Winter School
Geometry, Algebra and Combinatorics of Moduli Spaces and Configurations II

Milnor Fibre and Characteristic Variety of Line Arrangements

Oscar Papini
University of Pisa

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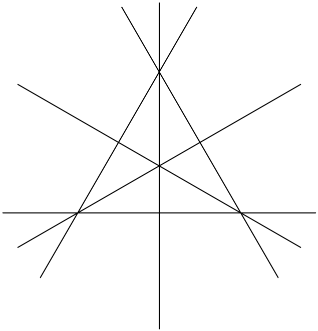
Definition (Hyperplane arrangement)

Let V be a finite-dimensional vector space over a field $\mathbb{K}$. A hyperplane arrangement $\mathcal{A}$ is a (finite) collection of affine hyperplanes of V. The same definition can be given for a projective hyperplane arrangement in a projective space.

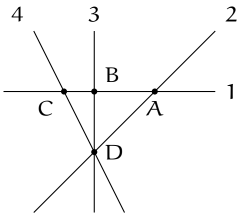


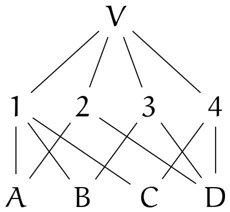
Basic definitions

- The complement of an arrangement $\mathcal{A}$ is the set
$$\mathcal{M}(\mathcal{A}) \coloneqq V \setminus \bigcup_{H \in \mathcal{A}} H.$$
- An arrangement $\mathcal{A}$ is central if
$$\bigcap_{H \in \mathcal{A}} H \neq \varnothing.$$
- The defining polynomial of an arrangement $\mathcal{A}$ is
$$Q_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_H$$
where α_H is a linear form defining H.

Definition (Intersection poset)

The intersection poset $L(\mathcal{A})$ of an arrangement $\mathcal{A}$ is the set of all non-empty intersections of hyperplanes of $\mathcal{A}$, partially ordered by reverse inclusion. It includes V as the intersection of zero hyperplanes.





A, B, C, D are the *singular points* of $\mathcal{A}$. For a singular point P, its *multiplicity* $m(P)$ is the number of lines passing through it.

Combinatorial properties

Definition (Combinatorial property)

We say that a property of an arrangement $\mathcal{A}$ is combinatorial if it depends only on the intersection poset $\mathrm{L}(\mathcal{A})$.

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- The cohomology ring $\mathrm{H}^*(\mathcal{M}(\mathcal{A}); \mathbb{C})$ *is* combinatorial (Orlik-Solomon algebra).
- The fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ *is not* combinatorial (Rybnikov counterexample).

Our setting

From now we will suppose that $\mathcal{A}$ is an arrangement of $n+1$ projective lines in $\mathbb{P}^2(\mathbb{R})$. The defining polynomial $Q_\mathcal{A}$ belongs to $\mathbb{R}[X, Y, Z]$ and it is homogeneous of degree $n+1$.

Since the topology of $\mathcal{M}(\mathcal{A})$ in $\mathbb{P}^2(\mathbb{R})$ is easy to describe, we will consider the *complexified* arrangement $\mathcal{A}_\mathbb{C}$, which is the arrangement in $\mathbb{P}^2(\mathbb{C})$ defined by $Q_\mathcal{A}$, and study the complement $\mathcal{M}(\mathcal{A}_\mathbb{C}) \subseteq \mathbb{C}^2$.

The Milnor fibre

Definition (Milnor fibre and geometric monodromy)

Let $\mathcal{A}$ be an arrangement of $n+1$ projective lines. Consider $Q = Q_{\mathcal{A}}$ as a map $Q\colon \mathbb{C}^3 \to \mathbb{C}$; it defines a fibration

$$Q|_{Q^{-1}(\mathbb{C}^*)} : Q^{-1}(\mathbb{C}^*) \to \mathbb{C}^*.$$

The fibre $F \coloneqq Q^{-1}(1)$ is the Milnor fibre of the arrangement. The map

$$\begin{aligned} h\colon \quad F &\to F \\ x &\mapsto \lambda x \end{aligned}$$

where $\lambda \coloneqq e^{2\pi i/(n+1)}$ is called geometric monodromy of the Milnor fibre.

