Invariant types in model theory

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to Federica

La perfezione è un falso, e rende pazzi
Elio e le Storie Tese

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Abstract

We study how the product of global invariant types interacts with the preorder of *domination*, i.e. semi-isolation by a small type, and the induced equivalence relation, *domination-equivalence*. We provide sufficient conditions for the latter to be a congruence with respect to the product, and show that this holds in various classes of theories. In this case, we develop a general theory of the quotient semigroup, the *domination monoid*, and carry out its computation in several cases of interest. Notably, we reduce its study in o-minimal theories to proving generation by 1-types, and completely characterise it in the case of Real Closed Fields. We also provide a full characterisation for the theory of dense meet-trees, and moreover show that the domination monoid is well-defined in certain expansions of it by binary relations.

We give an example of a theory where the domination monoid is not commutative, and of one where it is not well-defined, correcting some overly general claims in the literature. We show that definability, finite satisfiability, generic stability, and weak orthogonality to a fixed type are all preserved downwards by domination, hence are domination-equivalence invariants. We study the dependence on the choice of monster model of the quotient of the space of global invariant types by domination-equivalence, and show that if the latter does not depend on the former then the theory under examination is NIP.

Keywords. domination, domination-equivalence, model theory, neostability theory, invariant types, small-type semi-isolation

Mathematics Subject Classification. Primary: 03C45. Secondary: 03C64, 03C65

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Chapter 1

Introduction

Overture. To a sufficiently saturated model of a first-order theory one can associate a semigroup, that of *global invariant types* with the *tensor product*. This can be endowed with a preorder, which goes after the name of *domination*, and whose kernel is called *domination-equivalence*. This thesis studies the resulting quotient, starting from sufficient conditions for the tensor product to induce a well-defined operation on it. We show, correcting a remark in [HHM08], that this need not be always the case, develop a general theory of this object, which we dub *domination monoid*, provide tools to compute it, and do so in several cases of interest.

Let $S(\mathfrak{U})$ be the space of types in any finite number of variables over a model $\mathfrak{U}$ of a first-order theory that is κ-saturated and κ-strongly homogeneous for some large κ. For every set $A \subseteq \mathfrak{U}$, one has a natural action on $S(\mathfrak{U})$ by the group $\operatorname{Aut}(\mathfrak{U}/A)$ of automorphisms of $\mathfrak{U}$ that fix A pointwise. The space $S^{\mathrm{inv}}(\mathfrak{U})$ of *global invariant types* consists of those elements of $S(\mathfrak{U})$ which, for some *small* A, are fixed points of the action $\operatorname{Aut}(\mathfrak{U}/A) \curvearrowright S(\mathfrak{U})$. Each of these types has a canonical extension to bigger models $\mathfrak{U}_1 \succ \mathfrak{U}$, namely the unique one which is a fixed point of the action $\operatorname{Aut}(\mathfrak{U}_1/A) \curvearrowright S(\mathfrak{U}_1)$, and this allows us to define an associative product $\otimes$ on $S^{\mathrm{inv}}(\mathfrak{U})$. This is the semigroup which we are going to quotient.

We say that a global type p *dominates* a global type q, denoted by $p \geq_{\mathrm{D}} q$, when p together with a small set of formulas entails q. This is a preorder and we call the induced equivalence relation *domination-equivalence*, denoted by $\sim_{\mathrm{D}}$. We also look at *equidominance*, the refinement of domination-equivalence obtained by requiring that domination of p by q and of q by p can be witnessed by the same set of formulas. These notions have their roots in the work of Lascar, who in [Las75] generalised the Rudin–Keisler preorder on ultrafilters to types of a theory; this preorder was subsequently generalised to domination between stationary types in a stable theory.

Main results. Equidominance reached its current form in [HHM08], where it was used to prove a result of Ax–Kochen–Eršov flavour; namely the computation, in the case of Algebraically Closed Valued Fields, of the semigroup of invariant types modulo equidominance, which turns out to be commutative and to decompose in terms of value group and residue field. It was also claimed, without proof, that such a semigroup is well-defined and commutative in every complete first-order theory. The starting point of this research was to try to fill this gap by proving these claims. After attempting in vain to prove well-definedness of the quotient semigroup, I started to investigate sufficient conditions for it to hold. Eventually, a counterexample arose.

Theorem A (Theorem 5.3.8 and Corollary 5.3.2)**.** There is a ternary, ω-categorical, supersimple theory of SU-rank 2 with degenerate algebraic closure in which neither equidominance nor domination-equivalence are congruences with respect to the product of invariant types. Moreover, in the theory of the Random Graph, there are two types that do not commute modulo domination-equivalence.

On the positive side, we provide the following domination-equivalence invariants, in the general setting of an arbitrary first-order theory. We also show that independence of the number of domination-equivalence classes from the choice of monster model is incompatible with the Independence Property.

Theorem B (Theorem 2.3.7 and Theorem 2.3.16)**.** If $p_0 \geq_{\mathrm{D}} p_1$ and p_0 is definable, finitely satisfiable in some small model, generically stable, or weakly orthogonal to q, then so is p_1.

Theorem C (Theorem 6.2.9)**.** If there is only a bounded number of types modulo domination-equivalence, then T is NIP.

