.3.5 – Chebychev Inequality

We also have the Chebychev inequality:

$$\mu\{|f| > \lambda\} \leq \frac{\|f\|_{L^p}^p}{\lambda^p}.$$

Example 1.3.6. We can see that the inclusion is strict:indeed there exist $f \in L^{p,\infty} \setminus L^p$. As example,we can consider the function $f(x) = |x|^{-\frac{n}{p}}$.

Definition 1.3.7

Let us consider an operator T. We say that T is

- *Sublinear*: if for all f, g

$$|T(f+g)| \leq |T(f)| + |T(g)|$$

- *Strong* (p,p)*-continuous*: if

$$\|Tf\|_{L^p} \leq c\|f\|_{L^p}.$$

- *Weak* (p,p)*-continuous*: if

$$\mu\{|Tf| > \lambda\} \leq c\frac{\|f\|_{L^p}^p}{\lambda^p}.$$

Osservazione 1.3.1
It is clear that if T is strong (p,p) continuous, then it is also weak (p,p): indeed by Chebychev inequality we have

$$\mu\{|Tf| > \lambda\} \leq \frac{\|Tf\|_{L^p}^p}{\lambda^p} \leq \frac{c^p\|f\|_{L^p}^p}{\lambda^p}.$$

Theorem 1.3.8 – Marcinkiewitz theorem

Let us consider p_0, p_1 such that $1 \leq p_0 < p_1 < +\infty$. If $T : L^{p_0} + L^{p_1} \to \{$measurable functions$\}$ is sublinear, (p_0, p_0)-weak and (p_1, p_1)-weak continuous, then T is (p,p)-strong continuous for all $p \in (p_0, p_1)$.

Chapter 2

The Hardy-Littlewood-Sobolev inequality

The inequality we want to see is the following[1]: $\left\| \frac{1}{|x|^\alpha} * f \right\|_{L^q} \preccurlyeq \|f\|_{L^p}$.

Proposition 2.0.1

Given $f \in C_C^\infty$ and $C_\alpha = \pi^{-\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)$. If $0 \leq \alpha < d$ we have that

$$(C_\alpha |\xi|^{-\alpha} \hat{f})^\vee = C_{d-\alpha} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy.$$

Proof. Let's remember the definition of the *Gamma function*: $\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$, which has the property that $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$. We notice that if $0 < \alpha < d$ then

$$\begin{aligned} C_\alpha |\xi|^{-\alpha} &= \int_0^{+\infty} e^{-\pi|\xi|^2\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty e^{-t} \left[\frac{t}{\pi|\xi|^2}\right]^{\frac{\alpha}{2}-1} \frac{1}{\pi|\xi|^2} dt \\ &= \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} \pi^{\frac{\alpha}{2}} |\xi|^{-\alpha} \\ &= \Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{\alpha}{2}} |\xi|^{-\alpha}. \end{aligned}$$

Thanks to this equality we have that

$$\begin{aligned} (C_\alpha |\xi|^{-\alpha} \hat{f})^\vee &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty e^{-\pi|\xi|^2\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda f(y) e^{-2\pi i \xi\cdot y} dy e^{2\pi i \xi \cdot x} d\xi \\ &= \int_0^\infty \int_{\mathbb{R}^d} \lambda^{\frac{\alpha}{2}-1} f(y) e^{-\frac{\pi(x-y)^2}{\lambda}} \lambda^{-\frac{d}{2}} d\lambda dy \\ &= \int_{\mathbb{R}^d} f(y) \int_0^\infty \varepsilon^{1-\frac{\alpha}{2}} e^{-\pi(x-y)^2\varepsilon} \varepsilon^{\frac{d}{2}} \varepsilon^{-2} d\varepsilon dy \\ &= \int_{\mathbb{R}^d} f(y) \int_0^\infty \varepsilon^{\frac{d-\alpha}{2}-1} e^{-\pi(x-y)^2\varepsilon} d\varepsilon dy \\ &= \int_{\mathbb{R}^d} C_{d-\alpha} |x-y|^{-(d-\alpha)} dy \\ &= C_{d-\alpha} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy. \end{aligned}$$

[1] by $\preccurlyeq$ we mean that there is the inequality up to a constant, i.e. $\preccurlyeq \equiv \leq C$.

□

Osservazione 2.0.1
Since $f \in C_C^\infty$ we have that $\hat{f}$ is well defined, analytic and, for $|\xi| \to \infty$, all of its derivatives decay faster than the inverse of any polynomial[2] in ξ, hence $|\xi|^{-\alpha}\hat{f} \in L^1$. Since, a priori, $\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy$ decays only as $|x|^{\alpha-d}$ it is not in L^p for any $p \leq 2$ but, if $0 < \alpha < \frac{d}{2}$, thanks to the Hardy-Littlewood-Sobolev inequality it is a L^2 function, hence it admits Fourier transform. This gives us the relation:

$$C_\alpha |\xi|^{-\alpha} \hat{f} = C_{d-\alpha} \left(\int_{\mathbb{R}^d} \frac{\widehat{f(y)}}{|x-y|^{d-\alpha}} dy \right).$$

Idea We can study the Poisson equation: $-\Delta u = f$ on $\mathbb{R}^3$ where $f \in C_C^\infty$. We remember that[3]

$$\widehat{\frac{\partial u}{\partial x_i}} = \int \frac{\partial u}{\partial x_i}(x) e^{-2\pi i \xi \cdot x} dx = 2\pi \xi_i \int u(x) e^{-2\pi i \xi \cdot x}$$
$$\left(\widehat{\frac{\partial u}{\partial x_i}} \right)^2 = 4\pi^2 (\xi_i)^2 \int u(x) e^{-2\pi i \xi \cdot \alpha}.$$

So if we apply the Fourier transform to the Poisson equation we get:

$$-\widehat{\Delta u} = \hat{f} \Rightarrow 4\pi^2 |\xi|^2 \hat{u} = \hat{f} \Rightarrow \hat{u} = \frac{1}{4\pi^2} \frac{1}{|\xi|^2} \hat{f}.$$

Thanks to the previous formula, with $\alpha = 2$ and $d = 3$ we have:

$$u = \frac{1}{4\pi^2} \left(\frac{1}{|\xi|^2} \hat{f} \right)^\vee = \frac{1}{4\pi} \int \frac{f(y)}{|x-y|} dy$$

indeed:

$$\begin{aligned}
\left(\frac{1}{|\xi|^2} \hat{f} \right)^\vee &= \frac{C_1}{C_2} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \\
&= \sqrt{\pi} \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \\
&= \sqrt{\pi} \Gamma\left(\frac{1}{2}\right) \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \\
&= \sqrt{\pi} \sqrt{\frac{\pi}{\sin(\frac{\pi}{2})}} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \\
&= \pi \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy.
\end{aligned}$$

⌟

[2] This type of function will be really useful.
[3] you can think that u has compact support.

2.1 The maximal function

Definition 2.1.1 – Centered maximal function

Given $f \in L^1_{loc}(\mathbb{R}^d)$ we define the *centered maximal function* Mf as

$$\begin{aligned} Mf(x) &= \sup_{\delta>0} \operatorname{Avg}_{B(x,\delta)}|f| \\ &= \sup_{\delta>0} \frac{1}{\omega_d \delta^d} \int_{|y|<\delta} |f(x-y)|dy. \end{aligned}$$

Example 2.1.2. Let us compute Mf where $f = \chi_{[a,b]}(x)$. We start by noticing that, if $x > b$, then having for $\delta > 0$ $I = (x-\delta, x+\delta)$ it holds that

$$\frac{1}{2\delta}\int_I |f(y)|dy = \frac{\delta-(x-b)}{2\delta}.$$

If we consider $\delta = x - a$ then

$$Mf(x) = \begin{cases} \frac{b-a}{2|x-a|} & x \le a \\ 1 & a < x < b \\ \frac{b-a}{2|x-b|} & x \ge b \end{cases}.$$

Since $Mf \sim \frac{1}{|x|}$ then $Mf \notin L^1$.

Proposition 2.1.3

Given $f \in L^1_{loc}(\mathbb{R}^d)$ if $Mf \in L^1(\mathbb{R}^d)$ then $f \equiv 0$.

Proof. It is clear that $B(0,R) \subseteq B(x, \|x\|+R)$ for all $x \in \mathbb{R}^d$ and $R \in \mathbb{R}^+$. So for all $R \in \mathbb{R}^+$ it holds:

$$\begin{aligned} Mf(x) &\ge \frac{1}{\mu(B(x,\|x\|+R))} \int_{B(x,\|x\|+R)} |f(y)|dy \\ &\ge \frac{1}{\omega_d(\|x\|+R)^d} \int_{B(0,R)} |f(y)|dy. \end{aligned}$$

This means that if $Mf \in L^1$ then $\int_{B(0,R)} |f(y)|dy = 0$ which is $f = 0$ a.e. □

Osservazione 2.1.1

It holds that $\{Mf > \lambda\} \subseteq \mathbb{R}^d$ is an open subset, i.e. Mf is lower semi continuos.

Definition 2.1.4 – Non centered maximal function

We have

$$\begin{aligned} \tilde{M}_f(x) &= \sup_{\substack{\delta>0 \\ |y-x|<\delta}} \operatorname{Avg}_{B(y,\delta)}|f| \\ &= \sup_{\substack{\delta>0 \\ |y-x|<\delta}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|dy. \end{aligned}$$

Osservazione 2.1.2
It holds that $\{\tilde{M}_f > \lambda\} \subseteq \mathbb{R}^d$ is an open subset, i.e. $\tilde{M}_f$ is lower semi continuos.

Proof. If we consider $x \in \{\tilde{M}_f > \lambda\}$, there exists y such that

$$\frac{1}{B(y,R)} \int_{B(y,R)} |f(z)| dz > \lambda \ \forall x \in B(y,R).$$

This means that $B(y,R) \subseteq \{\tilde{M}_f > \lambda\}$, hence the thesis. □

Proposition 2.1.5

Given f we have

$$Mf \leq \tilde{M}f \leq 2^d Mf.$$

Proof. It is clear that $Mf \leq \tilde{M}_f$. Let see $\tilde{M}_f \leq 2^d M_f$. We start by seeing that, if $x \in B(y,R)$ then $B(y,R) \subseteq B(x,2R)$. It follows that:

$$\begin{aligned}\frac{1}{\mu(B(y,R))} \int_{B(y,R)} |f(z)| dz &\leq \frac{1}{\mu(B(x,R))} \int_{B(x,2R)} |f(z)| dz \\ &= \frac{2^d}{\mu(B(x,2R))} \int_{B(x,2R)} |f(z)| dz \leq 2^d Mf.\end{aligned}$$

□

Corollary 2.1.6

We have that

$$\mu\{Mf < \lambda\} \leq c \frac{\|f\|_{L^1}}{\lambda} \iff \mu\{\tilde{M}_f < \lambda\} \leq \tilde{c} \frac{\|f\|_{L^1}}{\lambda}.$$

Lemma 2.1.7 – Vitali covering lemma

Let $\{B_1, \ldots, B_k\}$ be a finite collection of balls in $\mathbb{R}^d$. THen there exists a subset $\{\tilde{B}_{j_1}, \ldots, \tilde{B}_{j_l}$ of pointwise disjoint balls such that

$$\bigcup_{i=1}^{k} B_i \subseteq \bigcup_{r=1}^{l} 3\tilde{B}_{j_r}.$$

Furthermore

$$\mu\left(\bigcup_{r=1}^{l} \tilde{B}_{j_r}\right) = \sum_{r=1}^{l} \mu(\tilde{B}_{j_r}) \geq \frac{1}{3^d} \mu\left(\bigcup_{i=1}^{k} B_i\right).$$

Proof. Omitted. □

Theorem 2.1.8 – Hardy-Littlewood

It holds that $1 \leq p \leq +\infty$

$$\|Mf\|_{L^p} \leq C_p \|f\|_{L^p}.$$

Proof. The thesis will follow thanks to Marcienkiewitz theorem. It is clear that $\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}$. We now want a $(1,1)$-weak estimate.

Let us call $E_\lambda = \{\tilde{M}f > \lambda\}$. The $(1,1)$-weak continuity will follow by an estimate of $\mu(E_\lambda)$. Since it is a measurable set we can find $K \subseteq E_\lambda$ compact such that $\mu(K)+\varepsilon = \mu(E_\lambda)$ and $\forall x \in K\ \exists B_x$ such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f(z)|dz > \lambda.$$

Since K is compact we can find B_{x_i} such that $K = \bigcup_{i=1}^{n} B_{x_i}$. For the Vitali covering lemma we also have

$$K \subseteq \cup B_i \subseteq 3\tilde{B}_j$$

where $\tilde{B}_j$ are disjoint. It follows that:

$$\begin{aligned}
\mu(K) \leq \mu\left(\bigcup_i B_i\right) &\leq 3^d \sum_j \sum \mu(\tilde{B}_j) \\
&\leq \sum_j 3^d \frac{1}{\lambda} \int_{\tilde{B}_j} f(y)dy \\
&\leq \frac{3^d}{\lambda} \int_{\cup_j \tilde{B}_j} f(y)dy \\
&\leq \frac{3^d}{\lambda} \|f\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

□

Definition 2.1.9 – Maximal operator

If we have a family of linear operators $\{T_n\}$ we define the *maximal operator* associeted to the family as

$$T^* f(x) := \sup_n |T_n f(x)|.$$

Lemma 2.1.10

Let $\{T_n\}$ be a family of linear operators, such that for each n

$$T_n : L^p \to \{\text{maximal functions}\}$$

such that T^* is of weak type (p,p). Then the following set is closed:

$$C = \{f \in L^p \text{ such that } \lim_{n\to+\infty} T_n f = f \text{ almost every } x\}.$$

Proof. Since L^p is a metric space we have that

$$C \text{ closed} \iff C \text{ sequentially closed.}$$

We want to prove that if $f_k \in C$ and $f_k \to f$ then $f \in C$. Given $\varepsilon > 0$ we have $\|f_k - f\| \leq \varepsilon$. Fix λ we want to prove that there exists $c \in \mathbb{R}^+$ such that:

$$\mu\left\{x : \limsup_{n\to+\infty} |T_n f - f| > \lambda\right\} \leq c\varepsilon.$$

Idea The following argument is really useful and many proofs use this basic idea: if we have that $x + y + z > \lambda$ then, worst case scenario, we surely have $\begin{cases} x > \frac{\lambda}{3} \\ y > \frac{\lambda}{3} \\ z > \frac{\lambda}{3} \end{cases}$. ⌟

Thanks to the linearity of T we can write:

$$\begin{aligned}
\mu\left\{x : \limsup_{n\to+\infty} |T_n f - f| > \lambda\right\} &= \mu\left\{x : \limsup_{n\to+\infty} |T_n f - T_n f_k + T_n f_k - f_k + f_k - f| > \lambda\right\} \\
&\leq \mu\left\{x : \limsup_{n\to+\infty} |T_n(f - f_k)| > \frac{\lambda}{3}\right\} + \\
&\quad + \mu\left\{x : \limsup_{n\to+\infty} |T_n f_k - f_k| > \frac{\lambda}{3}\right\} + \\
&\quad + \mu\left\{x : \limsup_{n\to+\infty} |f_k - f| > \frac{\lambda}{3}\right\} \\
&= A + B + C \preccurlyeq \frac{\varepsilon^p}{\lambda^p}
\end{aligned}$$

where the last estimate follows because:

$$\begin{aligned}
A &\leq \mu\left\{x : |T^*(f - f_k)| > \frac{\lambda}{3}\right\} \\
&\leq C\frac{\|f - f_k\|_{L^p}^p}{\lambda^p} \\
&\leq k\frac{\varepsilon^p}{\lambda^p} \text{ thanks to the } (p-p)\text{-weak continuity} \\
B &= 0 \text{ having } f_k \in C \\
C &\leq k\frac{\varepsilon^p}{\lambda^p} \text{ thanks to the Cebychev inequality}
\end{aligned}$$

□

Corollary 2.1.11 – Lebesgue differentiation theorem

If $f \in L^1$ then

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)dy = f(x)$$

for almost every x.