The Milnor fibre

The geometric monodromy induces a map

$$h_*: H_*(F; \mathbb{C}) \to H_*(F; \mathbb{C});$$

we will focus on the first homology group.

Proposition

There is a $\mathbb{C}[T^{\pm 1}]$*-module isomorphism*

$$H_1(F; \mathbb{C}) \simeq H_1(\mathcal{M}(\mathcal{A}_{\mathbb{C}}); \mathbb{C}[T^{\pm 1}])$$

where the action of T *on the left is given by the monodromy action, i.e.* $T \cdot [a] = h_1([a])$ *for* $[a] \in H_1(F; \mathbb{C})$.

$H_1(\mathcal{M}(\mathcal{A}_{\mathbb{C}}); \mathbb{C}[T^{\pm 1}])$ is an example of *local coefficients homology* (we'll come back on this later).

A-monodromicity

Since $\mathbb{C}[T^{\pm 1}]$ is a PID, and the monodromy action has order $n+1$, we have a decomposition

$$H_1(\mathcal{M}(\mathcal{A}_\mathbb{C});\mathbb{C}[T^{\pm 1}]) \simeq \bigoplus \mathbb{C}[T^{\pm 1}]/(\varphi_d)$$

where φ_d is the d-th cyclotomic polynomial, and $d \mid n+1$.

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Definition (A-monodromic arrangement)

An arrangement $\mathcal{A}$ of lines in $\mathbb{P}^2(\mathbb{R})$ is a-monodromic if

$$H_1(\mathcal{M}(\mathcal{A}_\mathbb{C}); \mathbb{C}[T^{\pm 1}]) \simeq \mathbb{C}^n \left[\simeq \left(\mathbb{C}[T^{\pm 1}]/(T-1) \right)^n \right].$$

This corresponds to the fact that the only eigenvalue of h_1 is 1, i.e. h_1 is trivial.

A conjecture

No general formula for the Milnor fibre homology is known (not even for the first Betti number!), nor it is known to what extent all this is combinatorial—there are some conjectures, though.

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Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{R})$. The *double point graph* $\Gamma(\mathcal{A})$ is the graph defined as follows:

- its vertex set is $\{H \mid H \in \mathcal{A}\}$;
- there is an edge $\{H_1, H_2\}$ iff $H_1 \cap H_2$ is a double point.

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Conjecture (Salvetti-Serventi '15/'17)

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Salvetti and Serventi proved this only assuming extra hypotheses on the graph $\Gamma(\mathcal{A})$.

Local systems

Definition (Local system)

Let $\mathcal{A}$ be an arrangement of $n+1$ lines in $\mathbb{P}^2(\mathbb{R})$, $M := \mathcal{M}(\mathcal{A}_\mathbb{C})$, and let R be a commutative ring with unity. A rank-1 abelian local system is a structure of $\pi_1(M)$-module on R.

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When $R = \mathbb{C}$, the action $\pi_1(M) \to \mathcal{A}ut(\mathbb{C}) \simeq \mathbb{C}^*$ factors through $H_1(M;\mathbb{Z})$, which is free abelian of rank $n+1$ generated by $\beta_1, \ldots, \beta_{n+1}$, where β_i is a loop around a complex line of $\mathcal{A}_\mathbb{C}$. In this case, the local system is defined by a choice of a non-zero complex number t_i for each β_i.

Local coefficient (co)homology

We will denote by $\mathbb{C}_{\mathbf{t}}$ the local system defined by $\mathbf{t} \coloneqq (t_1, \ldots, t_{n+1}) \in (\mathbb{C}^*)^{n+1}$, and with $H_*(M; \mathbb{C}_{\mathbf{t}})$ and $H^*(M; \mathbb{C}_{\mathbf{t}})$ respectively the homology and cohomology of M with coefficients in $\mathbb{C}_{\mathbf{t}}$.