The theorems above have already appeared in [Men20]. In theories where $\otimes$ is compatible with $\geq_{\mathrm{D}}$, it is possible to define the *domination monoid* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ to be the quotient of $(S^{\mathrm{inv}}(\mathfrak{U}), \otimes)$ by $\sim_{\mathrm{D}}$. Some of the results that first appear on these pages are concerned with the study of this monoid in o-minimal theories. In Theorem 4.1.27 we reduce the problem to obtaining a proof of generation by classes of 1-types, which is enough to secure the following application.

Theorem D (Theorem 4.2.37)**.** In the theory of Real Closed Fields, the domination monoid $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined and isomorphic to the semilattice of finite subsets $(\mathscr{P}_{\mathrm{fin}}(X), \cup)$, where X is the set of convex subrings of $\mathfrak{U}$ which are fixed setwise under the action of $\mathrm{Aut}(\mathfrak{U})$ by the pointwise stabiliser of some small set. In this isomorphism, weak orthogonality corresponds to disjointness.

This, together with the results in [EHM19], yields a similar characterisation for the theory of Real Closed Valued Fields. Another theory in which we compute $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is that of *dense meet-trees*. In Theorem 5.2.15 we also show that the domination monoid is well-defined in certain expansions of the latter studied in [EK19].

Theorem E (Theorem 5.2.12)**.** In the theory of dense meet-trees, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ has the form $\mathscr{P}_{\mathrm{fin}}(X) \oplus \bigoplus_\kappa \mathbb{N}$.

These are far from being the only examples we will encounter: in total, we provide full computations of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in a dozen theories or classes thereof. While some of these computations are implicit in the literature, some are, to the best of my knowledge, original.

Structure of thesis. In the rest of Chapter 1, we outline the structure of this document.

The theoretical core of this project consists of Chapter 2, in which we work in full generality. After setting up the context and providing the necessary definitions, we prove some general facts relating the product of invariant types with domination. In Example 2.1.30 we also encounter what may be the first instance (at least the first known to me) of a theory where domination differs from $\mathrm{F}^{\mathrm{s}}_\kappa$-isolation in the sense of Shelah. We introduce several notions that ensure compatibility of $\otimes$ with $\geq_{\mathrm{D}}$, the most general of which is called *stationary domination* (Definition 2.2.3). It is implied by stability, by elimination of quantifiers in a binary relational language, and more; see Figure 2.1. We then study the interaction of domination with properties of types, and prove Theorem B. Finally, we examine the role played by stably embedded sets, rephrase domination in different terms, and hint at a categorical[1] approach.

[1]In the sense of arrows, not in the sense of unique models.

Chapter 3 is a survey of some parts of classical stability theory relevant to the present endeavour. It also contains an analysis of the domination monoids of several stable theories but, I believe, essentially no new results.

The o-minimal case is investigated in Chapter 4. We reduce the problem of computing $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ (and showing it is well-defined) to proving it is generated by domination-equivalence classes of 1-types (Theorem 4.1.27). This approach is inspired by, but more general than, the characterisation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in [HHM08] for Divisible Ordered Abelian Groups, and fills a gap in the proof of the latter (Theorem 4.2.20). The chapter culminates in the proof of Theorem D.

After a chapter and two halves of theoretical development, Chapter 5 is almost completely dedicated to computations, and is home to Theorem A, Theorem E, and Theorem 5.2.15, as well as to easier examples illustrating various idiosyncrasies that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ may display. Amongst other theories, in Theorem 5.2.22 we review what happens in Algebraically Closed Valued Fields — the initial motivating example from [HHM08] — and see how Theorem D complements [EHM19] in computing $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in Real Closed Valued Fields.

As the notation $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ shows, in general this object depends on the choice of $\mathfrak{U}$; Chapter 6 studies this dependence. In Proposition 6.1.2 we prove that the natural map $\widetilde{\mathrm{Inv}}(\mathfrak{U}_0) \to \widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$ linking the domination monoids of $\mathfrak{U}_0 \prec \mathfrak{U}_1$ preserves several relations, and Corollary 6.1.9 gives sufficient conditions for its injectivity. We briefly return, in spirit, to Chapter 3 with another micro-survey of part of stability theory, then give our concluding proof, that of Theorem C.

Backmatter notwithstanding, Chapter 7 will populate our final pages with a modicum of history and a pinch of speculations and open problems. Several more questions are scattered throughout this work, as I have tried to ask them in their natural habitats.

The reader who has seen [Men20] before will probably like to know that the intersection of its complement with the current work mostly consists of, but is not included in, the union of Subsection 2.1.5 with everything from Subsection 2.3.3 to the end of Chapter 2, along with most of Section 3.2, the entirety of Chapter 4, and the first two sections of Chapter 5.

Modulo a very small number of exceptions we do not use results to be proven in later pages. Nevertheless, I have taken the liberty of introducing cross-references in various points of this thesis, and some of them point to later material. It is up to the reader whether to ignore them or to have a quick glance at chapters they have not read yet.

Chapters, and to a lesser extent sections within them, enjoy a fairly high degree of independence. After Section 2.1 it is in principle possible to read the rest of this thesis in any order, and readers who would like to be refreshed by an example are encouraged to quench their thirst in Section 3.2, Section 4.2, or Chapter 5 at any point, to visit Section 7.1 when tired of reading technical lemmas, and to look up in the Index any definition they may have missed as a consequence of a permutation of the order of reading. Having said that, there is a reason if chapters are assigned ordinals, and my personal recommendation for an element of S_7 would in this case be the identity.