Proof. Let us define the family of operators T_r where

$$T_r f(x) = \fint_{B(x,r)} f(y)dy$$

and consider the maximal operator $T^* f(x)$. We notice that T^* is $(1,1)$-weak, indeed

$$\left|\int_{B(x,r)} f(y)dy\right| \leq \int_{B(x,r)} |f(y)|dy$$

which means that $T^* f(x) \leq Mf(x)$. Having that Mf is $(1,1)$-weak, also $T^* f$ is. For the previous lemma the following is a closed set of L^1:

$$\{f \in L^1 : \lim_{r \to 0} T_r f = f \text{ a.e.}\}.$$

We conclude because for $f \in C_C^\infty$ it is clear that $\lim_{r \to 0} T_r f = f$ and they are a dense subset of L^1. $\square$

Proposition 2.1.12

Given $K \in L^1$ such that $K(x) = K(\|x\|)$ is a radial, non-increasing function we have that

$$|K * f(x)| \leq \|K\|_{L^1} Mf(x).$$

Proof. Let us suppose $K \in C_C^\infty$, then[4]:

$$\begin{aligned}
|K * f| &= \left| \int_{\mathbb{R}^d} K(y) f(x-y) dy \right| \\
&= \left| \int_0^{+\infty} K(r) \left(\int_{\Sigma_r} f(x-y) d\sigma_r \right) dr \right| \\
&= \left| - \int_0^{+\infty} K'(r) \left(\underbrace{\int_0^r \int_{\Sigma_s} f(x-y) d\sigma_s ds}_{\int_{B(x,r)} f(y) dy} \right) dr \right| \\
&= \left| - \int_0^{+\infty} K'(r) \frac{\mu(B(x,r))}{\mu(B(x,r))} \int_{B(x,r)} f(y) dy dr \right| \\
&\leq \left| \int_0^{+\infty} -K'(r) \mu(B(x,r)) Mf(x) dr \right| \\
&\leq Mf \left| \int_0^{+\infty} -K'(r) \mu(B(x,r)) dr \right| \\
&= Mf \left| \int_0^{+\infty} -K'(r) \int_0^r \left(\int_{\Sigma_s} 1 d\sigma_s \right) ds dr \right| \\
&\leq Mf \int_0^{\infty} |K(r)| \int_{\Sigma_r} d\sigma_r dr \\
&= Mf \|K\|_{L^1}
\end{aligned}$$

$\square$

[4]It is important to have $-K'(r) > 0$ so that we can consider Mf without any problem.

Corollary 2.1.13

If we consider $K_\varepsilon(x) = \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}\right)$ with $K \in L^1$, non increasing and radial. Then K^* is $(1,1)$-weak.

Proof. We have $\forall \varepsilon$

$$|K_\varepsilon * f(x)| \leq \|K_\varepsilon\|_{L^1} Mf \leq \|K\|_{L^1} Mf.$$

This implies

$$K^* f = \sup_\varepsilon |K_\varepsilon f| \leq \|K\|_{L^1} Mf.$$

□

Corollary 2.1.14

Assume $\int K dx = 1$ then $\forall f \in L^p$ with $1 \leq p < +\infty$ we have

$$K_\varepsilon * f \xrightarrow{\varepsilon \to 0} f \text{ a.e.}$$

Proof. Let us consider the maximal operator $K^* f = \sup_{\varepsilon > 0} |K_\varepsilon * f|$. Let us see that it is $(p-p)$-weak:

$$\mu\{K^f f > \lambda\} \leq \mu\{\|K\|_{L^1} Mf > \lambda\} \leq \frac{c\|f\|^p_{L^p} \|K\|^p_{L^1}}{\lambda^p}.$$

We conclude thanks to the lemma above. □

Osservazione 2.1.3

The previous result still holds, in a certain way, even if K which is not radial nor non-incresing: it is sufficient to consider a *radially symmetric majorant*, i.e. a function K_0 such that K_0 is radially simmetric, non increasing and

$$|K(x)| \leq |K_0(x)|.$$

Using the result on K_0 we get

$$|K * f| \leq \|K_0\|_{L^1} Mf(x).$$

2.1.1 Applications

The Lemma (2.1) has some non-obvious consequences in the theory of PDEs. Let us study the case of the *heat equation* and *Schrödinger's equation*.

Definition 2.1.15 – Heat Kernel

Given the *heat equation* with initial datum $f \in L^2(I)$, i.e.:

$$\begin{cases} \partial_t u = \Delta u \\ u(x) = f \end{cases}$$

we say u is a solution if

$$u = H_t * f,$$

where we definite the *heat kernel* as

$$H_t = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{-\pi|x|^2}{4\pi t}}.$$

Osservazione 2.1.4
We can show that a solution exists using the Fourier transform:

$$\begin{cases} \partial_t \hat{u} = -4\pi^2|\xi|^2\hat{u} \\ \hat{u}(\xi, \cdot) = \hat{f} \end{cases}.$$

This implies $\hat{u} = e^{-4\pi^2|\xi|^2 t}\hat{f}$ and so[5]

$$\begin{aligned} u &= (e^{-4\pi^2|\xi|^2 t})^{\vee} * f \\ &= \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{-\pi|x|^2}{4\pi t}} * f \\ &= H_t * f. \end{aligned}$$

Example 2.1.16. If u is the solution of the *heat equation* with initial datum $f \in L^2(\mathbb{R}^d)$ then $H_t * f \to f$ a.e.x.

Proof. We can see that $\|H_t\|_{L^1} = \|H_1\|_{L^1} = 1$, which means that H_t is scaling invariant. We notice also that $\sup_t \int |H_t|$ and $\forall \delta > 0$ $\int_{|x|>\delta} |H_t| \xrightarrow{t\to 0} 0$. Thus H_t is an *approximate identity*, i.e.

$$H_t * f \longrightarrow f \quad \text{in } L^2.$$

The pointwise convergence follows because

$$\sup |H_t * f| \leq Mf(x)$$

hence $C = \{f \in L^2 \text{ such that } \lim_{t\to 0} H_t f = f \text{ almost every } x\}$ is closed. The statement is clearly true for the C_C^∞ which are a dense, so the thesis follows for all $f \in L^2$. $\square$

Definition 2.1.17 – Schrödinger's kernel

Given the *Schrödinger's equation* with initial datum $g \in L^2(I)$, i.e.:

$$\begin{cases} \partial_t u = i\Delta u \\ u(x) = g \end{cases}$$

we say u is a solution if

$$u = S_t * g,$$

where we definite the *Schrödinger's kernel* as

$$S_t = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{-\pi|x|^2}{4\pi t}}.$$

---

[5] We consider initial datum $f \in L^2$, so that $\check{\hat{f}} = f$.

Osservazione 2.1.5
We can show that a solution exists using the Fourier transform:

$$\begin{cases} \partial_t \hat{u} = -4\pi^2 i|\xi|^2 \hat{u} \\ \hat{u}(\xi, \cdot) = \hat{g} \end{cases}$$

so $\hat{u} = e^{-4\pi^2 i|\xi|^2 t}\hat{g}$ and so

$$\begin{aligned} u &= (e^{-4\pi^2 i|\xi|^2 t})^{\vee} * g \\ &= \frac{1}{(4\pi t i)^{\frac{d}{2}}} e^{\frac{\pi|x|^2}{4t}} * g \\ &= S_t * g. \end{aligned}$$

Notice that in this case nor $e^{e^{-4\pi^2 i|\xi|^2 t}}$ or u are in L^1: the equalities follow by direct computations.

Example 2.1.18. If u is the solution of the *Schrödinger's equation* with initial datum $g \in L^2(\mathbb{R}^d)$ it is not true in general that $S_t * g \to g$ a.e.x.

Proof. The counterexample in $d = 1$ to this was given by König-Dalbert in 1984. If we add some stronger hypotesis, such as[6] $g \in H^s(\mathbb{R}^d)$ with $s \geq \frac{1}{4}$, thanks to Carleson theorem, we get that $S_t * g \to g$ a.e.x.
Another difference to the *heat equation* is the following: if we write

$$S_t * g := e^{it\Delta} g$$

then we have that this is a semigroup action on the initial datum, which is also an isometry[7] on L^2, i.e.

$$\|e^{it\Delta} g\|_{L^2} = \|g\|_{L^2}.$$

It is also true that

$$\|e^{it\Delta} g\|_{L^\infty} \lesssim \frac{1}{t^{\frac{d}{2}}} \|g\|_{L^1}.$$

By the Riesz-Thorin theorem we get the *dispersitive estimate*, i.e.:

$$\|e^{it\Delta} g\|_{L^q} \lesssim \frac{1}{t^{\frac{d}{2}\theta}} \|g\|_{L^p}.$$

The exponents p, q given by the theorem are such that

$$\begin{cases} \frac{1}{p} = \frac{1}{2} + \frac{\theta}{2} \\ \frac{1}{q} = \frac{1}{2} - \frac{\theta}{2} \end{cases} \Rightarrow \frac{1}{p} + \frac{1}{q} = 1.$$

where it is important that $p \in [1, 2]$ and $q \in [2, +\infty]$. Since $\theta = \frac{2}{p} - 1$ we get[8]

$$\begin{aligned} \|e^{it\Delta} g\|_{L^{p'}} &\lesssim \frac{1}{t^{\frac{d}{2}\theta}} \|g\|_{L^p} \\ &= \frac{1}{t^{\frac{d}{p} - \frac{d}{2}}} \|g\|_{L^p} \end{aligned}$$

□

[6] by H^s we denote the fractional Sobolev space.
[7] This is not true for the *heat equation*
[8] the exponent of t is positive because $p \leq 2$.

2.2 Hardy-Littlewood-Sobolev inequality

Prior to statement and proof of the Hardy- Littlewood-Sobolev inequality, we need the following:

Definition 2.2.1 – Volume and surface of balls

$\nu_d = \mu(\{\|x\|_{\mathbb{R}^d} \leq 1\}$ and ω_{d-1} = surface area (ossia $\|x\|_{\mathbb{R}^d} = 1$).

Proposition 2.2.2

For every d it holds that

$$\omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \quad \text{and} \quad \nu_d = \frac{\omega_{d-1}}{d}.$$

Proof. It is an easy computation[9]:

$$\begin{aligned}
\pi^{\frac{d}{2}} &= \int_{\mathbb{R}^d} e^{-|x|^2} dx = \omega_{d-1} \int_0^\infty e^{-r^2} r^{d-1} dr \\
&= \frac{\omega_{d-1}}{2} \int_0^\infty e^{-t} t^{\frac{d-1}{2}} t^{-\frac{1}{2}} dt \\
&= \frac{\omega_{d-1}}{2} \Gamma\left(\frac{d}{2}\right).
\end{aligned}$$

By definition we have

$$\nu_d = \int_{B(0,1)} 1 dx = \int_0^1 \omega_{d-1} r^{d-1} dr = \frac{\omega_{d-1}}{d}.$$

□

Definition 2.2.3

We define the *multiplicative operator* $|D|^s$ such that:

$$\widehat{|D|^s f} = (2\pi|\xi|)^s \hat{f}$$

This means[a], if we are allowed, that

$$|D|^s f = (2\pi|\xi|^s \hat{f})^\vee.$$

[a]Remembering that $\widehat{\partial_{x_i} f} = 2\pi i \xi_i \hat{f}$, we see, in the sense of *distributions*, that $|D|^2$ is a derivative operator.

Theorem 2.2.4 – Hardy-Littlewood-Sobolev inequality

For $0 < \gamma < d$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{d}$, with $1 < p, q < \infty$ we have

$$\left\| \frac{1}{|x|^\gamma} * f \right\|_{L^q} \leq C\|f\|_{L^p} \qquad \text{(HLS)}$$

[9]We used the following change of variables: $r^2 = t$, $dr = \frac{1}{2} t^{-\frac{1}{2}} dt$.

Osservazione 2.2.1
It is a generalization of Young inequality. Thanks to Example (1.3.6) we have that $|x|^{-\lambda} \in L^{p,\infty}$ for $p = \frac{d}{\lambda}$, indeed

$$\mu\left\{\frac{1}{|x|^\gamma} > \lambda\right\} = \mu\left\{\frac{1}{\lambda} > |x|^\gamma\right\} = \nu_d \frac{1}{\lambda^{\frac{d}{2}}}.$$

It follows that

$$\left\|\frac{1}{|x|^\gamma} * f\right\|_{L^q} \leq C\left\|\frac{1}{|x|^\gamma}\right\|_{L^{R,\infty}} \|f\|_{L^p}$$

which is like Young inequality with the exponents being

$$1 + \frac{1}{p} = \frac{1}{q} + \frac{\lambda}{d}.$$

Osservazione 2.2.2
Having $(|\xi|^{-k}\hat{f})^\vee = c\int \frac{f(y)}{|x-y|^{d-k}}$ it holds that

$$|D|^{-k} f = \frac{1}{|x|^{\frac{d-k}{\gamma}}} * f.$$

Hence (HLS) becomes:

$$\||D|^{\gamma-d} f\|_{L^q} \lesssim \|f\|_{L^p}.$$

Osservazione 2.2.3
The exponents p, q given in the statement of (HLS) are the only one possible: we see this thanks to a *scaling argument*. If the inequality is true $\forall f \in L^p$ then it true also for $f_\lambda = f(\frac{x}{\lambda})$, i.e.:

$$\lambda^{\smile}\||D|^{\gamma-d} f\|_{L^q} = \||D|^{\gamma-d} f_\lambda\|_{L^q} \lesssim \|f_\lambda\|_{L^p} = \lambda^{\frac{d}{p}}\|f\|_{L^p}.$$

We need to understand what $\smile$ is: remembering $\partial_{x_i} f_\lambda(x) = \frac{1}{\lambda}\partial_{x_i} f(\frac{x}{\lambda})$ we have that

$$|D|^\alpha f_\lambda(x) = |\lambda|^{-\alpha}|D|^\alpha f\left(\frac{x}{\lambda}\right).$$

This means that

$$\lambda^{-(\gamma-d)+\frac{d}{q}}\|f\|_{L^p} \leq \lambda^{\frac{d}{p}}\|f\|_{L^p}$$

which implies

$$d - \gamma + \frac{d}{q} = \frac{d}{p}.$$

We are finally ready to prove HLS.