Remark

1. We have

$$H(M; \mathbb{C}_{\mathbf{t}}) \simeq H(M; \mathbb{C}[T_1^{\pm 1}, \ldots, T_{n+1}^{\pm 1}])$$

   where the action of β_i on the right is given by multiplication by T_i.

2. The homology of the Milnor fibre is isomorphic to the homology of M with coefficients in the local system defined by $\beta_i \mapsto t$ for all $i = 1, \ldots, n+1$.

Definition (Characteristic variety)

Let $\mathcal{A}$ be an arrangement as before. The (first) characteristic variety is

$$\mathcal{V}(\mathcal{A}) \coloneqq \{\mathbf{t} \in (\mathbb{C}^*)^{n+1} \mid \dim H_1(\mathcal{M}(\mathcal{A}_\mathbb{C}); \mathbb{C}_\mathbf{t}) \geqslant 1\}.$$

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Theorem (Arapura '97)

$\mathcal{V}(\mathcal{A})$ *is a union of (eventually translated) subtori of* $(\mathbb{C}^*)^{n+1}$.

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Is $\mathcal{V}(\mathcal{A})$ combinatorial?

Resonance varieties

Let A be the Orlik-Solomon algebra associated with $\mathcal{A}$. Fix $a \in A^1$. Left-multiplication by a gives $A^\bullet$ the structure of a cochain complex.

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$$\mathcal{R}(\mathcal{A}) \coloneqq \{a \in A^1 \mid \dim H^1((A^\bullet, a\,\cdot\,); \mathbb{C}) \geqslant 1\}.$$

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Tangent Cone Theorem (Cohen-Suciu '99)

$\mathcal{R}(\mathcal{A})$ is the tangent cone of $\mathcal{V}(\mathcal{A})$ at $(1, \ldots, 1) \in (\mathbb{C}^*)^{n+1}$.

The "homogeneous part" of $\mathcal{V}(\mathcal{A})$ is combinatorial!

Local and non-local components

Denote the lines of $\mathcal{A}$ with $[n+1] := \{1, \ldots, n+1\}$ and a singular point with the subset of $[n+1]$ indicating the lines passing through it. Let $S \subseteq \wp([n+1])$ be the set of the singular points.

For each $P \in S$ with $\#(P) \geqslant 3$, there is a *local* component of $\mathcal{R}(\mathcal{A})$ given by

$$C(P) := \left\{ z \,\middle|\, \sum_{j=1}^{n+1} z_j = 0 \right\} \cap \bigcap_{j \notin P} \{ z \mid z_j = 0 \}$$

The *non-local* components admit a description in terms of *neighbourly partitions*.

Definition (Neighbourly partition)

A partition $\pi = (p_1 \mid \cdots \mid p_r)$ of $[n+1]$ is neighbourly if for all $i = 1, \ldots, r$ and for all $P \in S$

$$\#(p_i \cap P) \geqslant \#(P) - 1 \Rightarrow P \subseteq p_i.$$

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$$\#(p_i \cap P) \geqslant \#(P) - 1 \;\Rightarrow P \subseteq p_i.$$

If π is a neighbourly partition, define $C(\pi) \subseteq \mathbb{C}^{n+1}$ as

$$C(\pi) \coloneqq \left\{ z \;\middle|\; \sum_{j=1}^{n+1} z_j = 0 \right\} \cap \bigcap_{P \in \mathcal{P}} \left\{ z \;\middle|\; \sum_{j \in P} z_j = 0 \right\}$$

where $\mathcal{P} \coloneqq \{P \in \mathcal{S} \mid \nexists\, p \in \pi \text{ s.t. } P \subseteq p\}$.

Neighbourly partitions

Proposition

If $\dim(C(\pi)) \geqslant 2$, *then* $C(\pi)$ *is a non-local component of* $\mathcal{R}(\mathcal{A})$.