Chapter 2

The domination monoid

The context of this chapter is that of an arbitrary first-order theory T. We begin by defining our main objects of study, the preorder of *domination* and the *domination monoid* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, along with some sufficient conditions for the latter to be well-defined. Next, we deal with downward preservation of properties by domination, and with submonoids of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. We conclude by observing the situation from different perspectives.

2.1 Set-up

We lay down the fundamental definitions and facts for the work to come. In the last subsection, we also introduce a few variants.

2.1.1 The fine print

Conventions, notations, and abuses thereof are standard, and we now recall some of them. The reader familiar with the area may want to skip immediately to Definition 2.1.1, or even Definition 2.1.9. We refer to texts such as [CK90, Hod93, Kir19, Mar02, Poi00, TZ12] for model theory basics (and more).

In this thesis we only deal with consistent, complete first-order theories T, in a multi-sorted first-order language L, with infinite models, in the sense that each model of T has at least one infinite sort. If we define a property a theory may have, and then we say that a structure has it, we mean that its complete theory does. As customary, all mentioned inclusions between models of T are assumed to be elementary maps, and we call models of T which are saturated enough *monster* models; we denote them by $\mathfrak{U}$, $\mathfrak{U}_0$, etc. The reader who is happy to assume that arbitrarily large strongly inaccessible cardinals exist, may take $\mathfrak{U}$ to denote a model of strongly inaccessible size $\kappa(\mathfrak{U}) > \beth_\omega(|T|)$ which is $\kappa(\mathfrak{U})$-saturated, and let *small* mean "of size strictly less than $\kappa(\mathfrak{U})$".

In the absence of large cardinals, formally speaking, monster models will come with cardinals, i.e. we consider pairs $(\mathfrak{U}, \kappa(\mathfrak{U}))$ such that $\mathfrak{U}$ is $\kappa(\mathfrak{U})$-saturated and $\kappa(\mathfrak{U})$-strongly homogeneous for a large enough strong limit $\kappa(\mathfrak{U})$, say $\kappa(\mathfrak{U}) > \beth_\omega(|T|)$. When we say that A is *small* we mean $|A| < \kappa(\mathfrak{U})$. *Large* means "not small". So $\mathfrak{U}$ may be for instance λ^+-saturated and λ^+-strongly homogeneous for $\lambda^+ > \kappa(\mathfrak{U})$, but sets of size λ are not considered small. This is done in order to be able to assume that $\kappa(\mathfrak{U})$ is a strong limit without extra set-theoretical hypotheses. In practice, we will not mention $\kappa(\mathfrak{U})$ for most of the time, in order not to burden our notation excessively.

The letter A usually represents a small subset of $\mathfrak{U}$, the letters M, N small elementary substructures. If A is small and included in $\mathfrak{U}$ we denote this by $A \subset^+ \mathfrak{U}$, or $A \prec^+ \mathfrak{U}$ if additionally $A \prec \mathfrak{U}$. If a model is denoted e.g. by N, and not by $\mathfrak{U}$ or variations (e.g. $\mathfrak{U}_1$), the notation $A \subset^+ N$ means that N is $|A|^+$-saturated and $|A|^+$-strongly homogeneous, and similarly for $M \prec^+ N$.

Parameters and variables are tacitly allowed to be finite tuples unless otherwise specified. Concatenation of tuples is denoted by juxtaposition, and so is union of sets, e.g. $AB = A \cup B$. Coordinates of a tuple are indicated with subscripts, starting with 0, so for instance $a = (a_0, \ldots, a_{|a|-1})$, where $|a|$ denotes the length of a. Since we work in multi-sorted logic, variables and parameters come with a sort; we abuse the notation and terminology, and the length $|a|$ will also denote the tuple of sorts of each a_i. So, for example, if the sorts of T include H_0 and H_1 and $a = (a_0, a_1)$ where $a_i \in H_i(\mathfrak{U})$ for $i < 2$, and we write $|a| = |b|$, then $b_0 \in H_0(\mathfrak{U})$ and $b_1 \in H_1(\mathfrak{U})$. Another common way in which we are going to abuse the notation will be to write e.g. $a \in \mathfrak{U}$ instead of $a \in \mathfrak{U}^{|a|}$. In the example above, $\mathfrak{U}^{|a|} = H_0(\mathfrak{U}) \times H_1(\mathfrak{U})$. A *sequence* is a function with domain a totally ordered set, not necessarily $\mathbb{N}$. To avoid confusion when dealing with a sequence of tuples, indices are written as superscripts, as in $(a^i)_{i \in I}$. Tuples and sequences may be sometimes treated as sets, as in $a^0_0 \in a^0 \in (a^i)_{i \in I}$. Lowercase Latin letters towards the end of the alphabet, e.g. x, y, z, usually denote tuples of variables, while letters such as a, b, c usually denote tuples of elements of a model. We write e.g. $x = a$ instead of $\bigwedge_{i < |x|} x_i = a_i$, and definable functions may be tuples too, e.g. we may write $y = f(x)$ instead of $(y_0, \ldots, y_{|y|-1}) = (f_0(x), \ldots, f_{|y|-1}(x))$. If $|x| = 1$ we write x instead of x_0.