Proof of HLS inequality.

$$\begin{aligned}
\frac{1}{|x|^\gamma} * f &= \int \frac{1}{|y|^\gamma} f(x-y)dy \\
&= \int_{|y|\leq R} \frac{1}{|y|^\gamma} f(x-y)dy + \int_{|y|>R} \frac{1}{|y|^\gamma} f(x-y)dy \\
&= f * \left(\frac{1}{|y|^\gamma}\chi_{B_R}\right) + \int_{|y|>R} \frac{1}{|y|^\gamma} f(x-y)dy.
\end{aligned}$$

We now proceed to estimate separately the two integrals. The first one is an easy computation[10]:

$$\begin{aligned}\left|f * \frac{1}{|y|^\gamma}\chi_{B_R}\right| &\leq Mf\|\frac{1}{|y|^\gamma}\chi_{B_R}\|_{L^1}\\ &= Mf\int_{B_R}\frac{1}{|y|^\gamma}dy\\ &= Mf\omega_{d-1}\int_0^R\frac{r^{d-1}}{r^\gamma}dr = \frac{\omega_{d-1}}{d-\gamma}R^{d-\gamma}\\ &\preccurlyeq Mf(x)R^{d-\gamma}.\end{aligned}$$

For the second integral we see that[11]:

$$\begin{aligned}\int_{|y|>R}\frac{1}{|y|^\gamma}f(x-y)dy &\leq \|f\|_{L^p}\left\|\frac{1}{|y|^\gamma}\chi_{B_R^C}\right\|_{L^{p'}}\\ &= \|f\|_{L^p}\left(\omega_{d-1}\int_R^\infty\frac{r^{d-1}}{r^{\gamma p'}}dr\right)^{\frac{1}{p'}}\\ &\sim \|f\|_{L^p}R^{\frac{d-\gamma p'}{p'}}\end{aligned}$$

where we remember that, having $q \neq +\infty$, the is finite[12] since

$$1 < 1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{d} \Rightarrow \frac{\gamma}{d} > \frac{1}{p'}.$$

We proved we have the following estimate:

$$\left|\frac{1}{|\cdot|^\gamma} * f(x)\right| \lesssim MfR^{d-\gamma} + \|f\|_{L^p}R^{\frac{d}{p'}-\gamma}$$

Having different exponents we want to find R such that $MfR^{d-\gamma} = \|f\|_{L^p}R^{\frac{d}{p'}-\gamma}$. It is clear that

$$\frac{Mf}{\|f\|_{L^p}} = R^{\frac{d}{p'}-d} = R^{-\frac{d}{p}} \Rightarrow R = \left(\frac{Mf}{\|f\|_{L^p}}\right)^{-\frac{p}{d}}.$$

With this choice of R we have:

$$\left|\frac{1}{|\cdot|^\gamma} * f(x)\right| \lesssim \|f\|_{L^p}^{1-\frac{p}{q}}Mf(x)^{\frac{p}{q}}.$$

which implies that its L^q norm is:

$$\left\|\frac{1}{|\cdot|^\gamma} * f(x)\right\|_{L^q} \lesssim \|f\|_{L^p}^{1-\frac{p}{q}}\|Mf\|_{L^p}^{\frac{p}{q}}.$$

The Hardy-Littlewood-Sobolev inequality follows sinche Mf is (p,p) weak, hence

$$\begin{aligned}\left\|\frac{1}{|\cdot|^\gamma} * f(x)\right\|_{L^q} &\lesssim \|f\|_{L^p}^{1-\frac{p}{q}}\|Mf\|_{L^p}^{\frac{p}{q}}\\ &\preccurlyeq \|f\|_{L^p}^{1-\frac{p}{q}}\|f\|_{L^p}^{\frac{p}{q}}\\ &\preccurlyeq \|f\|_{L^p}.\end{aligned}$$

We observe that this proof gives a better estimate than the HLS since we have a control with the maximal function. $\square$

[10] We stress that it is pointwise

[11] We notice that R is elevated to a negative power.

[12] we need to have $d - 1 - \gamma p' < -1$.

2.2.1 Applications

Thanks to the Hardy-Littlewood-Sobolev inequality we can see if, given the energy E to a problem, we have *coercivity*. By this term we mean the possibility that, if $\|\nabla u\|_{L^p} \to +\infty$ then $E(u) \to +\infty$.

Example 2.2.5. Given the problem $-\Delta\phi = |u|^2$, which energy is

$$E(u) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy$$

there exists p such that $|E(u)| \preccurlyeq \|u\|_{L^p}$.

Proof. If we are in $\mathbb{R}^3$ we know the solution to this problem is

$$\Phi = \frac{1}{4\pi}\int \frac{|u(y)|^2}{|x-y|}dy.$$

Remembering that $\int \frac{|u(x)|^2|u(y)|^2}{|x-y|} = \left(\frac{1}{x} * |u|^2\right)|u|^2$ then thanks to HLS we see that p has to be such that

$$1 + \frac{1}{p'} = \frac{1}{p} + \frac{1}{3} \Rightarrow 2 - \frac{1}{p} = \frac{1}{p} + \frac{1}{3} \Rightarrow p = \frac{6}{5}.$$

This means that

$$\begin{aligned} E(u) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy &= \int_{\mathbb{R}^3}\left(\frac{1}{x} * |u|^2\right)|u|^2dx \\ &\leq \|u^2\|_{L^p}\left\|\frac{1}{x} * |u|^2\right\|_{L^{p'}} \\ &\preccurlyeq \|u\|^2_{L^p}\|u\|^2_{L^p} \\ &= \|u^2\|^2_{L^{\frac{6}{5}}} = \|u\|^4_{L^{\frac{12}{5}}}. \end{aligned}$$

The fact that there is a fourth power should not shock us, as a matter of fact we could immediatly see that, for homogeneity of HLS, we need to have something like $\|u\|^4_{L^p}$, whichever p is. In conclusion we have that, for $p = \frac{12}{5}$ we have

$$|E(u)| \preccurlyeq \|u\|_{L^p}.$$

$\square$

Example 2.2.6. What if we consider the same problem for any dimension d? The energy in this case is:

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{d-2}}dxdy.$$

Proof. In this case the solution is, where C is a constant we do not care about, the following:

$$\Phi = C\int \frac{|u(y)|^2}{|x-y|^{d-2}}dy.$$

The exponent given by HLS is

$$1 + \frac{1}{p'} = \frac{1}{p} + \frac{d-2}{d} \Rightarrow p = \frac{2d}{d+2}.$$

So we have

$$\begin{aligned}\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{d-2}} dx dy &\leq \|u^2\|_{L^p} \|u^2\|_{L^{p'}} \\ &\lesssim \|u^2\|^2_{L^{\frac{2d}{d+2}}} = \|u\|^4_{L^{\frac{4d}{d+2}}}.\end{aligned}$$

□

We now see whether or not we have *coercivity*: in order to this we need the Sobolev inequality, stated in 3.1.3, which tells us that in $\mathbb{R}^3$ we have $\|u\|_{L^6} \preccurlyeq \|\nabla u\|_{L^2}$. In general, if we have an energy E we can write $E = K + U$ where K is the kinetic energy and U is the potential one. In this case we want to study

$$\inf_{\substack{\|u\|_{L^2}=1 \\ u \in H^1}} \frac{1}{2} \left\{ \int |\nabla u|^2 - \frac{1}{4} \iint \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy > +\infty \right\}.$$

In this case we have[13] $K = \int |\nabla u|^2$ and $U = \frac{1}{4} \iint \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy$. We can see that:

$$U \lesssim \|u\|^4_{L^{\frac{12}{5}}} \lesssim \|u\|^{\alpha}_{L^2} \|u\|^{\beta}_{L^6} \lesssim \|u\|^{\alpha}_{L^2} \|\nabla u\|^{\beta}_{L^2}.$$

We have *coercivity* if $\beta < 2$. Indeed by Hölder we have

$$\|u\|_{L^p} \leq \|u\|^{\theta}_{L^2} \|u\|^{1-\theta}_{L^6}.$$

where

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{6} = \frac{3\theta + 1 - \theta}{6} \Rightarrow \frac{2\theta}{6} = \frac{1}{p} - \frac{1}{6} = \frac{6-p}{6p} \Rightarrow \theta = \frac{6-p}{2p}.$$

If $p = \frac{12}{5}$ we get $\theta = \frac{3}{4}$, hence

$$U \lesssim \|u\|^3_{L^2} \|\nabla u\|_{L^2}.$$

□

Example 2.2.7. Given the problem[a] $i\partial_t u = -\Delta u - \frac{1}{|x|} u$, prove that

$$\inf_{\|u\|_{L^2}=1} \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{2} \int \frac{|u|^2}{|x|} dx > -\infty.$$

[a] It arises from quantum mechanics

Proof. Using the Hardy inequality (3.1.1):

$$\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx \right)^{\frac{1}{2}} \leq \frac{2}{d-2} \|\nabla u\|_{L^2}$$

we have $\inf > -\infty$. □

[13] Physically this are the kinetic and potential energy associated to an electric charge

Chapter 3

The Hilbert and Riesz Transform

3.1 Schwartz class and Distributions

Thanks to Plancherel equality and Riesz-Thorin theorem we are able to define the Fourier transform for fuctions $f \in L^p$ with $1 \leq p \leq 2$. The aim of this chapter is to generalize this definition, in order to have a Fourier transform for more objects, which will be the *tempered distributions*.

3.1.1 Schwartz class of functions

The idea will be, given some linear operator T acting on a class of function $\mathcal{S}$, to define $\hat{T}$ such that

$$\langle \hat{T}, u \rangle = \langle T, \hat{u} \rangle.$$

It is immediate that we need to define the class of function $\mathcal{S}$ in order that, if $u \in \mathcal{S}$, also $\hat{u} \in \mathcal{S}$.

Let us now fix some notation[1]: given $x \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multiindex, where $\alpha_i \geq 0 \ \forall i$ and $|\alpha| = \sum_{i=1}^d \alpha_i$, we write

$$x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$$

$$\partial^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d} f$$

It is immediate to observe that:

Proposition 3.1.1

For every $x \in \mathbb{R}^d$ and α multiindex the following hold:

1. $$|x|^\alpha \leq C_{d,\alpha} |x|^{|\alpha|}.$$

2. For all $k \in \mathbb{Z}_+$
$$|x|^k \leq C_{k,d} \sum_{|\alpha|=k} |x^\alpha|$$

Proof. This inequalities follow in the same way: let us considere the following maps:

$$\begin{array}{rccc} \xi_1: & \mathbb{S}^{d-1} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & |x^\alpha| \end{array}.$$

[1] we will use the same notation as in Grafakos

$$\begin{array}{rccc} \xi_2 : & \mathbb{S}^{d-1} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sum_{|\alpha|=k} |x^\alpha| \end{array} .$$

Both of them have the minimum which is greater then 0 for costruction, by omogeneity we get the thesis. □

Osservazione 3.1.1
In the first inequality, it is not possible to have an inequality where we have two different multiindexes α, β:

$$|x^\alpha| \precsim |x|^{|\beta|}.$$

In order to see this it is sufficient to do a *rescaling argument*: if it true for x it should also hold for λx, where $\lambda \in \mathbb{R}$.

Proposition 3.1.2 – Leibnitz rule

For all $f, g \in \mathbb{R}^d$ we have

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_d}{\beta_d} \partial^\beta f \partial^{\alpha-\beta} g$$

where $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ $\forall 1 \leq j \leq d$.

Definition 3.1.3 – Schwartz function

$f \in C^\infty$ is in the Schwartz class $\mathcal{S}$ if $\forall \alpha, \beta$ multiindex we have

$$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f| < +\infty.$$

This will be a class of test functions.

Osservazione 3.1.2
We will often use this equivalent definition: given $f \in C^\infty$ the following holds:

$$f \in \mathcal{S} \iff |\partial^\beta f(x)| \leq C_{\beta,N} \frac{1}{\lfloor 1 + |x| \rfloor^N} \qquad \forall \beta \forall n.$$

This means that Schwartz functions decay faster then every polynomial.

Definition 3.1.4 – Convergence in Schwartz's class

We say that given $f_n \in \mathcal{S}$, $f \in \mathcal{S}$ then $f_n \xrightarrow{\mathcal{S}} f$ if $\forall \alpha, \beta$

$$\rho_{\alpha,\beta}(f_n - f) \overset{n\to\infty}{\longrightarrow} 0.$$

Proposition 3.1.5

For $p \in [1, +\infty]$ if $f_n \xrightarrow{\mathcal{S}} f$ then[a] $f_n \xrightarrow{L^p} f$ and

$$\|\partial^\beta f\|_{L^p} \leq C_{d,p} \sum_{|\alpha| \leq \left\lfloor \frac{d+1}{p} \right\rfloor + 1} \rho_{\alpha,\beta}(f).$$

[a] This implies that the convergence in $\mathcal{S}$ is stronger than convergence in all L^p.

Proof. Let us prove that, given the estimate, than the $\mathcal{S}$-convergence implies the L^p-convergence:

$$\|f_n - f\|_{L^p} \preccurlyeq \sum_{|\alpha| \leq \left\lfloor \frac{d+1}{p} \right\rfloor + 1} \rho_{\alpha,\beta}(f_n - f) \to 0$$

because is a finite sum of numbers going to 0.
Let us prove the estimate:

$$\begin{aligned}
\|\partial^\beta f\|_{L^p} &= \left(\int_{\mathbb{R}^d} |\partial^\beta f|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{|x|<1} |\partial^\beta f|^p dx + \int_{|x|\geq 1} |\partial^\beta f|^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\nu_d \left(\sup_{x\in\mathbb{R}^d} |\partial^\beta f| \right)^p + C \sup_{x\geq 1} |x|^{d+1} |\partial^\beta f|^p \right)^{\frac{1}{p}} \\
&\leq \left(\nu_d \left(\sup_{x\in\mathbb{R}^d} |\partial^\beta f| \right)^p + C \sup_{x\geq 1} |x|^{\left\lfloor \frac{d+1}{p} \right\rfloor + 1} |\partial^\beta f|^p \right)^{\frac{1}{p}} \\
&\preccurlyeq \rho_{\alpha+\beta}(f) + \sum_{|\alpha| = \left\lfloor \frac{d+1}{p} \right\rfloor + 1} \rho_{\alpha,\beta}(f)
\end{aligned}$$

where the last inequality follows from the property of multindexes and where we also used the fact that

$$\int_{|x|\geq 1} |\partial^\beta f|^p dx = \int_{|x|\geq 1} |x|^{d+1} |x|^{-d-1} |\partial^\beta f|^p dx \leq C \sup_{x\geq 1} |x|^{d+1} |\partial^\beta f|^p.$$

□

We can see that the $\mathcal{S}$ class is closed under many operations such as convolution and fourier transform.

Proposition 3.1.6

Given $f, g \in \mathcal{S}$ we have that $f * g \in \mathcal{S}$.