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If π is a partition of a *subset* $B \subseteq [n+1]$, define *support* of π, $\operatorname{supp}(\pi)$, the set B.

Proposition

Let $\mathcal{B} \subseteq \mathcal{A}$ *be a subarrangement and let* π *be a neighbourly partition for* $\mathcal{B}$ *such that* $\dim(C(\pi)) \geqslant 2$*. Then*

$$C(\pi) \cap \bigcap_{j \notin \operatorname{supp}(\pi)} \{z_j = 0\}$$

is a non-local component of $\mathcal{R}(\mathcal{A})$*. All non-local components of* $\mathcal{R}(\mathcal{A})$ *arise from subarrangements of* $\mathcal{A}$ *this way.*

Combinatorics of the characteristic variety

For the homogeneous part of the characteristic variety $\mathcal{V}(\mathcal{A})$, we define *ideals* of $\mathbb{C}[T_1^{\pm 1}, \dots, T_{n+1}^{\pm 1}]$ such that their varieties are the components of $\mathcal{V}(\mathcal{A})$.

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- If $P \in \mathcal{S}$ with $\#(P) \geqslant 3$, define

$$\mathcal{I}(P) := \left(\prod_{j=1}^{n+1} T_j - 1 \right) + \left(T_j - 1 \mid j \notin P \right);$$

  this corresponds to a local component of $\mathcal{V}(\mathcal{A})$.

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  where $\mathcal{P} \coloneqq \{P \in \mathcal{S} \mid \nexists\, p \in \pi \text{ s.t. } P \subseteq p\}$.

Proposition

Let $\mathcal{B} \subseteq \mathcal{A}$ be a subarrangement and let π be a neighbourly partition for $\mathcal{B}$ such that $\dim(\mathcal{I}(\pi)) \geqslant 2$. Then the component passing through $(1, \ldots, 1)$ of the variety in $(\mathbb{C}^*)^{n+1}$ defined by the ideal

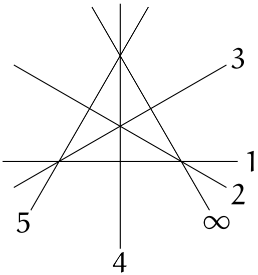
$$\mathcal{I}(\pi) + (T_j - 1 \mid j \notin \operatorname{supp}(\pi))$$

is a non-local component of $\mathcal{V}(\mathcal{A})$. All non-local components of $\mathcal{V}(\mathcal{A})$ passing through $(1, \ldots, 1)$ arise from subarrangements of $\mathcal{A}$ this way.

Example: A_3

Note: in all examples, we actually compute the characteristic variety of the affine arrangement $\mathfrak{a}\mathcal{A}$ of n lines in $\mathbb{R}^2$.

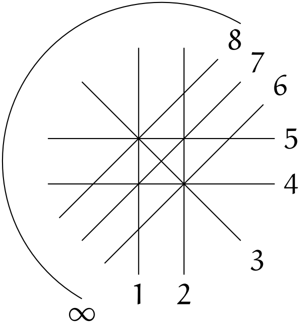
$$\mathcal{V}(\mathcal{A}) = \{(\mathbf{t}, t_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid \mathbf{t} \in \mathcal{V}(\mathfrak{a}\mathcal{A}),\ t_1 \cdots t_{n+1} = 1\}$$



$1\,4 \,|\, 2\,5 \,|\, 3\,\infty$

- 4 local components
- 1 non-local component, given by $\mathcal{I}(1\,4 \,|\, 2\,5 \,|\, 3\,\infty)$

Example: B_3



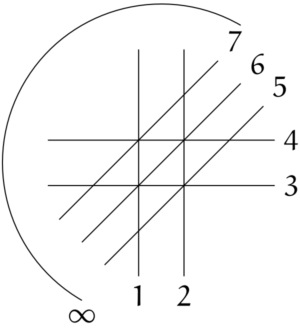
$1\,2\,|\,3\,7\,|\,4\,5$
$\vdots$ } 11 part.
$3\,\infty\,|\,4\,8\,|\,5\,6$