A *partial type* is a not necessarily complete type in finitely many variables. *Type over* B means "complete type over B in finitely many variables". Types are usually denoted by letters like p, q, r, and partial types by letters such as π or Φ, but π is also used to denote various projection maps. For $\Phi(x, y)$ a partial type, the notation $\exists y\ \Phi(x, y)$ means $\{\exists y\ \varphi(x, y) \mid \varphi(x, y) \in \Phi(x, y)\}$. We sometimes write e.g. p_x in place of $p(x)$ and denote with $S_x(B)$ the space of types in variables x. When T is single-sorted, we also use the notation $S_n(B)$ for the space of types in n variables. A *global type* is a complete type over $\mathfrak{U}$.

When mentioning realisations of global types, or supersets of a monster, we implicitly think of them as living inside a bigger monster model, which usually goes unnamed but is sometimes denoted e.g. by $\mathfrak{U}_1$. Similarly, implications are to be understood modulo the elementary diagram $\mathrm{ED}(\mathfrak{U}_*)$ of an ambient monster model $\mathfrak{U}_*$ containing everything we mention, including e.g. $\mathfrak{U}_1$. For example $\vDash \varphi(a)$ means $\mathfrak{U}_* \vDash \varphi(a)$, and if $c \in \mathfrak{U}_1$ and $p \in S(\mathfrak{U}c)$ then $(p \restriction \mathfrak{U}) \vdash p$ is a shorthand for $(p \restriction \mathfrak{U}) \cup \mathrm{ED}(\mathfrak{U}_*) \vdash p$. We sometimes take deductive closures implicitly, as in "$\{x = a\} \in S_x(\mathfrak{U})$".

A partial type Φ is *realised in* A iff there is $a \in A$ such that $\vDash \Phi(a)$. If we say that a type $p(x) \in S_x(B)$ is *realised*, it means that it is realised in B. Equivalently, for some $b \in B$, it contains $x = b$.

Formulas are usually denoted by lowercase Greek letters. When we say *L-formula*, we mean without parameters; we sometimes write "$L(\emptyset)$-formula" for emphasis. On the contrary, *definable* means "$\mathfrak{U}$-definable"; if we want to only allow parameters from A, we say "A-definable", "definable over A", etc. In formulas, (tuples of) variables will be separated by commas or semicolons. The distinction is purely cosmetic, to help readability, and usually it means we regard the variables on the left of the semicolon as "object variables" and the ones on the right as "parameter variables", e.g. we may write $\varphi(x, y; w) \in L$, $\varphi(x, y; d) \in p(x) \otimes q(y)$. The distinction between a partial type Φ and the set it defines is sometimes blurred. Its set of realisations in B is denoted by $\Phi(B)$.

The definitions of *stable*, *simple*, NIP, etc. are standard and we will not state them; see for instance [TZ12] and [Sim15]. Forking independence is denoted by $\mathop{\smile\!\!\!\!\!\!|}\ $. Definable and algebraic closure (in the model-theoretic sense) are denoted respectively by dcl and acl. The set of finite subsets of X is denoted by $\mathscr{P}_{\mathrm{fin}}(X)$. The set of natural numbers is denoted by ω, or by $\mathbb{N}$ when we think of it as an ordered monoid. It always contains 0.

The meaning of *I* and *we* is as in [Hod93, Note on notation].

2.1.2 The product of invariant types

We briefly recall some standard results on invariant types and fix some more notation. The material in this subsection is well-established, and references are e.g. [Sim15, Section 2.2] or [Poi00, Chapter 12].

Definition 2.1.1.

1. Let $A \subseteq B$. A type $p \in S_x(B)$ is *A-invariant* (or *invariant over A*, or *does not split over A*) iff for all $\varphi(x;y) \in L$ and $a \equiv_A b$ in $B^{|y|}$ we have $p(x) \vdash \varphi(x;a) \leftrightarrow \varphi(x;b)$. A global type $p \in S_x(\mathfrak{U})$ is *invariant* iff it is A-invariant for some $A \subset^+ \mathfrak{U}$. Such an A is called a *base* for p.

2. If $p \in S_x(\mathfrak{U})$ is A-invariant and $\varphi(x;y) \in L(A)$, write

$$(d_p\varphi(x;y))(y) := \{\operatorname{tp}_y(b/A) \mid \varphi(x;b) \in p, b \in \mathfrak{U}\}$$

   The map $\varphi \mapsto d_p\varphi$ is called the *defining scheme* of p over A.

3. We denote by $S^{\text{inv}}_x(\mathfrak{U}, A)$ the space of global A-invariant types in variables x, with A small, and by $S^{\text{inv}}_x(\mathfrak{U})$ the union of all $S^{\text{inv}}_x(\mathfrak{U}, A)$ as A ranges among small subsets of $\mathfrak{U}$. We denote by $S_{<\omega}(B)$, or just by $S(B)$, the union of all spaces of types over B in a finite tuple of sorts. Similarly for, say, $S^{\text{inv}}_{<\omega}(\mathfrak{U})$.

In other words, $p \in S(\mathfrak{U})$ is A-invariant if and only if, whenever $a \vDash p$ and $b, c \in \mathfrak{U}$, if $b \equiv_A c$ then $b \equiv_{Aa} c$.

If we say that a type p is invariant, and its domain is not specified and not clear from context, it is usually a safe bet to assume that $p \in S(\mathfrak{U})$. Similarly if we say that a tuple has invariant type without specifying over which set.

The following remarks are standard. By the first one, the name “invariant” is appropriate: a global type is A-invariant if and only if it is invariant under a certain action of $\operatorname{Aut}(\mathfrak{U}/A)$. The second says that, in every theory, invariant types are ubiquitous: every consistent formula is satisfied by an element with invariant global type.