Proof. For any $x, y \in \mathbb{R}^d$ and $N \in \mathbb{N}$ we have:

$$\frac{1}{(1+|x-y|)^N} \leq \frac{(1+|y|)^N}{(1+|x|)^N}.$$

It is sufficient to prove this for $N = 1$ and then it will follow for induction. When $N = 1$ we can see that

$$1 + |x| = 1 + |x - y + y| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|).$$

If we now compute $f * g(x)$ we have:

$$\begin{aligned}
|f * g| &= \left|\int f(x-y)g(y)dy\right| \\
&\leq \int \frac{1}{(1+|x-y|)^N} \frac{dy}{(1+|y|)^{N+d+1}} \\
&\preccurlyeq \frac{1}{(1+|x|)^N} \int \frac{(1+|y|)^N}{(1+|y|)^{N+d+1}} dy \\
&= \frac{1}{(1+|x|)^N} \int \frac{1}{(1+|y|)^{d+1}} < +\infty.
\end{aligned}$$

We now have to check all the derivatives $\partial^\beta(f * g)$, but this is easy by observing that:

$$\partial^\beta(f * g) = \partial^\beta f * g.$$

□

Lemma 3.1.7

For any $f \in \mathcal{S}$ and any α multiindex we have:

1. $\widehat{\partial^\alpha f} = (2\pi i \xi)^\alpha \hat{f}(\xi)$
2. $\partial^\alpha \hat{f}(\xi) = (\widehat{-2\pi i x})^\alpha f(\xi)$

Proof. For semplicity we will assume we only have one derivative, i.e. $\alpha = (1, 0, \ldots, 0)$.
Proof of 1):

$$\begin{aligned}
\widehat{\partial^\alpha f} &= \int_{\mathbb{R}^d} \partial^\alpha f(x) e^{-2\pi i \xi \cdot x} dx \\
&= (-1)^{|\alpha|} \int (-2\pi i \xi)^\alpha f(x) e^{-2\pi i \xi \cdot x} dx \\
&= (2\pi i \xi)^\alpha \hat{f}(\xi).
\end{aligned}$$

The integration by parts is possible because, considering $R \to +\infty$, the following holds:

$$\int_{B_R} \partial_{x_1} f e^{-2\pi i \xi \cdot x} dx = -\int_{B_R} f(-2\pi i \xi_1) e^{-2\pi i \xi \cdot x} dx + \int_{\partial B_R} f e^{-2\pi i \xi \cdot x)} \nu_1 d\sigma$$

Idea This trick is quite common for Schwartz functions which decay really fast. We want to see that the second term goes to zero so it's done: it is quite obvious because f decays faster then any polynomial. ⌟

Proof of 2):

$$\begin{aligned}
\partial_{\xi_1} \hat{f}(\xi) &= \lim_{h \to 0} \frac{\hat{f}(\xi + h e_1) - \hat{f}(\xi)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \int f(x) \left[e^{-2\pi i (\xi + h e_1) \cdot x} - e^{-2\pi i \xi \cdot x}\right] dx \\
&= \lim_{h \to 0} \int f(x) e^{-2\pi i \xi \cdot x} \frac{e^{-2\pi i (h x_1) - 1}}{h} \\
&= \int f(x) \lim() \\
&= (\widehat{-2\pi i x})^\alpha f.
\end{aligned}$$

where the limit passed inside because

$$e^{-2\pi i\xi\cdot x}\frac{e^{-2\pi i(hx_1)-1}}{h}-(-2\pi i x_1)e^{-2\pi i\xi\cdot x}=H(x)$$

needs to go to 0 in L^1. We need to see $|H(x)|\leq C|x|$ in order to apply dominance convergence, which follows from

$$\sup|g'|\geq\left|\frac{g(h)-g(0)}{h}\right|.$$

□

Theorem 3.1.8

If $f\in\mathcal{S}$ then $\hat{f}\in\mathcal{S}$.

Proof. We want to see that $x^\alpha\partial^\beta f$ in L^∞ for any α,β. The idea is to see that something like this holds:

$$\|x^\alpha\partial^\beta\hat{f}\|_{L^\infty}=\|\hat{\smile}\|_{L^\infty}\leq\|\smile\|_{L^1}$$

Let us understand what the $\smile$ is: by the previous formulas we have

$$\|x^\alpha\partial^\beta\hat{f}\|_{L^\infty}=\|\widehat{\partial^\alpha(x^\beta f)}\|_{L^\infty}\frac{(2\pi)^{|\beta|}}{2\pi^{|\alpha|}}\leq C\|\partial^\alpha(x^\beta f)\|_{L^1}<+\infty.$$

□

Example 3.1.9. For $d=1$ let $g=\chi_{[0,b]}(x)$. We want to compute $\hat{g}$.

$$\hat{g}(\xi)=\int_a^b g(x)e^{-2\pi i\xi x}dx=\left[\frac{e^{-2\pi i\xi x}}{-2\pi i\xi}\right]_a^b=\frac{e^{-2\pi i\xi a}-ie^{-2\pi i\xi b}}{2\pi i\xi}.$$

Example 3.1.10. For any d let us consider $g=\prod_{i=1}^d\chi_{[a_i,b_i]}(x_i)$. So

$$\hat{g}(\xi)=\prod_{i=1}^d\left[\frac{e^{-2\pi i\xi_i a_i}-ie^{-2\pi i\xi_i b_i}}{2\pi i\xi_i}\right].$$

Is it true that $\hat{g}$ goes like $\frac{1}{|x|}$? We see that if $\xi\in\mathbb{R}^d\backslash\{0\}$ then there exists i_0 such that $|\xi_{i_0}|>\frac{|\xi|}{\sqrt{d}}$. If this were not the case we would have

$$|\xi|^2=\sum x_i^2<\sum\frac{|\xi|^2}{d}=|\xi|^2$$

which is absurd. So in the previous computation we have:

$$|\hat{g}(\xi)|\leq\frac{2}{2\pi}\frac{\sqrt{d}}{|\xi|}\prod_{i\neq i_0}(b_i-a_i)$$

where we used the Lagrange's estimate to write $\frac{e^{-2\pi i\xi_i a_i}-ie^{-2\pi i\xi_i b_i}}{2\pi i\xi_i}\leq(b_i-a_i)$.

Theorem 3.1.11

If $f \in L^1$ we have $\hat{f}(\xi) \to 0$ for $|\xi| \to +\infty$.

Proof. We know that for each $f \in L^1$ there exist g simple function such that, for $\varepsilon > 0$, $\|f - g\|_{L^1} \leq \frac{\varepsilon}{2}$. We see that

$$|\hat{f}(\xi)| = |\hat{f}(\xi) - \hat{g}(\xi) + \hat{g}(\xi)| \leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| \leq \|f - g\|_{L^1} + \frac{C}{|\xi|}.$$

The thesis follows because for $|\xi| > M_0$ we have $|\hat{g}| \leq \frac{\varepsilon}{2}$, which gives $|\hat{f}(\xi)| \leq \varepsilon$. □

Proposition 3.1.12 – Hardy inequality

Given $d \geq 3$ for all $f \in C_C^\infty(\mathbb{R}^d \backslash \{0\})$ we have:

$$\left(\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx \right)^{\frac{1}{2}} \leq \frac{2}{d-2} \|\nabla f\|_{L^2}.$$

Proof of Hardy. The best costant is never achieved (non capito come si vede)
The reason why we ask $f \in C^\infty(\mathbb{R}^d \backslash \{0\})$ is that we dont want problems when integrating by parts. Remember[2]

$$\frac{1}{|x|^2} = -\frac{1}{2} x \cdot \nabla \left(\frac{1}{|x|^2} \right)$$

So if we define $R(f) = \sum x_i \frac{\partial f}{\partial x_i}$ we have the following properties:

$$\frac{1}{|x|^2} = -\frac{1}{2} R \left(\frac{1}{|x|^2} \right)$$

$$\left| \sum x_i \partial_{x_i}(f) \right|^2 \leq \sum_{i=1}^d x_i^2 \sum |\partial_{x_i} f|^2 = \sum x_i^2 \nabla f$$

Let's prove the inequality:

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx &= \int |f|^2 \left(-\frac{1}{2} x \cdot \nabla \left(\frac{1}{|x|^2} \right) \right) dx \\
&= \frac{1}{2} \int \sum_{i=1}^d \partial_{x_i}(x_i |f|^2) \cdot \frac{1}{|x|^2} dx \\
&= \frac{d}{2} \int \frac{|f|^2}{|x|^2} + \frac{1}{2} \int \frac{1}{|x|^2} \sum x_i [f_{x_i} \bar{f} + f \bar{f}_{x_i}] dx
\end{aligned}$$

where we use the fact that, having $|f|^2 = f\bar{f}$, it follows that $\partial_{x_i}(f\bar{f}) = f_{x_i}\bar{f} + f\bar{f}_{x_i}$. For now we have that:

$$\frac{d-2}{2} \int \frac{|f|^2}{|x|^2} \leq \frac{1}{2} \int \frac{R(f)\bar{f}}{|x|^2} + \frac{R(\bar{f})f}{|x|^2} dx.$$

[2] $\partial_{x_i} \frac{1}{|x|^2} = \frac{-2}{|x|^3} \frac{x_i}{|x|} = \frac{-2x_i}{|x|^4}$ e quindi $\sum x_i \cdot \frac{-2x_i}{|x|^4} = \frac{-2}{|x|^2}$.

By applying Cauchy-Schwartz we get

$$\int \left| \frac{R(f)\bar{f}}{|x|^2} dx \right| = \left(\int \frac{|R(f)|^2}{|x|^2} \right)^{\frac{1}{2}} \left(\int \frac{|\bar{f}|^2}{|x|^2} \right)^{\frac{1}{2}}$$

and in conclusion

$$\begin{aligned} \frac{d-2}{2} \left(\int \frac{|f|^2}{|x|^2} \right)^{\frac{1}{2}} &\leq \left(\int \frac{|R(f)|^2}{|x|^2} dx \right)^{\frac{1}{2}} \\ &\leq \left(\int |\nabla f|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

□

Osservazione 3.1.3
From this inequality we have:

$$\int \frac{|f|^2}{|x|^2} dx \preccurlyeq \|\nabla f\|_{L^2}^2 = \||\xi| \hat{f}\|_{L^2}.$$

It is possible to generalize and get,$\forall s, 0 < s < \frac{d}{2}$

$$\int \frac{|f|^2}{|x|^{2s}} \preccurlyeq \||\xi|^s \hat{f}\|_{L^2}.$$

Proposition 3.1.13 – Heisenberg's inequality

For any $f \in s$ we have

$$\|f\|_{L^2}^2 \leq \frac{4\pi}{d} \inf_{y \in \mathbb{R}^d} \left(\int |x-y|^2 |f(x)|^2 dx \right)^{\frac{1}{2}} \inf_{z \in \mathbb{R}^d} \left(\int |\xi - z|^2 |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

Osservazione 3.1.4
In some books you can find a different statement, which usually is

$$\|f\|_{L^2}^2 \leq C \|xf\|_{L^2} \|\nabla f\|_{L^2}.$$

The problem with this formulation is that, its *physical meaning* is not really clear. If you consider f as the position of a particle and $\hat{f}$ as its frequency, you cannot have a peak for both of them because, if this were the case, you would have both of the inf $= 0$. Let us remember then having a peak means that we know with certainty the quantity rapresented by those functions.

Proof of Heisenberg. Let us see it for $d = 1$.

$$\|f\|_{L^2}^2 = \int f\bar{f} \partial_x (x-y) dx = -\int \partial_x (f\bar{f})(x-y) dx = -\int (f_x \bar{f} + f \bar{f}_x)(x-y) dx$$

By cauchy schwartz

$$\|f\|_{L^2}^2 \leq 2 \left(\int |f_x|^2 dx \right)^{\frac{1}{2}} \left(\int |\bar{f}_x||^2 dx \right)^{\frac{1}{2}} = (*)$$

e visto che

$$(\int |f_x|^2 dx)^{\frac{1}{2}} = (\int 4\pi^2 |\xi|^2 |\hat{f}|^2 d\xi)^{\frac{1}{2}}$$

allora

$$(*) = 4\pi(\int |\xi|^2 |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}} (\int |f|^2 (x-y)^2)^{\frac{1}{2}}$$

e ci posso anche mettere un $\inf_y$. By remembering that $\widehat{fe^{-2\pi ixz}} = \hat{f}(\xi - z)$ allora si può rifare lo stesso giochetto come prima per il primo pezzo con $\hat{f}$ così da avere $\int_z$.
Exercise to generalize to every d. $\square$

3.1.2 Distributions

Today we introduce distributions and then we will prove the Sobolev inequality.
Notation: $C_0^\infty = \{$ smooth function with compact support$\}$. We have $C_0^\infty \subset \mathcal{S} \subset C^\infty$. These three spaces of test functions will generate three dual spaces:

$$(\mathcal{S})' = \mathcal{S}' = \{\text{tempered distributions}\}$$

$$(C_0^\infty)' = D' = \{\text{distributions}\}$$

$$(C^\infty)' = \mathcal{E}' = \{\text{distribution with compact support}\}$$

Taking the dual reverses the inclusions, hence $\mathcal{E}' \subset \mathcal{S}' \subset D'$. We recall what the convergence in these spaces is:

- $f_k \overset{\mathcal{S}}{\to} f$ if $\forall \alpha, \beta$ $\rho_{\alpha,\beta}(f_k - f) \to 0$.
- $f_k \overset{C_0^\infty}{\to} f$ if $\forall \alpha$ we have $\|\partial^\alpha (f_k - f)\|_{L^\infty} \to 0$ where $supp f_k \subset B$ forall k with B compact.
- $f_k \overset{C^\infty}{\to} f$ if $\forall \alpha, \forall N > 0$ we have $\sup |\partial^\alpha f_k - f| \to 0$.

f sta in un duale qualsiasi se $T(f_k) \to T(f)$ quando $f_k \to f$ nello spazio di cui considero il duale.

Example 3.1.14. For any $\varphi \in C_0^\infty$ with $d = 1$ we consider $\varphi_k = \frac{1}{k}\varphi(x-k)$. We can see that $\varphi_k \overset{C^\infty}{\to} 0$ whereas $\varphi_k \overset{\mathcal{S}}{\not\to} 0$. The first one follows because φ è a supporto compatto, quindi esiste k tale che $\varphi_k = 0$ essnedo fuori dal supporto e quindi ok. Vediamo perchè non vale l'altra: assume that is converges to 0, then we have

$$\rho_{1,0} = \sup_{x \in \mathbb{R}} \left| \frac{x\varphi(x-k)}{k} - 0 \right|$$

Claim is that $\rho_{1,0} \overset{k \to +\infty}{\to} 0$, if we consider $x = k$ then we have $\varphi(0)$ which can be whatever we want, so absurd. Quindi è quite strong come convergenza.

Proposition 3.1.15

A linear functional T acting on $\mathcal{S}$, so $T : \mathcal{S} \to \mathbb{C}$, is a *tempered distribution* if and only if there exists m, k such that

$$|\langle T, f\rangle| \leq c \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq k}} \rho_{\alpha,\beta}(f) \quad \forall f \in \mathcal{S}$$

Proof. Omessa. □

Example 3.1.16 (Dirac mass at zero). We denote the dirac mass at the point a with δ_a. We define δ_a as the operator such that

$$\langle \delta_a, \varphi\rangle = \varphi(a) \quad \forall \varphi \in C^\infty.$$

We have $\delta_0 \in \mathcal{E}'$, indeed if we have $\varphi_k \xrightarrow{C^\infty} \varphi$ then

$$\langle \delta_0, \varphi_k\rangle \to \langle \delta_0, \varphi\rangle$$

Because $\mathcal{E}' \subset \mathcal{S}' \subset D'$, it is true that $\delta_0 \in \mathcal{S}$ and $\delta_0 \in D'$.