$3\,7\,\infty\,|\,1\,|\,2\,|\,4\,|\,5$
$1\,5\,6\,|\,2\,|\,4\,|\,7\,|\,\infty$
$2\,4\,8\,|\,1\,|\,5\,|\,7\,|\,\infty$
$1\,5\,6\,|\,2\,4\,8\,|\,3\,7\,\infty$

- 7 local components
- 11 non-local components given by A_3 subarrangements
- 1 non-local component given by $\mathcal{I}(1\,5\,6\,|\,2\,4\,8\,|\,3\,7\,\infty)$

The three partitions of the form $x\,x\,x\,|\,x\,|\,x\,|\,x\,|\,x$ give rise to a zero-dimensional ideal, so they don't contribute to $\mathcal{V}(\mathcal{A})$.

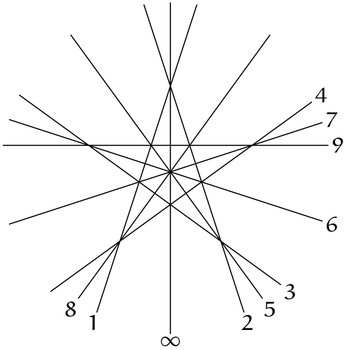
Example: B_3 deleted



$1\,4 \mid 2\,3 \mid 6\,\infty$
$1\,5 \mid 2\,6 \mid 3\,\infty$
$1\,6 \mid 2\,7 \mid 4\,\infty$
$1\,\infty \mid 3\,7 \mid 4\,6$
$2\,\infty \mid 3\,6 \mid 4\,5$
$1\,4\,5 \mid 2 \mid 3 \mid 6 \mid \infty$
$2\,3\,7 \mid 1 \mid 4 \mid 6 \mid \infty$

- 7 local components
- 5 non-local components given by A_3 subarrangements
- 1 translated component with equations (in $\mathcal{V}(a\mathcal{A})$)

$$t_6+1,\quad t_2-t_3,\quad t_1-t_4,\quad t_5t_7-1,\quad t_4t_7+t_3,$$
$$t_3t_5+t_4,\quad t_4^2-t_5,\quad t_3t_4+1,\quad t_3^2-t_7$$



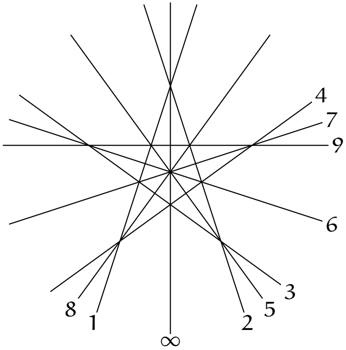
$$\left.\begin{array}{c} 1\,5\,|\,2\,7\,|\,3\,\infty \\ \vdots \\ 3\,7\,|\,4\,6\,|\,9\,\infty \end{array}\right\}\ \text{10 part.}$$

$$1\,6\,|\,2\,7\,|\,3\,8\,|\,4\,5\,|\,9\,\infty\ (\star)$$

- 11 local components
- 10 non-local components given by A_3 subarrangements
- 4 translated components with equations (in $\mathcal{V}(a\mathcal{A})$)

$$t_7 - t_8,\quad t_6 - t_8,\quad t_5 - t_8,\quad t_4 - t_9,\quad t_3 - t_9,\quad t_2 - t_9,$$
$$t_1 - t_9,\quad t_8 t_9 + t_9^2 + t_8 + t_9 + 1,\quad t_8^2 - t_9,\quad t_9^3 - t_8$$