Remark 2.1.2. By $|A|^+$-strong homogeneity of $\mathfrak{U}$, a global $p \in S_x(\mathfrak{U})$ is A-invariant if and only if it is a fixed point of the pointwise stabiliser $\operatorname{Aut}(\mathfrak{U}/A)$ of A under the usual action of $\operatorname{Aut}(\mathfrak{U})$ on $S_x(\mathfrak{U})$, defined by

$$f \cdot p := \{\varphi(x; f(d)) \mid \varphi(x;y) \in L(\emptyset), \varphi(x;d) \in p\}$$

If $a \in \mathfrak{U}_1 \succ \mathfrak{U}$ and $a \vDash p$, then for every $f_1 \in \mathrm{Aut}(\mathfrak{U}_1/A)$ extending f and fixing $\mathfrak{U}$ setwise we have $\mathrm{tp}(f_1(a)/\mathfrak{U}) = f \cdot p$.

Remark 2.1.3. For every $A \subset^+ \mathfrak{U}$ and tuple of variables x, the set $S^{\mathrm{inv}}_x(\mathfrak{U}, A)$ is closed in $S_x(\mathfrak{U})$. On the other hand, $S^{\mathrm{inv}}_x(\mathfrak{U})$ is dense in $S_x(\mathfrak{U})$.

Proof. A global type $p(x)$ is A-invariant if and only if it extends the partial type $\{\varphi(x;a) \leftrightarrow \varphi(x;b) \mid a \equiv_A b, \varphi(x;y) \in L\}$, and this proves the first part. For the second part, let $\varphi(x)$ be a consistent $L(\mathfrak{U})$-formula, and let $M \prec^+ \mathfrak{U}$ be any small model containing the parameters in φ. Since M is a model, the set $\varphi(M)$ is nonempty, hence is contained in an ultrafilter D on $M^{|x|}$. Define $p_D(x) := \{\psi(x) \in L(\mathfrak{U}) \mid \psi(\mathfrak{U}) \cap M \in D\}$. Since D is an ultrafilter we have $p_D(x) \in S_x(\mathfrak{U})$, and we now show that p_D is M-invariant. If this is not the case, then there are $\psi(x;y) \in L(\emptyset)$ and $a \equiv_M b$ in $\mathfrak{U}^{|y|}$ such that $p_D(x) \vdash \psi(x;a) \triangle \psi(x;b)$, where $\triangle$ denotes exclusive disjunction. By definition, the set of realisations of $\psi(x;a) \triangle \psi(x;b)$ in M is in D, hence is nonempty and contains a point $m \in M$. But $\vDash \psi(m;a) \triangle \psi(m;b)$ contradicts the fact that $a \equiv_M b$. □

Proposition 2.1.4. Let $A \subset^+ \mathfrak{U} \subseteq B$ and $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A)$.

1. There is a unique $p \mid B$ extending p to an A-invariant type over B, given by requiring, for $\varphi(x;y) \in L$ (equivalently, $\varphi(x;y) \in L(A)$) and $b \in B$,

$$\varphi(x;b) \in p \mid B \iff \mathrm{tp}(b/A) \in (d_p\varphi(x;y))(y)$$

2. For all $\varphi(x,y;w) \in L(\emptyset)$ (equivalently, $\varphi(x,y;w) \in L(A)$), $d \in \mathfrak{U}$ and $q \in S_y(\mathfrak{U})$, the following are equivalent.
   (a) For some (equivalently, all) $b \vDash q$ we have $\varphi(x,b;d) \in p \mid \mathfrak{U}b$.
   (b) For some (equivalently, all) $b \in \mathfrak{U}$ such that $b \vDash q \restriction Ad$ we have $\varphi(x,b;d) \in p$.
   (c) $q \in \pi^{-1}\big((d_p\varphi(x,y;d))(y)\big)$, where $\pi\colon S_y(\mathfrak{U}) \to S_y(Ad)$ is the restriction map $q \mapsto q \restriction Ad$.

3. If $A \subseteq A_1 \subset^+ \mathfrak{U}$, then $p \in S^{\mathrm{inv}}(\mathfrak{U}, A_1)$, and $p \mid B$ is the unique A_1-invariant extension of p.

Proof. Immediate from the definitions. □

If B contains both $\mathfrak{U}$ and C, we also use $p \mid C$ to denote $(p \mid B) \restriction C$.

The proposition above will be used tacitly throughout this thesis. To begin with, it ensures that the following operation is well-defined, i.e. does not depend on $b \vDash q$ and on whether we regard p as A-invariant or A_1-invariant.

Definition 2.1.5. Let $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A)$ and $q \in S_y(\mathfrak{U})$. Define the *product*, or *tensor product* $p(x) \otimes q(y) \in S_{xy}(\mathfrak{U})$ as follows. Fix $b \vDash q$. For each $\varphi(x, y) \in L(\mathfrak{U})$, define

$$\varphi(x, y) \in p(x) \otimes q(y) \iff \varphi(x, b) \in p \mid \mathfrak{U}b$$

We define inductively $p^{(1)} := p(x^0)$ and $p^{(n+1)} := p(x^n) \otimes p^{(n)}(x^{n-1}, \dots, x^0)$.

Some authors denote by $p(x) \otimes q(y)$ what we denote by $q(y) \otimes p(x)$. The reasons for this will be explained in Subsection 7.1.3.

Proposition 2.1.6.

1. If $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$, then $p \otimes q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$.