Idea per analisi armonica più importante $\mathcal{S}'$, per pde D'. ⌟

Example 3.1.17. if $f \in L^p$ with $1 \leq p \leq +\infty$ then if we consider

$$\langle T_f, \varphi\rangle = T_f(\varphi) = \int_{\mathbb{R}^d} f\varphi dx$$

we have $T_f \in D'$.

$$|\langle T_f, \varphi\rangle| \leq \|\varphi\|_{L^{p'}} \|f\|_{L^p}$$

so if $\varphi_k \xrightarrow{\mathcal{S}} \varphi$ then $\varphi_k \xrightarrow{L^p} \varphi$ and

$$|\langle T_f, \varphi_j - \varphi\rangle| \leq \|\varphi_j - \varphi\|_{L^{p'}} \|f\|_{L^p}.$$

Example 3.1.18. If we have $|g| \precsim (1+|x|)^k$ for some k, where g is a measurable function

$$\begin{aligned} \left|\int_{\mathbb{R}^d} g\varphi dx\right| &= |T_g(\varphi)| \precsim \int_{\mathbb{R}^d} (1+|x|)^k |\varphi(x)| dx \\ &= \int (1+|x|)^m (1+|x|)^{k-m} |\varphi(x)| dx \\ &\overset{*}{\precsim} \sup_{x\in\mathbb{R}^d} (1+|x|)^m |\varphi(x)| \int (1+|x|)^{k-m} dx \end{aligned}$$

dove $*$ segue perchè φ è schwartz. Se $k-m < d$ the function integrable. So we take m such that $m > k-d$ and we have that $T_g(\varphi)$ is controlled by a finite number of seminorms, so it is a tempered distribution.

We see now differentiation of distributions and the fourier transform of tempered distribution.

Definition 3.1.19 – Differentiation of distribution

We define[a] per $\varphi \in C_0^\infty$

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$$

[a] il $(-1)^{|\alpha|}$ is for rispettare the integration by parts.

Definition 3.1.20 – Fourier Transform and Antitransform of tempered distribution

$\varphi \in \mathcal{S}$ allora[a]

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle$$

(well defined because is $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$). We also have

$$\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$$

[a] anche qui rispetti il caso in cui T is a function

Idea Let us compute $\widehat{\partial^\alpha \delta_0}$ so we can prendere confidenza. ⌟

Proposition 3.1.21

It holds

$$\widehat{\partial^\alpha \delta_0} = (2\pi i x)^\alpha.$$

Proof. We want to see that

$$\langle \widehat{\partial^\alpha \delta_0}, \varphi \rangle = \int (2\pi i x)^\alpha \varphi dx$$

Idea Distributions *act* on functions. So you consider the integral. ⌟

$$\begin{aligned}
\langle \widehat{\partial^\alpha \delta_0}, \varphi \rangle &= \langle \partial^\alpha \delta_0, \hat{\varphi} \rangle \\
&= (-1)^{|\alpha|} \langle \delta_0, \partial^\alpha \hat{\varphi} \rangle \\
&= (-1)^{|\alpha|} \langle \delta_0, \widehat{(-2\pi i x)^\alpha \varphi} \rangle \\
&= \langle \delta_0, \widehat{(2\pi i x)^\alpha \varphi} \rangle \\
&= \widehat{(2\pi i x)^\alpha \varphi}(0) = \int (2\pi i x)^\alpha \varphi(x) e^{-2\pi i 0 \cdot x}.
\end{aligned}$$

In particular $\hat{\delta_0} = 1$. □

Idea There are distributions whose fourier transform is a function. So the next definition should not impress ourselves: ⌟

So we can consider the homogeneous sobolev space

$$\mathring{H}^s = \{\varphi \in \mathcal{S}', \hat{\varphi} \in L^1_{loc} s.t. \int |\xi|^{2s} |\hat{\varphi}|^2 d\xi < +\infty\}.$$

Esercizio 3.1.1
Prove that
$$\widehat{\delta_a} = e^{-2\pi i a\cdot x}$$

3.1.3 Sobolev Inequality

We are set on $\mathbb{R}^d$. We consider, given $\varphi \in \mathcal{S}$, the operator D such that
$$\widehat{|D|^s\varphi} = (2\pi i|\xi|)^s\hat{\varphi}$$

Osservazione 3.1.5
remember that $\widehat{\partial_{x_j}\varphi} = 2\pi i\xi_j\hat{\varphi}$ and $\hat{\Delta\varphi} = -4\pi^2|\xi|^2\hat{\varphi}$ then
$$|D|^2 = -\Delta$$

Theorem 3.1.22 – Sobolev Inequality

For $1 < p < +\infty$ and $f \in \mathcal{S}$ such that $|D|^s f \in L^p$ then
$$\|f\|_{L^q} \preccurlyeq \||D|^s f\|_{L^p}$$
where $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$.

Osservazione 3.1.6
The noble example, the one really common to use, is for $p = 2$: $\|f\|_{L^q} \preccurlyeq \||D|f\|_{L^2}$ with $s = 1$. So we have $q = \frac{2d}{d-2}$ (we need $d \geq 3$). So we have
$$\|f\|_{L^{\frac{2d}{d-2}}} \leq \||D|f\|_{L^2}$$
By plancherel we have that
$$\||D|f\|_{L^2} = \|\nabla f\|_{L^2}$$
infatti abbiamo
$$\|\widehat{|D|f}\|^2_{L^2} = \|\widehat{\nabla f}\|_{L^2}$$
che esplicitato è
$$\int 4\pi^2|\xi|^2|\hat{f}|^2 = \int 4\pi^2|\xi|^2|\hat{f}|^2 = \int |\nabla f|^2$$
In general for any p, where we dont have plancherel, we have an estimate with the hilbert/riesz transform which gives
$$\||D|u\|_{L^p} \preccurlyeq \|\nabla u\|_{L^p}$$

let us see that the exponents have to be those by a scaling argument: if you define $f_\lambda = f(\frac{x}{\lambda})$ so
$$\|f_\lambda\|_{L^q} = \lambda^{\frac{d}{q}}\|f\|_{L^q}$$
we also have that
$$|D|^s f_\lambda = \lambda^{-s}|D|f(\frac{x}{\lambda})$$

so

$$\||D|^s f_\lambda\|_{L^p} = \lambda^{-s+\frac{d}{p}} \||D|^s f\|_{L^p}$$

so we need to have

$$\frac{d}{q} = -s + \frac{d}{p}$$

Proof of Sobolev's theorem. We recall that[3]

$$\|f\|_{L^q} = \sup_{\substack{\|g\|_{L^{p'}}=1 \\ g\in\mathcal{S}}} |\langle f, g\rangle|$$

Ma allora possiamo usare plancherel

$$= \sup_{\substack{\|g\|_{L^{p'}}=1 \\ g\in\mathcal{S}}} |\langle \hat{f}, \hat{g}\rangle| = \sup_{\substack{\|g\|_{L^{p'}}=1 \\ g\in\mathcal{S}}} |\langle (2\pi|\xi|)^s \hat{f}, (2\pi|\xi|^{-s}\hat{g}\rangle$$

Chiaramente $(2\pi|\xi|)^s\hat{f}$ è una distribuzione temperata (diminuisce più di polinomi). Ho quindi bisogno che $(2\pi|\xi|^{-s}\hat{g} \in S$, ma per questo serve che $\hat{g} = 0$ in un intorno dell'origine.

Idea Domanda: le funzioni con questa proprietà sono dense nella classe di Schwartz? Se si allora il sup può passare a quelle. ⌟

if this were the case we would have

$$\sup_{g\in F} |\langle |D|^s f, |D|^{-s} g\rangle|$$

where $F = \{g \in \mathcal{S}, \hat{g} = 0 \text{ in a neigherbood of the origin}\}$. Let us consider $\varphi(t) = \begin{cases} 1 & |t| \le 1 \\ 0 & |t| > 2 \end{cases}$ and

$$\hat{g}_\varepsilon(\xi) = \hat{g}(\xi)\left[1 - \varphi\left(\frac{|\xi|}{\varepsilon}\right)\right]$$

Clearly $\hat{g}_\varepsilon \in F$. So we just have to prove there is convergence in L^p: we have

$$g - g_\varepsilon = \left(\hat{g}\varphi\left(\frac{|\xi|}{\varepsilon}\right)\right)^\vee = \varepsilon^d g * \check{\varphi}(\varepsilon\cdot)](x)$$

So[4]

$$\|g - g_\varepsilon\|_{L^p} = \varepsilon^d \|g * \check{\varphi}(\varepsilon\cdot)(x)\|_{L^p} \le \varepsilon^d \|g\|_{L^1} \varepsilon^{-\frac{d}{p}} \|\check{\varphi}_{L^p} = \varepsilon^{d(1-\frac{1}{p})} \|g\|_{L^1} \|\check{\varphi}\|_{L^p}$$

So, by remembering $\widehat{|D|^s f} = (2\pi|\xi|)^s\hat{f}$ we have

$$\|f\|_{L^q} = \sup_{\substack{g\in F \\ \|g\|_{L^{p'}}}} \langle |D|^s f, |D|^{-s} g\rangle \le \sup_{\substack{\|g\|_{L^{q'}}=1 \\ g\in F}} \|\||D|^s f\|_{L^p} \||D|^{-s} g\|_{L^{p'}}$$

(we applied holder). We use now HLS to see

$$\||D|^{-s} g\|_{L^{p'}} \preccurlyeq \|g\|_{L^{q'}} = 1$$

[3] ci possiamo ridurre a $\mathcal{S}$ perchè è denso in $L^{p'}$.

[4] remember $\|f(\varepsilon x)\|_{L^p} = \varepsilon^{-\frac{d}{p}} \|f\|_{L^p}$

How do we apply HLS? we can see that if $0 < s < d$

$$|D|^{-s}g = c\frac{1}{|x|^{d-s}} * g$$

and we want to see

$$\|\frac{1}{|x|^{d-s}} * g\|_{L^{p'}} \preccurlyeq \|g\|_{L^{q'}}$$

Remembering

$$\|\frac{1}{|x|^{\alpha}} * g\|_{L^{p'}} \preccurlyeq \|g\|_{L^{q'}}$$

where

$$1 + \frac{1}{p'} = \frac{1}{q'} + \frac{\alpha}{d}$$

with our α we have $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. □

Osservazione 3.1.7
curiosità: Lieb ha trovato la costante migliore per

$$\|\frac{1}{|x|^{\alpha}} * u\|_{L^q} \leq C\|u\|_{L^p}$$

Da questa possiamo ottenere per dualità

$$\|u\|_{L^{\frac{2d}{d-2s}}} \leq \tilde{C}\||D|^s u\|_{L^2}$$

3.2 The Hilbert and Riesz transform

Let us understand the reason why we start to care about the Riesz transform. Thanks to the Sobolev inequality, when $s = 1$, we have that

$$\|u\|_{L^q} \preccurlyeq \left\||D|u\right\|_{L^p}.$$

If $p = 2$ we see that $\||D|u\| = \|\nabla u\|$, indeed since $\widehat{\partial_{x_i} u} = 2\pi i \xi_i \hat{u}$ we have

$$\widehat{|D|u} = 2\pi|\xi|\hat{u}.$$

Unfortunately this is not the case for any p, but we will see that for $1 < p < +\infty$ we have:

$$\||D|u\|_{L^p} \preccurlyeq \|\nabla u\|_{L^p}.$$

In order to arrive to such an inequality we first try to write[5] $|D|u$ in terms of ∇u.

Idea Remembering how the Fourier transform behaves for differentiation we see that:

$$2\pi|\xi| = \frac{2\pi|\xi|^2}{|\xi|} = \sum_{i=1}^{d} \frac{2\pi\xi_i^2}{|\xi|} = \sum_{j=1}^{d}(2\pi i\xi_j)\left(\frac{-i\xi_j}{|\xi|}\right).$$

If we can define an operator T_j such that

$$\widehat{T_j\varphi} = -i\frac{\xi_j}{|\xi|}\hat{\varphi} \Rightarrow T_j\varphi = \left(-i\frac{\xi_j}{|\xi|}\hat{\varphi}\right)^{\vee}$$

then it will be clear that

$$T_j(\partial_{x_k}\varphi) = \left(-\frac{i\xi_j}{|\xi}2\pi i\xi_k\hat{\varphi}\right)^{\vee} = \partial_{x_k}(T_j\varphi).$$

So we have that the operations do operations do commute and we have a multiplier for the operator T_j. ⌟

Since the multipliers coincide seen through Fourier we have that

$$|D|\varphi = \sum_{j=1}^{d} T_j(\partial_{x_j}\varphi).$$

This means that the L^p norm can be estimated[6] as follows:

$$\begin{aligned}\||D|\varphi\|_{L^p} &= \left\|\sum_{j=1}^{d} T_j(\partial_{x_j}\varphi)\right\|_{L^p} \\ &\le \sum_{j=1}^{d} \|T_j(\partial_{x_j}\varphi)\|_{L^p} \\ &\overset{3.3}{\preccurlyeq} \sum_{j=1}^{d} \|\partial_{x_j}\varphi\|_{L^p}.\end{aligned}$$

Thus we have an equivalent norm to $\|\nabla\varphi\|_{L^p}$.

[5]We have that $|D|$ is a *global operator*, whereas ∇ it's local. One can talk about *pseudodifferential operator*.

[6]The Calderón-Zygmund theorem (3.3) has a central role.

Idea The right operator T_j will be the *Riesz transform* R_j acting on Schwartz's functions. Since

$$R_j\varphi = \left(-i\frac{\xi_j}{|\xi|}\hat{\varphi}\right)^\vee = \left(-i\frac{\xi_j}{|\xi|}\right)^\vee * \varphi$$

We will need to give a sense, with the language of tempered distributions, to

$$\left(-i\frac{\xi_j}{|\xi|}\right)^\vee.$$

⌟

Another reason why the Riesz transform is so important is given by the study of PDEs. When we consider the problem $-\Delta u = f$ we have that $\frac{\partial^2 u}{\partial x^2}$ have the same regularity as f. Can we say something about the mixed partials? We will have that

$$\frac{\partial^2}{\partial x_i \partial x_j} = R_i(R_j(\Delta))$$

hence the regularity of the homogeneous derivatives gives the regularity of the mixed ones if the operator R_j is continuous, i.e.:

$$\|R_j(\Delta u)\|_{L^p} \preccurlyeq \|\Delta u\|_{L^p}.$$

This will be a consequence of the *Schauder's estimates*.

3.2.1 Hilbert Transform

Let us start from the case $d = 1$. This means that $\frac{-i\xi_j}{|\xi|} = -isign(\xi)$.

Definition 3.2.1 – Hilbert transform

Given $\varphi \in \mathcal{S}$ we define the *Hilbert transform* as the tempered distribution such that

$$\widehat{H\varphi} = -isign(\xi)\hat{\varphi}.$$

Idea Remembering the good properties of the Fourier transform related to the convolution operator we want to find a kernel $\smile$ such that

$$H\varphi = \smile.$$

⌟

Definition 3.2.2 – Principal value

We consider $\omega_0 \in \mathcal{S}'$ such that

$$\langle \omega_0, \varphi \rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\varepsilon|\le|x|\le 1}\frac{\varphi(x)}{x}dx + \frac{1}{\pi}\int_{|x|>1}\frac{\varphi(x)}{x}dx = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x|\ge\varepsilon}\frac{\varphi(x)}{x}dx$$

This is also called the *principal value* of $\frac{1}{x}$.