The R(10) arrangement

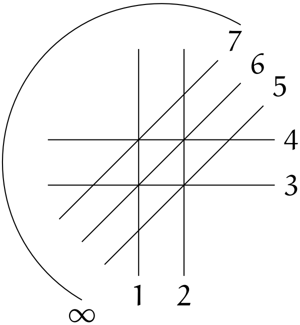


$$1\,5\,|\,2\,7\,|\,3\,\infty \quad \vdots \quad 3\,7\,|\,4\,6\,|\,9\,\infty \; \Big\} \text{ 10 part.}$$

$$1\,6\,|\,2\,7\,|\,3\,8\,|\,4\,5\,|\,9\,\infty \quad (\star)$$

The ideal $\mathcal{I}(\star)$ associated with the partition $(\star)$ is zero-dimensional, so it doesn't contribute to the variety. *But its primary decomposition has 7 ideals, among which there is the ideal that defines the translated components!*

B_3 deleted – Part II



$1\,4 \mid 2\,3 \mid 6\,\infty$
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$1\,4\,5 \mid 2 \mid 3 \mid 6 \mid \infty$
$2\,3\,7 \mid 1 \mid 4 \mid 6 \mid \infty$

If we try to compute the primary decompositions of the two ideals given by the $x\,x\,x \mid x \mid x \mid x \mid x$ partitions, we obtain nothing of interest.

Sum of partitions

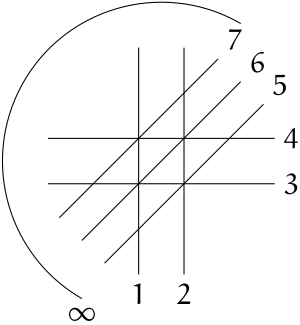
Definition (Sum of partitions)

Let π_1 and π_2 be partitions with *a priori* different supports $\mathrm{supp}(\pi_1)$ and $\mathrm{supp}(\pi_2)$. Define the sum of π_1 and π_2 as the partition $\pi_1 + \pi_2$ such that

1. $\mathrm{supp}(\pi_1 + \pi_2) = \mathrm{supp}(\pi_1) \cup \mathrm{supp}(\pi_2)$;
2. it is the finest partition such that each block of π_1 and π_2 is contained in a block of $\pi_1 + \pi_2$.

$$\begin{aligned} \pi_1 &= \quad 1\,4\,5\,|\,2\,|\,3\,|\,6\,|\,\infty \\ \pi_2 &= \quad 2\,3\,7\,|\,1\,|\,4\,|\,6\,|\,\infty \end{aligned}$$

$$\pi_1 + \pi_2 = \quad 1\,4\,5\,|\,2\,3\,7\,|\,6\,|\,\infty$$



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$2\,\infty \mid 3\,6 \mid 4\,5$

$1\,4\,5 \mid 2 \mid 3 \mid 6 \mid \infty$

$2\,3\,7 \mid 1 \mid 4 \mid 6 \mid \infty$

The primary decomposition of $\mathcal{I}(1\,4\,5 \mid 2\,3\,7 \mid 6 \mid \infty)$ is $I_1 \cap I_2$, where

$$I_1 = \begin{pmatrix} t_6 + 1, \quad t_2 - t_3, \quad t_1 - t_4, \quad t_5 t_7 - 1, \quad t_4 t_7 + t_3, \\ t_3 t_5 + t_4, \quad t_4^2 - t_5, \quad t_3 t_4 + 1, \quad t_3^2 - t_7 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} t_6 - 1, \quad t_2 - t_3, \quad t_1 - t_4, \quad t_5 t_7 - 1, \quad t_4 t_7 - t_3, \\ t_3 t_5 - t_4, \quad t_4^2 - t_5, \quad t_3 t_4 - 1, \quad t_3^2 - t_7 \end{pmatrix}$$

- I have other examples of translated components appearing in primary decompositions of ideals associated with iterated sum of partitions. Unfortunately I don't have a criterion to select which ideals actually belong to the characteristic variety. But I am working on it!
- I used a chain complex introduced by Gaiffi and Salvetti in order to compute the characteristic varieties. The algorithm requires some time (it took more than two weeks for $R(10)$) and becomes unfeasible for arrangements with > 10 lines.
- Notice that the monodromy of the Milnor fibre can be retrieved from the characteristic variety (just put all t_i's equal to t).

Thank you!

Thank you for your attention.