2. The operation $\otimes$ is associative on $S^{\mathrm{inv}}(\mathfrak{U})$.

Proof. ① Suppose that $d \equiv_A \tilde{d}$ are tuples from $\mathfrak{U}$, $\varphi(x, y; w) \in L(\emptyset)$ and $p(x) \otimes q(y) \vdash \varphi(x, y; d)$. Let $b \vDash q$. By definition, $p \mid \mathfrak{U}b \vdash \varphi(x, b; d)$. Since q is A-invariant, $bd \equiv_A b\tilde{d}$. Therefore, $p \mid \mathfrak{U}b \vdash \varphi(x, b; \tilde{d})$, hence $p \otimes q \vdash \varphi(x, y; \tilde{d})$.

② Let $p_x, q_y, r_z \in S^{\mathrm{inv}}(\mathfrak{U}, A)$, and fix $c \vDash r$, $b \vDash q \mid \mathfrak{U}c$, and $a \vDash p \mid \mathfrak{U}bc$. By definition, $(a, b, c) \vDash p \otimes (q \otimes r)$, and we need to check that $(a, b) \vDash (p \otimes q) \mid \mathfrak{U}c$. By Proposition 2.1.4, if $\varphi(x, y, c; d) \in (p(x) \otimes q(y)) \mid \mathfrak{U}c$ then there is $\tilde{c} \equiv_{Ad} c$ in $\mathfrak{U}$ such that $\varphi(x, y, \tilde{c}; d) \in p \otimes q$. Since $(a, b) \vDash p \otimes q$, we have $\vDash \varphi(a, b, \tilde{c}; d)$. Since, $\mathrm{tp}(b/\mathfrak{U}c)$ is A-invariant, $b\tilde{c}d \equiv_A bcd$, and because $\mathrm{tp}(a/\mathfrak{U}bc)$ is A-invariant we have $\vDash \varphi(a, b, c; d)$. □

Example 2.1.7.

1. If T is stable, then by definability of types $S^{\mathrm{inv}}(\mathfrak{U}) = S(\mathfrak{U})$. Moreover $p(x) \otimes q(y)$ can be described as $\mathrm{tp}(a, b/\mathfrak{U})$ where $a \vDash p$, $b \vDash q$, and $a \mathop{\smile\kern-0.9em\mid}_{\mathfrak{U}} b$. Note that, by forking symmetry, $p(x) \otimes q(y) = q(y) \otimes p(x)$. In fact it can be shown, by using the equivalence of instability with (7) in [She90, Theorem II.2.13] (see also [Sim15, Lemma 2.59]), together with [She90, Lemma VII.4.1], that $p \otimes q = q \otimes p$ for all $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ if and only if T is stable.

2. If T is the theory DLO of dense linear orders with no endpoints, and $p(x) := \mathrm{tp}(+\infty/\mathfrak{U})$ is the type of a point larger than $\mathfrak{U}$, then $p(x) \otimes p(y) = p(x) \cup p(y) \cup \{x > y\}$. Note that this is different from $p(y) \otimes p(x)$, as the latter proves $y > x$.

2.1.3 Domination

Definition 2.1.8. If $p(x), q(y) \in S(B)$ and $A \subseteq B$, we write

$$S_{pq}(A) := \{r \in S_{xy}(A) \mid r \supseteq (p \upharpoonright A) \cup (q \upharpoonright A)\}$$

In situations like those above we implicitly assume, for convenience and with no loss of generality, that x and y share no common variable.[1]

Definition 2.1.9. Let $p \in S_x(\mathfrak{U})$ and $q \in S_y(\mathfrak{U})$.

1. We say that p *dominates* q, and write $p \geq_{\mathrm{D}} q$, iff there are some small A and some $r \in S_{xy}(A)$ such that

   - $r \in S_{pq}(A)$, and
   - $p(x) \cup r(x, y) \vdash q(y)$.

   In this case, we say that r is a *witness* to, or *witnesses* $p \geq_{\mathrm{D}} q$.

2. We say that p and q are *domination-equivalent*, and write $p \sim_{\mathrm{D}} q$, iff $p \geq_{\mathrm{D}} q$ and $q \geq_{\mathrm{D}} p$.

3. We say that p and q are *equidominant*, and write $p \equiv_{\mathrm{D}} q$, iff there are some small A and some $r \in S_{xy}(A)$ such that

   - $r \in S_{pq}(A)$,
   - $p(x) \cup r(x, y) \vdash q(y)$, and
   - $q(y) \cup r(x, y) \vdash p(x)$.

   In this case, we say that r is a *witness* to, or *witnesses* $p \equiv_{\mathrm{D}} q$.

So $p \equiv_{\mathrm{D}} q$ if and only if both $p \geq_{\mathrm{D}} q$ and $q \geq_{\mathrm{D}} p$ hold, and both statements can be witnessed by the same r. To put it differently, a direct definition of $p \sim_{\mathrm{D}} q$ can be obtained by replacing, in the last clause of the definition of

[1] I will refrain from pointing out too many of these pedantries from now on.

$p \equiv_{\mathrm{D}} q$, the small type r with another small type $r' \in S_{pq}(A)$, possibly different from r. The last two relations are in general distinct; see Example 2.1.15.

Note that we are not requiring $p \cup r$ to be a complete global type in variables xy; in other words, domination is "small-type semi-isolation", as opposed to "small-type isolation". See Subsection 2.1.5 for the latter, and in particular Example 2.1.30 for a theory where it differs from domination.