Osservazione 3.2.1
With the previous definition it in not clear why $\omega_0 \in \mathcal{S}'$. We can rewrite $\langle \omega_0, \varphi \rangle$ as follows:

$$\left\langle PV\left(\frac{1}{x}\right), \varepsilon \right\rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi}{x} dx.$$

This second form gives more an idea of why $\omega_0 \in \mathcal{S}'$.

Proof.

Idea We will start to use a very simple, but poweful trick: if we have an odd function f on a even interval I then

$$\int_I f(x) dx = 0.$$

Dealing with computations where integral appear we can add the quantity we need to proceed, for example to use Lagrange theorem, without any problem. ⌟

We have that $\omega_0 \in \mathcal{S}'$ if $\langle \omega_0, \varphi \rangle$ is controlled by a finite number of seminorms. This follows because

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi}{x} dx &= \int_{|\varepsilon| \leq |x| \leq 1} \frac{\varphi(x)}{x} dx + \int_{|x|>1} \frac{x\varphi}{x^2} dx \\
&= \int_{|\varepsilon| \leq |x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x} + \int_{|x|>1} \frac{x\varphi}{x^2} dx \\
&\leq 2\|\varphi'\|_{L^\infty} + \|x\varphi\|_{L^\infty}.
\end{aligned}$$

□

Definition 3.2.3 – Truncated Hilbert Transform

We consider the *truncated Hilbert transform* as

$$H^\varepsilon \varphi = \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} dy = \frac{1}{\pi} \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y} dy.$$

Osservazione 3.2.2
It is clear that we can rewrite the Hilbert's transform as

$$H\varphi = \lim_{\varepsilon \to 0} H^\varepsilon \varphi = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y} dy.$$

This form is handier because it is in terms of a *singular integral.*

Idea Achtung! In this case the denominator is elevated to the power of 1 which is also the dimension in which we are working. This works because we do **not** have modules. Having an odd function we are able to prove a (p,p)-estimate. In the case of higher dimension we will have modules, hence the denominator will be of a different power, which will be estimate thanks to (HLS). ⌟

Proposition 3.2.4

The multiplier ω_0 is such that

$$\hat{\omega}_0 = -isign\xi.$$

This clearly gives us $H\varphi = \omega_0$.

Proof.

$$\begin{aligned}
\langle \widehat{\omega_0}, \varphi \rangle &= \langle \omega_0, \hat{\varphi} \rangle \\
&= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|\xi| \geq \varepsilon} \frac{\hat{\varphi(x)}}{\xi} \\
&= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|\xi| \geq \varepsilon} \frac{1}{\xi} \int_{\mathbb{R}} \varphi(x) e^{-2\pi i \xi x} dx d\xi \\
&= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(x) \int_{|\xi| \geq \varepsilon} \frac{1}{\xi} e^{-2\pi i \xi x} d\xi dx \\
&= \lim_{\varepsilon \to 0} \frac{-i}{\pi} \int_{\mathbb{R}} \varphi(x) \int_{|\xi| \geq \varepsilon} \frac{\sin 2\pi \xi x}{\xi} d\xi dx \\
&= \int_{\mathbb{R}} \varphi(-isign(x)) dx \\
&= \langle \varphi, -isign(x) \rangle
\end{aligned}$$

Where we used the following identity:

$$\int_{-\infty}^{\infty} \frac{\sin(bx)}{x} dx = \pi sign(b).$$

This follows because:

1. we have that
$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \pi.$$
Indeed if we consider $I(a) = \int_0^{+\infty} \frac{sin(x)}{x} e^{-ax} dx$, this is equal to prove $I(0) = \frac{\pi}{2}$.
$$\begin{aligned}
I'(a) &= \int_0^{+\infty} \frac{\sin(x)}{x} e^{-ax}(-x) = -\int_0^{+\infty} \sin(x) e^{-ax} dx \\
&= \cancelto{0}{\left[\frac{e^{-ax}}{-a} \sin(x) \right]_0^{+\infty}} + \frac{1}{a} \int_0^{+\infty} e^{-ax} \cos(x) dx \\
&= -\frac{1}{a} \left[\frac{e^{-ax}}{x} \cos(x) \right]_0^{+\infty} - \frac{1}{a} \int_0^{+\infty} e^{-ax} \sin(x) dx \\
&= -\frac{1}{a^2} - \frac{1}{a^2} I'(a).
\end{aligned}$$
This gives the following ODE:
$$(1 + \frac{1}{a^2}) I' = -\frac{1}{a^2} \Rightarrow -I' = -\frac{1}{1 + a^2}$$

which solution is

$$I(a) = -\arctan(a) + C.$$

We see that sending $a \to +\infty$ we have $I(a) \to 0$ and $\arctan(a) \to \frac{\pi}{2}$, hence $C = \frac{\pi}{2}$.

2. We just need the following change of variables: $bx = x'$, indeed:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\sin(x')}{\frac{x'}{b}} \frac{1}{b} dx' &= \left[\int_{-\infty}^{+\infty} \frac{\sin(x)}{x} dx\right] sign(b) \\ &= \pi sign(b).\end{aligned}$$

□

We now show a really useful lemma for any operator T which, will have a nice consequence for the Hilbert transform.

Lemma 3.2.5

If[a] $\sup_{\omega>0} \|S_\omega\|_{L^p \to L^p} < +\infty$ we have

$$\|S_\omega f\|_{L^p} \preccurlyeq c_\omega \|f\|_{L^p}.$$

As a direct consequence we have that

$$\|S_\omega(f) - f\|_{L^p} \overset{\omega \to +\infty}{\longrightarrow} 0.$$

[a] By $\|S_\omega\|$ we consider the operator norm of $S_\omega : L^p \to L^p$, which is $\sup_{\substack{f \in L^p \\ \|f\|_{L^p}=1}} \|S_\omega f\|_{L^p}$.

Proof of Lemma. We consider

$$A = \{f \in L^1 \cap L^p : \hat{f} \text{ has compact support}\}$$

We have that A is dense in L^p, indeed

(non immediato, ci deve ripensare) and that for any function in A we have $S_\omega(f) = f$ for $\omega >> 1$.

$$B = \{f \in L^p : \lim_{\omega \to +\infty} S_\omega(f) = f \ in \ L^p\}$$

is closed in L^p. (same spirit of the maximal operator proof)
We want to prove that $f_n \in B$ e $f_n \to f$ in L^p allora $f \in B$. Let $\|f_n - f\| \leq \varepsilon$. We want to prove $\forall \varepsilon$:

$$\limsup_{\omega} \|S_\omega(f) - f\| \leq \varepsilon$$

Aggiungo e tolgo

$$\begin{aligned}\|S_\omega(f) - f\| &= \|S_\omega(f) - S_\omega(f_n) + S_\omega(f_n) - f_n + f_n - f\| \\ &\leq \|S_\omega(f) - S_\omega(f_n)\| + \|S_\omega(f_n) - f_n\| + \|f_n - f\| \\ &= \|S_\omega(f - f_n)\| + \|S_\omega(f_n) - f_n\| + \|f_n - f\| \\ &\leq C\|f - f_n\|_{L^p} + 0 + \varepsilon \\ &\leq (1 + C)\varepsilon.\end{aligned}$$

□

Vedremo che

$$\|Hf\|_{L^p} \precsim \|f\|_{L^p}$$

for $1 < p < \infty$. Da questo segue

Corollary 3.2.6

Take $S_\omega(f) = (\chi_{[-\omega,\omega]}(\xi)\hat{f})^\vee$. Is it true that $S_\omega \overset{\omega\to\infty}{\longrightarrow} f$ in L^p?

Proof of corollary. For $p = 2$ it is trivial thanks to Placherel inequality, for any p it's not so immediate. it is true for $1 < p < \infty$. We write

$$S_\omega(f) = \int_{|\xi|<\omega} \hat{f}(\xi)e^{2\pi i\xi x}d\xi.$$

Let us see that $\sup \|S_\omega\|_{L^p\to L^p} < +\infty$ then we conclude thanks to the previous lemma.

We introduce the *Riesz Projector*

$$\begin{array}{rccc} P: & L^p & \longrightarrow & L^p \\ & f & \longmapsto & \frac{1}{2}\left[f + iHf\right] \end{array}.$$

We see

$$\widehat{Pf} = \frac{1}{2}\left[\hat{f} + sign(\xi)\hat{f}\right] = \chi_{[0,+\infty)}(\xi)\hat{f}.$$

Idea Voglio vedere $\chi_{[0,+\infty]} = \chi_{[-\omega,+\infty)} - \chi_{[\omega,+\infty)}$ so we'll have $\chi_{[-\omega,\omega]}$. caratteristica 0 infinito come caratteristica (- omega, infito) - caratteristica (omega , infinito) così avremo caratteristica dell'intervallo. chiaramente la caratteristica 0 , infinito è p-p continua , dobbiamo vedere che lo è anche la traslazione in trasformata. ⌟

Claim:

$$S_\omega(f) = e^{-2\pi i\omega x}P(e^{2\pi i\omega x}f) - e^{2\pi i\omega x}P(e^{-2\pi i\omega x}f)$$

è vero, mostralo per esercizio. (questo è l'idea di traslazione per avere solo intervallo - omega, omega).

We see that

$$\|S_\omega(f)\|_{L^p} \leq \|P(e^{2\pi i\omega x}f)\|_{L^p} + \|P(e^{-2\pi i\omega x}f)\|_{L^p} \leq C\|e^{2\pi i\omega x}f\|_{L^p} + C\|e^{-2\pi i\omega x}f\|_{L^p}$$

e l'esponenziale sparisce da norma. (per controllare norma P in ultimo passaggio hai usato stima p-p per trasformata hilbert) □

(parentesi: se si fa con cubo allora il risultato resta vero grazie a trasformata di riesz. se invece si usa la caratteristica con disco il risultato crolla perchè non si ha una $p - p$ stima e questo è l'argomento di un seminario)

3.2.2 Riesz Transform

Definition 3.2.7 – Riesz trasform

We define con $\varphi \in \mathcal{S}$ (con il pedice j indichiamo la $j-$esima direzione)

$$\langle \omega_j, \varphi\rangle = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{y_j}{|y|^{d+1}}\varphi(y)dy$$

and the Riesz trasform

$$R_j\varphi = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{x_j-y_j}{|x-y|^{d+1}}\varphi(y)dy$$

Proposition 3.2.8

It holds

$$\widehat{R_j\varphi} = \frac{-i\xi_j}{|\xi|}\hat{\varphi}(\xi).$$

Proof.

$$\begin{aligned}
\langle\hat{\omega}_j,\varphi\rangle &= \langle\omega_j,\hat{\varphi}\rangle \\
&= \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\lim_{\varepsilon\to 0}\int_{|\xi|>\varepsilon}\int\frac{\xi_j}{|\xi|}\int_{\mathbb{R}^d}\varphi(x)e^{-2\pi i x i\cdot\xi} \\
&= \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\varphi(x)\underbrace{\int_{\frac{1}{\varepsilon}>|\xi|>\varepsilon}\frac{\xi_j}{|\xi|}e^{-2\pi i x\cdot\xi}}_{=-i\frac{x_j}{|x|}}
\end{aligned}$$

Ricordando che $\xi = r\theta$ con $r > 0$ e $\theta\in\mathbb{S}^{d-1}$ vediamo che vale l'uguaglianza segnata:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\frac{1}{\varepsilon}>|\xi|>\varepsilon}\frac{\xi_j}{|\xi|}e^{-2\pi i x\cdot\xi} &= \lim\int_{\mathbb{S}^{d-1}}\int_{\frac{1}{\varepsilon}>r>\varepsilon}\frac{r\theta_j e^{-2\pi i r\theta\cdot x}}{r^{d+1}}r^{d-1}drd\sigma \\
&= \lim\int_{\mathbb{S}^{d-1}}\int_{\frac{1}{\varepsilon}>r>\varepsilon}\frac{\theta_j e^{-2\pi i r\theta\cdot x}}{r}drd\sigma \\
&= -i\lim\int_{\mathbb{S}^{d-1}}\int_{\frac{1}{\varepsilon}>r>\varepsilon}\frac{\theta_j\sin(2\pi r\theta\cdot x)}{r}drd\sigma \\
&= -\frac{i\pi}{2}\int_{\mathbb{S}^{d-1}}\theta_j sign(\theta\cdot x)d\sigma \\
&= \frac{i\pi}{2}\left(\frac{\Gamma(\frac{d+1}{2})}{\pi^{d+1}}\right)^{-1}.
\end{aligned}$$

dove l'ultimo uguale lo vediamo la prossima volta ∎

3.3 Calderón-Zygmund's theory

Definition 3.3.1 – Calderón-Zygmund kernel

We define the *Calderón-Zygmund kernel* as the function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ such that:

1. $|K(x)| \leq \beta |x|^{-d}$ per ogni $x \neq 0$.
2. *Hörmander's condition*: there exists B such that $\forall y \neq 0$
$$\int_{|x| \geq 2|y|} |K(x) - K(x-y)| dx \leq B$$
3. *Zero-Average on shells*: We have $\forall R, S > 0$:
$$\int_{R<|x|<S} K(x) dx = 0.$$

Osservazione 3.3.1
These three condition will not always be used. We will specify when this is the case or not.

Definition 3.3.2 – Singular integrals

Given a Calderón-Zygmund kernel, we define an operator by means of its principal value. The *singular integral operator* with kernel K is defined as follows:
$$Tf := \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} K(x-y) f(y) dy.$$

Osservazione 3.3.2
If K is such that $|\nabla K| \preccurlyeq \frac{1}{|x|^{d+1}}$ for all $x \neq 0$, then the Hörmander's condition holds.

Proof. DA SCRIVERE $\square$

Theorem 3.3.3 – Calderón-Zygmund's Theorem

We have
$$\|Tf\|_{L^p} \preccurlyeq \|f\|_{L^p}.$$

Idea The proof of this theorem will follow the next steps:

1. We prove the $L^2 - L^2$ strong continuity.
2. We prove the $L^1 - L^1$ weak continuity.
3. By using the Marcinkiewicz theorem we have $L^p - L^p$ strong continuity for $1 < p \leq 2$.
4. By duality we get $L^p - L^p$ strong continuity for $2 \leq p < +\infty$.

$\lrcorner$

Definition 3.3.4

For any $r, s \in \mathbb{R}^+$ we can consider the *truncation kernel* $K_{r,s}$ defined as:

$$K_{r,s}(x) = K(x)\chi_{r<|x|<s}(x).$$

This truncation will be useful in order to avoid singulary at 0 and at infinity.