Lemma 2.1.10. Suppose $r_0(x,y)$ witnesses $p_0(x) \geq_{\mathrm{D}} p_1(y)$ [resp. $p_0(x) \equiv_{\mathrm{D}} p_1(y)$] and $r_1(y,z)$ witnesses $p_1(y) \geq_{\mathrm{D}} p_2(z)$ [resp. $p_1(y) \equiv_{\mathrm{D}} p_2(z)$]. Then $\exists y\ (r_0(x,y) \cup r_1(y,z))$ is consistent, and any of its completions witnesses $p_0(x) \geq_{\mathrm{D}} p_2(z)$ [resp. $p_0(x) \equiv_{\mathrm{D}} p_2(z)$].

Proof. We prove it for $\equiv_{\mathrm{D}}$ first, as the proof for $\geq_{\mathrm{D}}$ is even easier. Suppose that $r_i \in S_{p_i p_{i+1}}(A_i)$, for $i < 2$. By hypothesis and compactness, for every formula $\varphi(z) \in p_2$ there are formulas $\psi(y,z) \in r_1$, $\theta(y) \in p_1$ and $\chi(x,y) \in r_0$ such that $p_0 \cup \{\chi(x,y)\} \vdash \theta(y)$ and $\{\theta(y) \wedge \psi(y,z)\} \vdash \varphi(z)$. If we let $\sigma_\varphi(x,z) \coloneqq \exists y\ (\chi(x,y) \wedge \psi(y,z))$, then $p_0(x) \cup \{\sigma_\varphi(x,z)\} \vdash \varphi(z)$. Moreover, we have $\sigma_\varphi(x,z) \in L(A_0A_1)$. Analogously, for each $\delta(x) \in p_0$ we can find $\rho_\delta(z,x) \in L(A_0A_1)$ such that $p_2(z) \cup \{\rho_\delta(z,x)\} \vdash \delta(x)$, obtained in the same way *mutatis mutandis*. It is now enough to show that the set of formulas

$$\Phi \coloneqq p_0(x) \cup r_0(x,y) \cup p_1(y) \cup r_1(y,z) \cup p_2(z)$$

is consistent, as this will in particular entail consistency of $\exists y\ (r_0(x,y) \cup r_1(y,z))$; since the latter contains

$$\{\sigma_\varphi \mid \varphi \in p_2\} \cup \{\rho_\delta \mid \delta \in p_0\}$$

it will have a completion to a type in $S_{p_0p_2}(A_0A_1)$ witnessing $p_0 \equiv_{\mathrm{D}} p_2$. To see that Φ is consistent, in a larger monster $\mathfrak{U}_1$ let $(a,b) \vDash p_0 \cup r_0$ and $(\tilde{b}, \tilde{c}) \vDash p_1 \cup r_1$. Since $\operatorname{tp}(b/\mathfrak{U}) = p_1 = \operatorname{tp}(\tilde{b}/\mathfrak{U})$, there is $f \in \operatorname{Aut}(\mathfrak{U}_1/\mathfrak{U})$ such that $f(\tilde{b}) = b$, and then $(a, b, f(\tilde{c})) \vDash \Phi$.

For $\geq_{\mathrm{D}}$, use the same proof but ignore the formulas ρ_δ. □

Corollary 2.1.11. The relations $\geq_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ are respectively a preorder and an equivalence relation on $S_{<\omega}(\mathfrak{U})$. Hence, $\sim_{\mathrm{D}}$ is an equivalence relation too.

Proof. Transitivity is Lemma 2.1.10, and the rest is obvious. □

As we are interested in the interaction of these notions with $\otimes$, we restrict our attention to quotients of $S^{\mathrm{inv}}(\mathfrak{U})$. Note that, by the following lemma, whether or not $p \in S^{\mathrm{inv}}(\mathfrak{U})$ only depends on its equivalence class.

Lemma 2.1.12. If $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A)$ and $r \in S_{xy}(B)$ are such that $p \cup r$ is consistent and $p \cup r \vdash q \in S_y(\mathfrak{U})$, then q is invariant over AB. In particular, if $p \geq_{\mathrm{D}} q$ and p is invariant, then so is q.

Proof. Suppose that $\varphi(y; w) \in L(\emptyset)$ and $d \in \mathfrak{U}$ are such that $q(y) \vdash \varphi(y; d)$, and let $\tilde{d} \in \mathfrak{U}$ be such that $\tilde{d} \equiv_{AB} d$. Let $\rho(x, y) \in r$ be such that $p(x) \vdash \forall y\ (\rho(x, y) \to \varphi(y; d))$. By AB-invariance of p and the fact that $\rho \in L(AB)$ we have $p(x) \vdash \forall y\ (\rho(x, y) \to \varphi(y; \tilde{d}))$. Therefore, $q \vdash \varphi(y; \tilde{d})$. □

Anyway, q will not be in general A-invariant: for instance, as we will see in the proof of point 3 of Proposition 2.1.27, for every p and every realised q we have $p \geq_{\mathrm{D}} q$, and it is enough to take q realised in $\mathfrak{U} \setminus \mathrm{dcl}(A)$ to get a counterexample. In fact, q does not even need to be domination-equivalent to an A-invariant type: see Counterexample 5.1.11.