Lemma 3.3.5 – Strong $L^2 - L^2$ continuity

Given a Calderón-Zygmund kernel K it holds that

$$\|K_{r,s} * f\|_{L^2} \preccurlyeq \|f\|_{L^2} \text{ uniformly in } r, s.$$

Remembering that $K * f = \lim\limits_{\substack{r\to 0\\ s\to+\infty}} K_{r,s} * f$, then we have:

$$\|K * f\|_{L^2} \leq C\|f\|_{L^2}.$$

Proof. In this proof we suppose $f \in \mathcal{S}$. If we consider

$$K_\varepsilon * f = \int_{\varepsilon<|y|<\frac{1}{\varepsilon}} K(y)f(x-y)dy$$

the $L^2 - L^2$ estimate will follow from the uniform convergence and the fact that $\{K_\varepsilon * f\}$ is a Cauchy sequence in L^2.
$\underline{\{K_\varepsilon * f\} \text{ is a Cauchy sequence}}$: given $\varepsilon_2 < \varepsilon_1$ we want to study

$$\|K_{\varepsilon_2} * f - K_{\varepsilon_1} * f\|_{L^2_x} = \left\| \int_{\varepsilon_2<|y|<\frac{1}{\varepsilon_2}} K(y)f(x-y)dx - \int_{\varepsilon_1<|y|<\frac{1}{\varepsilon_1}} K(y)f(x-y)dx \right\|_{L^2_x}.$$

We now compute the two integrals separately. We first note that:

$$\begin{aligned}
\|K\chi_{\frac{1}{\varepsilon_1}<|y|<\frac{1}{\varepsilon_2}}\|_{L^2} &= \left(\int_{\frac{1}{\varepsilon_1}<|y|<\frac{1}{\varepsilon_2}} |K(x)|^2 dx\right)^{\frac{1}{2}} \\
&\preccurlyeq \left(\int \frac{1}{|x|^{2d}}\right)^{\frac{1}{2}} \\
&= \left(\omega_{d-1}\int \frac{r^{d-1}}{r^{2d}}\right)^{\frac{1}{2}} \approx \varepsilon_1^{\frac{d}{2}}.
\end{aligned}$$

By Young's interpolation theorem we have

$$\begin{aligned}
\left\| \int_{\frac{1}{\varepsilon_1}<|y|<\frac{1}{\varepsilon_2}} K(y)f(x-y)dx \right\|_{L^2} &\leq \left\| K\chi_{\frac{1}{\varepsilon_1}<|y|<\frac{1}{\varepsilon_2}} \right\|_{L^2} \|f\|_{L^1} \\
&\preccurlyeq \|f\|_1.
\end{aligned}$$

For the second integral we use the third condition in the definition[7] of the Calderón-Zygmund kernel, i.e.

[7]the trick is the following: having and integral equal to 0 we can add the quantity we need in order to proceed in the computation without changing it.

$$\int_{\varepsilon_2<|y|<\varepsilon_1} K(y)f(x-y)dx = \int_{\varepsilon_2<|y|<\varepsilon_1} K(y)(f(x-y)-f(x))dx.$$

Having $f \in \mathcal{S}$ we can write $f(x-y)-f(x) = \int_0^1 \langle \nabla f(x-ty), y \rangle dt$ and, by Minkowski's theorem, it follows that:

$$\begin{aligned}\left\| \int_{\varepsilon_2<|y|<\varepsilon_1} K(y) \int_0^1 \nabla f(x-ty)\cdot y dt dy \right\| &\leq \int_0^1 \int_{\varepsilon_2<|y|<\varepsilon_1} |y||K(y)| \|\nabla f(x-ty)\|_{L^2_x} dy dt \\ &\preccurlyeq \|\nabla f(x)\|_{L^2} \int_0^1 \int_{\varepsilon_2\leq|y|\leq\varepsilon_1} \frac{1}{|y|^{d-1}} \\ &= \|\nabla f\| \omega_{d_1} \int_0^1 \int_{\varepsilon_2}^{\varepsilon_1} \frac{r^{d-1}}{r^{d-1}} \\ &\preccurlyeq \varepsilon_1\end{aligned}$$

where we used the first condition, i.e $|y||K(y)| \leq \frac{1}{|y|^{d-1}}$ and the fact that the norm is translation invariant.
<u>Uniform estimate</u>: For this part of the proof we will specialize ourself to the case of $r = \frac{1}{\varepsilon}$ and $s = \varepsilon$, so the kernel will be

$$K_\varepsilon = K\chi_{\frac{1}{\varepsilon}\leq|x|\leq\varepsilon}$$

One can see that, if K is a Calderón-Zygmund kernel, then also K_ε is one. If we want an uniform estimate on K_ε, thanks to Plancherel, we can check it on $\hat{K}_\varepsilon$.

$$\begin{aligned}\hat{K}_\varepsilon &= \int_{\mathbb{R}^d} K_\varepsilon e^{-2\pi i\xi x} dx \\ &= \int_{|x|\leq\frac{1}{|\xi|}} K_\varepsilon e^{-2\pi i\xi x} dx + \int_{|x|\geq\frac{1}{|\xi|}} K_\varepsilon e^{-2\pi i\xi x} dx \\ &= I + II.\end{aligned}$$

We now estimate the two integrals separately. We can rewrite[8]:

$$\begin{aligned}|I| &= \left| \int_{\frac{1}{\varepsilon}<|x|<\frac{1}{|\xi|}} K(x)(e^{-2\pi i\xi\cdot x}-1)dx \right| \\ &\leq \int_{\frac{1}{\varepsilon}<|x|<\frac{1}{|\xi|}} |K(x)| 2\pi|\xi||x| dx \\ &= 2\pi|\xi| \int_{\frac{1}{\varepsilon}<|x|<\frac{1}{|\xi|}} |K(x)||x| dx \\ &\leq 2\pi|\xi| \int_{\frac{1}{\varepsilon}}^{\frac{1}{|\xi|}} \frac{1}{|x|^{d-1}} \\ &\approx 2\pi|\xi| \frac{1}{|\xi|} \approx 1.\end{aligned}$$

[8]We use the trick on shell and the Lagrange estimate with $F(x) = e^{-2\pi i\xi\cdot x}$.

Idea The estimate of II starts with a computation that, apparently is nonsense, but that it will be handy since it allows us to use the Hörmander condition, which we still have not used. We can write

$$\begin{aligned} II &= \int_{\frac{1}{|\xi|}\leq|x|<\varepsilon} K(x)e^{-2\pi i\xi\cdot x} \\ &= -\int_{\frac{1}{|\xi|}\leq|x|<\varepsilon} K(x)e^{-2\pi i\xi\cdot\left(x+\frac{\xi}{2|\xi|^2}\right)}dx \\ &= -\underbrace{\int_{\frac{1}{|\xi|}<|x-\frac{\xi}{2|\xi|^2}|<\varepsilon} e^{-2\pi i\xi\cdot x}K\left(x-\frac{\xi}{2|\xi|^2}\right)dx}_{B}. \end{aligned}$$

So we have an equality like $II = -B$. We can also write $2II = II - B$. The key passege will be to add and substract the right quantity A so that we can use Hörmander and have

$$2II = (II - A) + (-B + A).$$

One can see that the right A is:

$$A = \int_{\frac{1}{|\xi|}\leq|x|\leq\varepsilon} K\left(x-\frac{\xi}{2|\xi|^2}\right)e^{-2\pi i\xi\cdot x}dx.$$

┘

We can estimate the first term:

$$\begin{aligned} |II - A| &= \left|\int_{\frac{1}{|\xi|}\leq|x|\leq\varepsilon}\left(K(x)-K(x-\frac{\xi}{2|\xi|^2})\right)e^{-2\pi i\xi\cdot x}dx\right| \\ &\leq \int_{\frac{1}{|\xi|}\leq|x|\leq\varepsilon}\left|K(x)-K\left(x-\frac{\xi}{2|\xi|^2}\right)\right|dx \\ &\preccurlyeq 1 \end{aligned}$$

where we use the Hörmander condition[9] on $|x| > \frac{1}{|\xi|} = 2|y|$.

For the other term we can see instead that:

$$A - B = \int_{\frac{1}{|\xi|}<|x|} K_\varepsilon\left(x-\frac{\xi}{2|\xi|^2}\right)e^{-2\pi i\xi\cdot x}dx - \int_{\frac{1}{|\xi|}<\left|x-\frac{\xi}{2|\xi|^2}\right|} K_\varepsilon\left(x-\frac{\xi}{2|\xi|^2}\right)e^{-2\pi i\xi\cdot x}dx$$

If we explicit the inequality in the domain of the integral we have[10]:

$$\left|x-\frac{\xi}{2|\xi|^2}\right| \geq |x| - \left|\frac{\xi}{2|\xi|}\right| = |x| - \frac{1}{2|\xi|} \geq \frac{1}{2|\xi|}.$$

[9] The change of variables is: $\frac{\xi}{2|\xi|^2} = y$. This gives $|y| = \frac{1}{2|\xi|}$.

[10] since $\frac{1}{|\xi|} < |x|$ then $-\frac{1}{2|\xi|} > -\frac{|x|}{2}$.

$$\begin{aligned}\left|\int_{|x-\frac{\xi}{2|\xi|^2}|\leq|\xi|\leq|x|\leq s} K\left(x-\frac{\xi}{2|\xi|^2}\right) e^{-2\pi i\xi\cdot x}dx\right| &\leq \int_{\frac{1}{2|\xi|}\leq|x-\frac{\xi}{2|\xi|^2}\leq\frac{1}{|\xi|}} |K(x-\frac{\xi}{2|\xi|^2})e^{-2\pi i\xi\cdot x}|dx\\ &= \int_{\frac{1}{2|\xi|}\leq|x|\leq\frac{1}{|\xi|}} |K(x)|dx\\ &\leq \int_{\frac{1}{2|\xi|}}^{\frac{1}{|\xi|}} \frac{r^{d-1}}{r^d}dr = \log(2)\end{aligned}$$

which does not depend on ξ.

Conclusion: Using Plancherel we can see that

$$\|K_\varepsilon * f\|_{L^2} = \|\widehat{K_\varepsilon}\hat{f}\|_{L^2} \leq \sup_{\varepsilon>0}|\hat{K}_\varepsilon\|\hat{f}\|_{L^2} \leq \sup_{\varepsilon>0}|\hat{K}_\varepsilon\|f\|_{L^2} \preccurlyeq \|f\|_{L^2}$$

Remembering that, in the sense of the principal valute, we have

$$Tf = \int K(x-y)f(y)dy$$

then we have the $L^2 - L^2$ estimate:

$$\|Tf\|_{L^2} \preccurlyeq \|f\|_{L^2} \quad \forall f \in \mathcal{S}.$$

By a density argument this follows also $\forall f \in L^p$. $\square$

Osservazione 3.3.3
For the Hilbert and the Riesz transform this estimate is obvious. It follows directly from the Plancherel formula.

Lemma 3.3.6 – Weak $L^1 - L^1$ continuity

Given the kernel K and the estimate $\|Tf\|_{L^2} \preccurlyeq \|f\|_{L^2}$ we have that:

$$\mu\{|K * f| > \lambda\} \preccurlyeq \frac{\|f\|_{L^1}}{\lambda}.$$

Osservazione 3.3.4
This lemma actually holds if we know that $\|Tf\|_{L^2} \preccurlyeq \|f\|_{L^2}$ and if have a kernel M which satisfies the Hörmander condition. In our case the $L^2 - L^2$ estimate is guaranteed thanks to the Calderón-Zygmund kernel.

In order to prove this lemma first we need to see the following:

Lemma 3.3.7 – Calderón-Zygmund Decomposition Lemma

If we have $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$ we can write[a] $f = g + b$, with this two properties:

$$\begin{cases} |g| \leq \lambda \\ b = \sum_{\mathcal{Q}\in Q} \chi_{\mathcal{Q}} f \end{cases}$$

where Q is a family of disjoint cubes, with the property:

$$\lambda \leq \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |f|dx \leq 2^d\lambda.$$

Furthermore,

$$\left|\bigcup_{\mathcal{Q}} \mathcal{Q}\right| \leq \frac{1}{\lambda}\|f\|_1.$$

[a] g denotes good function and b bad function.

Proof. For all $\ell \in \mathbb{Z}$ we define D^ℓ a family of dyadic cubes:

$$D^\ell = \left\{\prod_{i=1}^{d}\left[2^\ell m_i, 2^\ell(m_i)+1)\right) \quad m_1, \dots, m_d \in \mathbb{Z}\right\}.$$

If the cube $\mathcal{Q} \in D^\ell$ and $\mathcal{Q}' \in D^{\ell'}$ there are two possibilities:

1. $\mathcal{Q} \cap \mathcal{Q}' = \emptyset$
2. $\mathcal{Q} \subset \mathcal{Q}'$ or $\mathcal{Q}' \subset \mathcal{Q}$.

Fix λ then for $f \in L^1$ we have that there exists $\ell_0, \forall \mathcal{Q} \in D^{\ell_0}$ such that[11]

$$\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |f|dx \leq \lambda.$$

For any cube in D^{ℓ_0} we take the[12] *"children"* $\mathcal{Q}'$, i.e. $\mathcal{Q}' \in D^{\ell_0 - 1}$. With the children it may happen that

$$\frac{1}{|\mathcal{Q}'|}\int_{\mathcal{Q}'} |f|dx < \lambda \quad \text{or} \quad \frac{1}{|\mathcal{Q}'|}\int_{\mathcal{Q}'} |f|dx \geq \lambda.$$

If the first condition happens, we say $\mathcal{Q}'$ is a *good set* and iterate the construction, otherwise $\mathcal{Q}'$ is a *bad set*. We define $\mathcal{B}$ as the family of bad sets. We now ask ourselves: if $\mathcal{Q} \in \mathcal{B}$ do we have an upper bound for the integral?

$$\lambda \leq \frac{1}{|\mathcal{Q}'|}\int_{\mathcal{Q}'} |f|dx = \frac{2^d}{|\mathcal{Q}|}\int_{\mathcal{Q}'} |f|dx \leq 2^d\lambda.$$

Idea If we take a point which $\notin \mathcal{B}$, it means that each cube $\mathcal{Q}$ which surrounds it, has integral $< \lambda$. The thesis will follow thanks to Lebesgue theorem. ⌟

What about the measure of the bad family?

$$\left|\bigcup_{\mathcal{Q}\in\mathcal{B}} \mathcal{Q}\right| \leq \sum_{\mathcal{Q}\in\mathcal{B}} |\mathcal{Q}| \leq \sum \frac{1}{\lambda}\int_{\mathcal{Q}} |f|dx \leq \frac{1}{\lambda}\|f\|_{L^1}.$$

Let us take $x_0 \in \mathbb{R}^d \setminus \bigcup_{\mathcal{Q}\in\mathcal{B}} \overline{\mathcal{Q}}$. It means that there exists $\mathcal{Q}_i$ such that $|\mathcal{Q}_i| \to 0$, $x_0 \in \mathcal{Q}_i$ and

$$\frac{1}{|\mathcal{Q}_i|}\int_{\mathcal{Q}_i} |f|dx \leq \lambda$$

Then for lebesgue theorem we have $|f(x_0)| \leq \lambda$ a.e. We can define the good function $g := f - \sum_{\mathcal{Q}} \chi_{\overline{\mathcal{Q}}} f$ and have the thesis. □

[11] Having $f \in L^1$ and that $|\mathcal{Q}| = 2^{d\ell_0}$ we can determine ℓ_0 such that $\frac{\|f\|_{L^1}}{\lambda} < 2^{d\ell_0}$.

[12] It is a cube with half the side length of the previous one.

Proof of Weak $L^1 - L^1$ continuity. Let us define

$$f_1 = \begin{cases} f & \mathbb{R}^d \setminus \bigcup_{\mathcal{Q}\in\mathcal{B}} \mathcal{Q} \\ \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} f dx & \textit{otherwise} \end{cases} \qquad f_2 = \begin{cases} 0 & \mathbb{R}^d \setminus \bigcup_{\mathcal{Q}\in\mathcal{B}} \mathcal{Q} \\ f - \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} f dx & \mathcal{Q}\in\mathcal{B} \end{cases}$$

We have that $f = f_1 + f_2$, where

1. $\|f_1\|_{L^\infty} \leq 2^d\lambda$, indeed if $x \in \mathbb{R}^d \setminus \bigcup_{\mathcal{Q}\in\mathcal{B}} \mathcal{Q}$ we have that $|f| < \lambda$, otherwise $\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} f dx < 2^d\lambda$.

2. $\|f_1\|_{L^1} \leq \|f\|_{L^1}$, indeed

$$\int_{\mathcal{Q}} \left| \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} f dx \right| dy \leq \int_{\mathcal{Q}} |f| dx$$

3. $\|f_2\|_{L^1} \leq 2\|f\|_{L^1}$, it is similar to the previous one.

In order to prove the $(1-1)$-weak condition we can write:

$$\mu\{|K_\varepsilon * f| > \lambda\} \leq \underbrace{\mu\left\{|K_\varepsilon * f_1| > \frac{\lambda}{2}\right\}}_{I} + \underbrace{\mu\left\{|K_\varepsilon * f_2| > \frac{\lambda}{2}\right\}}_{II}$$

Knowing $\int |f_1|^2 dx \leq 2^d\lambda\|f_1\|_{L^1}$, thanks to the $(2-2)$-continuity and to the chebychev inequality we have:

$$I \leq C\frac{\|f_1\|_{L^2}^2}{\lambda^2} \preccurlyeq \frac{\|f\|_{L^1}}{\lambda}.$$

Idea In order to estimate II it will be cunning to consider bigger cubes $\mathcal{Q}^*$ in order to separate points but having still control. ⌟

Given $\mathcal{Q}\in\mathcal{B}$ which side is of length ℓ, we consider $\mathcal{Q}^*$ such that its size is $\ell^* = 2\sqrt{d}\ell$. We can write

$$\begin{aligned} II &\leq \mu\left(\bigcup_{\mathcal{Q}\in\mathcal{B}} \mathcal{Q}^*\right) + \mu\left\{\mathbb{R}^d \setminus \cup\, \mathcal{Q}^*, |K_\varepsilon * f_2| > \frac{\lambda}{2}\right\} \\ &\leq \sum_{\mathcal{Q}\in\mathcal{B}} \mu(\mathcal{Q}^*) + \mu\left\{\mathbb{R}^d \setminus \cup\, \mathcal{Q}^*, |K_\varepsilon * f_2| > \frac{\lambda}{2}\right\} \\ &= \sum (2\sqrt{d})^d \mu(\mathcal{Q}) + \mu\left\{\mathbb{R}^d \setminus \cup\, \mathcal{Q}^*, |K_\varepsilon * f_2| > \frac{\lambda}{2}\right\} \\ &\preccurlyeq \frac{\|f\|_{L^1}}{\lambda} + \mu\left\{\mathbb{R}^d \setminus \cup\, \mathcal{Q}^*, |K_\varepsilon * f_2| > \frac{\lambda}{2}\right\}. \end{aligned}$$

We have the thesis if we prove that

$$\underbrace{\mu\left\{\mathbb{R}^d \setminus \cup\, \mathcal{Q}^*, |K_\varepsilon * f_2| > \frac{\lambda}{2}\right\}}_{C} \preccurlyeq \frac{\|f\|_{L^1}}{\lambda}$$

If we consider $f_{\mathcal{Q}} = (f - \fint_{\mathcal{Q}} |f| dx)\chi_{\mathcal{Q}}$, by Chebychev we have

$$C \preccurlyeq \frac{1}{\lambda} \int_{\mathbb{R}^d \setminus \cup \mathcal{Q}^*} |K_\varepsilon * f_2| dx \preccurlyeq \sum_{\mathcal{Q}} \frac{1}{\lambda} \int_{\mathbb{R}^d \setminus \mathcal{Q}^*} |K_\varepsilon * f_{\mathcal{Q}}| dx$$

We can find $y_{\mathcal{Q}}$ such that each of its neighbourhood is contained in $\mathcal{Q}$. Then we have[13]:

$$\begin{aligned} K_\varepsilon * f_{\mathcal{Q}} &= \int_{\mathcal{Q}} K_\varepsilon(x-y) f_{\mathcal{Q}}(y) dy \\ &= \int_{\mathcal{Q}} [K_\varepsilon(x-y) - K_\varepsilon(x-y_{\mathcal{Q}})] f_{\mathcal{Q}}(y) dy. \end{aligned}$$

We have proved that

$$C \preccurlyeq \frac{1}{\lambda} \sum_{\mathcal{Q}} \int_{\mathbb{R}^d \setminus \mathcal{Q}^*} \left| \int_{\mathcal{Q}} [K_\varepsilon(x-y) - K_\varepsilon(x-y_{\mathcal{Q}})] f_{\mathcal{Q}}(y) dy \right| dx.$$

We conclude thanks to the Hörmander condition and Fubini:

$$\begin{aligned} C &\preccurlyeq \frac{1}{\lambda} \sum_{\mathcal{Q}} \int_{\mathcal{Q}} \int_{\mathbb{R}^d \setminus \mathcal{Q}^*} |[K_\varepsilon(x-y) - K_\varepsilon(x-y_{\mathcal{Q}})| |f_{\mathcal{Q}}|(y) dx dy \\ &\preccurlyeq \frac{1}{\lambda} \sum_{\mathcal{Q} \in \mathcal{B}} \int_{\mathcal{Q}} |f_{\mathcal{Q}}| dx \\ &\leq \frac{1}{\lambda} \|f\|_{L^1}. \end{aligned}$$

□

Proof of Theorem. For the previous lemmas we get the $L^2 - L^2$ strong continuity and the $L^1 - L^1$ weak continuity. By applying the Marcinkiewitz theorem we have the $L^p - L^p$ strong continuity for $1 < p \leq 2$. We show now the $L^p - L^p$ strong continuity for $2 \leq p < \infty$ by a *duality argument*: given T, let us consider the *adjoint operator* T^*:

$$\langle Tf, g \rangle = \langle f, T^* g \rangle.$$

So T^* is defined as

$$T^* g = \int K^*(x-y) g(y) dy$$

where

$$K^* = \overline{K(-x)}.$$

By definition of the $\|\cdot\|_{L^p}$ we have

$$\|Tf\|_{L^p} = \sup_{\|g\|_{L^{p'}}=1} |\langle Tf, g \rangle| = \sup_{\|g\|_{L^{p'}}=1} |\langle f, T^* g \rangle|.$$

So we have

$$|\langle f, T^* g \rangle| \leq \|f\|_{L^p} \|T^* g\|_{L^{p'}} \preccurlyeq \|f\|_{L^p} \|g\|_{L^{p'}}$$

and

$$\|Tf\|_{L^p} \preccurlyeq \|f\|_{L^p}.$$

□

[13] $f_{\mathcal{Q}}$ has zero average for definition

3.4 Littlewood-Paley theory

We now present the last result seen in this course, which was proved at the beginning of the 1960s and is nowadays one of the most common tool in harmonic analysis.

Lemma 3.4.1 – Paley-Littlewood decomposition

There exists $\psi \in C^\infty(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d \backslash \{0\}$ such that

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1 \quad \forall x \neq 0$$

For every $x \neq 0$ at max two of the terms overlaps. We can also take ψ radially symmetric. (è una funzione che si concentra quello che ti interessa)

Osservazione 3.4.1
This function ψ can be seen as a *locator*, i.e. when we consider $\psi \hat{f}$ we are localizing the function to a specific frequency. Indedd we can see that

$$\hat{f} = \sum_{j=-\infty}^{\infty} \psi(2^{-j}x)\hat{f} = \sum_{j=-\infty}^{+\infty} \hat{f}_j.$$

Proof. Let us consider $\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases}$ such that $\chi \in C^\infty$. We define $\psi(x) = \chi(x) - \chi(2x)$. This function is such that[14]:

$$\begin{aligned} \sum_{j=-N}^{N} \psi(2^{-j}x) &= \psi(2^N x) + \psi(2^{N-1}x) + \cdots + \psi(2^{-N}x) \\ &= \chi(2^{-N}x) - \chi(2^{N+1}x) \end{aligned}$$

Passing to the limit $N \to +\infty$ we have the thesis since $\chi(0) - \chi(\infty) = 1$. $\square$

The following result has a proof which is a prototype use of the locator we just defined.

Theorem 3.4.2 – Mikhlin- Hörmander

Let $m : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ such that $\forall |\gamma| \leq d+2$ it holds

$$|\partial^\gamma m(z)| \preccurlyeq |\xi|^{-|\gamma|}.$$

Then for all $1 < p < \infty$

$$\left\| \left(m(\xi)\hat{f}\right)^\vee \right\|_{L^p} \preccurlyeq \|f\|_{L^p}.$$

Idea We proved that having a Calderón-Zygmund kernel K gives us the estimate $\|Tf\|_{L^p} \preccurlyeq \|f\|_{L^p}$. In this statement it is as if we were considering m such that $\hat{K} = m$, hence we are unloading the requests on the kernel as decay conditions on the multiplier. ⌟

[14] We have a telescopic sum.

Proof. If we consider $\gamma = 0$ then it clear that $m \in L^\infty$ and so, for Plancherel:

$$\left\| \left(m(\xi)\hat{f} \right)^\vee \right\|_{L^2} = \|m(\xi)\hat{f}\|_{L^2} \leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

This gives as a $(2,2)$-estimate. As we saw in Observation (3.3.4) we just need to prove that the Hörmander condition holds. Let us write[15]:

$$\begin{aligned} m(\xi) &= \sum_{N \in 2^{\mathbb{Z}}} \psi\left(\frac{x}{N}\right) m(\xi) \\ &= \sum_{N \in 2^{\mathbb{Z}}} \psi_N(\xi) m(\xi) \\ &= \sum_{N \in 2^{\mathbb{Z}}} m_N(\xi). \end{aligned}$$

Observation (3.3.2) guarantees us that, if $|\nabla K| \preccurlyeq \frac{1}{|x|^{d+1}}$, then the Hörmander condition holds. But we remember that $K = \check{m}$, hence:

$$|\nabla K| \leq \sum_{N \in 2^{\mathbb{Z}}} |\nabla(\check{m}_N)|.$$

Let us estimate $\|x^\alpha \nabla \check{m}_N\|_{L^\infty}$. Remembering that $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$ and that $\partial^\alpha \hat{f} = \left(\widehat{(-2\pi i x)}|x|^\alpha f\right)$, we have that

$$\|x^\alpha \nabla \check{m}_N\|_{L^\infty} \preccurlyeq \|\partial^\alpha \xi m_N\|_{L^1}.$$

It holds[16]:

$$\begin{aligned} |\partial^\alpha(\xi m_N)| = |\partial^\alpha(\xi \psi_N(\xi) m(\xi))| &\preccurlyeq \sum_{\alpha_1 + \alpha_2 = \alpha} |\partial^{\alpha_1}(\xi m(\xi))||\partial^{\alpha_2}(\psi_N(\xi))| \\ &= \sum_{\alpha_1 + \alpha_2 = \alpha} |\partial^{\alpha_1}(\xi m)| \left| \partial^{\alpha_2} \psi\left(\frac{\xi}{N}\right) \right| \\ &\preccurlyeq \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{1}{|\xi|^{|\alpha_1| - 1}} N^{-|\alpha_2|} \left| \partial^{\alpha_2} \psi\left(\frac{x}{N}\right) \right| \\ &\preccurlyeq \sum_{\alpha_1 + \alpha_2 = \alpha} |\xi|^{1 - |\alpha_1|} N^{-|\alpha_2|} \left| \partial^{\alpha_2} \psi\left(\frac{\xi}{N}\right) \right|. \end{aligned}$$

Idea The advantage of using m_N is that the function is now localized, i.e. $\int_{\mathbb{R}^d} = \int_{|\xi| \sim N}$. ⌟

[15] This is a standard abbreviation to write the dyadic numbers

[16] we remember that $|\partial^{\alpha_1} m| \preccurlyeq \frac{1}{|\xi|^{|\alpha_1}}$

Proceeding in the computation we get:

$$\begin{aligned}
\|x^\alpha \nabla \check{m}_N\|_{L^\infty} &\leq \sum_{\alpha_1+\alpha_2=\alpha} \int_{\mathbb{R}^d} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|} |\partial^{\alpha_2}\psi\left(\frac{\xi}{N}\right) d\xi \\
&\leq \sum_{\alpha_1+\alpha_2=\alpha} \int_{|\xi|\sim N} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|} |\partial^{\alpha_2}\psi\left(\frac{\xi}{N}\right) d\xi \\
&= \sum_{\alpha_1+\alpha_2=\alpha} \int_0^N r^{1-|\alpha_1|} r^{d-1} dr N^{-|\alpha_2|} \\
&\preccurlyeq \sum_{\alpha_1+\alpha_2=\alpha} N^{-|\alpha_2|} N^{d+1-|\alpha_2|} \\
&= N^{d+1-|\alpha|}.
\end{aligned}$$

Let us now consider two cases: $\alpha = 0$ and $\alpha = d+2$:

$$\begin{aligned}
\alpha = 0 \quad &\Rightarrow |\nabla \check{m}_N| \leq N^{d+1} \\
\alpha = d+2 \quad &\Rightarrow |x^{d+2} \nabla \check{m}_N| \leq N^{d+1-d-2} = \frac{1}{N} \\
&\Rightarrow |\nabla \check{m}_N| \preccurlyeq \frac{1}{N}\frac{1}{x^{d+2}}
\end{aligned}$$

Thus:

$$|\nabla \check{m}_N| \leq \min\left(N^{d+1}, \frac{1}{N}\frac{1}{|x|^{d+2}}\right).$$

In conclusion[17]:

$$\begin{aligned}
|\nabla K| \leq \sum_{N\in 2^{\mathbb{Z}}} |\nabla(\check{m}_N)| &= \sum_{N \geq \frac{1}{|x|}} |\nabla(\check{m}_N)| + \sum_{N \leq \frac{1}{|x|}} |\nabla(\check{m}_N)| \\
&\leq \sum_{N \geq \frac{1}{|x|}} \frac{1}{N}\frac{1}{|x|^{d+2}} + \sum_{N \leq \frac{1}{|x|}} N^{d+1} \\
&\preccurlyeq \frac{1}{|x|^{d+1}}.
\end{aligned}$$

□

Another classical use of this locator is in the PDEs theory, where it ensures the validity of a chain rule for fractional derivatives.

[17] For dyadic sums it is true that $\sum_{N \geq N_0} N = \frac{2}{N_0}$ and $\sum_{N \leq N_0} N = 2N_0$.