Definition 2.1.13. Let $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ be the quotient of $S^{\mathrm{inv}}(\mathfrak{U})$ by $\sim_{\mathrm{D}}$, and $\overline{\mathrm{Inv}}(\mathfrak{U})$ the quotient of $S^{\mathrm{inv}}(\mathfrak{U})$ by $\equiv_{\mathrm{D}}$. The partial order induced by $\geq_{\mathrm{D}}$ on $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ will, with abuse of notation, still be denoted by $\geq_{\mathrm{D}}$. We call $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$ the *domination poset* and $\overline{\mathrm{Inv}}(\mathfrak{U})$ the *equidominance quotient*.

The reason we do not equip $\overline{\mathrm{Inv}}(\mathfrak{U})$ with the relation induced by $\geq_{\mathrm{D}}$ is that it is not antisymmetric in general, since $\equiv_{\mathrm{D}}$ may differ from $\sim_{\mathrm{D}}$.

If $p \cup r \vdash q$, by passing to a suitable extension of r there is no harm in enlarging its domain, provided it stays small. This sort of manipulation, which we used for example in the proof of Lemma 2.1.10, will from now on be done tacitly. Moreover, if p, q are invariant, we will usually assume that the domain of r is a base for both p and q. Sometimes, we say that r witnesses domination even if it is not complete. In that case, we mean that any of its completions does. Similarly, we sometimes just write e.g. "put in r the formula $\varphi(x, y)$".

Remark 2.1.14. In [HHM08], the name *domination-equivalence* is used to refer to $\equiv_{\mathrm{D}}$ (no mention is made of $\geq_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$). The reason for this change in terminology is to ensure consistency with the notions with the same names classically defined for stable theories, which coincide with the ones just defined (see Subsection 3.1.1). As $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ carries a poset structure, and is better behaved than $\overline{\mathrm{Inv}}(\mathfrak{U})$ (see e.g. Proposition 5.1.6), we mostly focus on the former.

Example 2.1.15.

1. In every strongly minimal theory any two global types are domination-equivalent, equivalently equidominant, precisely when they have the same dimension over $\mathfrak{U}$. See Subsection 3.2.1.

2. In DLO, if $p(x) \coloneqq \operatorname{tp}(+\infty/\mathfrak{U})$, then $p(x) \equiv_{\mathrm{D}} p(y) \otimes p(z)$, as can be easily seen by using some r containing the formula $x = z$.

3. In an arbitrary theory, let f be a definable function with domain $\varphi(x)$ and codomain $\psi(y)$, or rather a tuple of definable functions if $|y| > 1$. If $p(x) \in S(\mathfrak{U})$ is such that $p(x) \vdash \varphi(x)$, the *pushforward* f_*p is the global type $\{\theta(y) \in \mathfrak{U} \mid p \vdash \theta(f(x))\}$. For all such p and f, we have $p \geq_{\mathrm{D}} f_*p$, witnessed by any small type containing $y = f(x)$. If f is a bijection, then $p \equiv_{\mathrm{D}} f_*p$.

4. Equidominance differs from domination-equivalence in the theory DLOP of a DLO with a dense-codense predicate P (see Subsection 5.1.1). For a stable example, see Example 3.2.34. The reason the two equivalence relations may differ is that, even if there are r_0 and r_1 such that $p \cup r_0 \vdash q$ and $q \cup r_1 \vdash p$, we may still have that the union $r_0 \cup r_1$ is inconsistent.

2.1.4 Interaction

We start our investigation of the compatibility of $\otimes$ with $\geq_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ with two easy lemmas. While the first one will not be needed until later, the second one will be used repeatedly.

Lemma 2.1.16. If $A \subseteq B \subseteq C$, $p_x, q_y \in S(C)$, and $r \in S_{pq}(A)$ are such that $p \cup r \vdash q$, then $(p \upharpoonright B) \cup r \vdash (q \upharpoonright B)$.

Proof. Let $\psi(y) \in q \upharpoonright B$. By hypothesis and compactness there is $\rho(x, y) \in r$ such that $p \vdash \forall y\ (\rho(x, y) \to \psi(y))$. As $A \subseteq B$, this formula is in $p \upharpoonright B$. □

It might be tempting to drop the hypothesis $A \subseteq B$ and try to prove that $(p \upharpoonright B) \cup (r \upharpoonright A \cap B) \vdash (q \upharpoonright B)$. This is false: in the theory of infinite sets, suppose $A = \{a_0, a_1\}$, $B = \{a_0, b\}$ and $C = AB$, with a_0, a_1, b pairwise distinct. Let $p_x, q_y \in S(C)$ be respectively $\{x = a_0\}$ and $\{y = a_1\}$, and let $r \in S_{xy}(A)$ be $\{x = a_0 \wedge y = a_1\}$. Clearly, $p \cup r \vdash q$, but $(p \upharpoonright B) \cup (r \upharpoonright A \cap B) \nvdash y \neq b$.

Lemma 2.1.17. If $p_x, q_y \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ and $r \in S_{pq}(A)$ are such that $p \cup r \vdash q$, then for all $B \supseteq \mathfrak{U}$ we have $(p \mid B) \cup r \vdash (q \mid B)$.

Proof. Let $\varphi(y; w)$ be an $L(\emptyset)$-formula and $b \in B$ be such that $\varphi(y; b) \in (q \mid B)$. Pick any $\tilde{b} \in \mathfrak{U}$ such that $\tilde{b} \equiv_A b$. By definition of $q \mid B$ we have $\varphi(y; \tilde{b}) \in q$, so by hypothesis and compactness there is an $L(A)$-formula $\rho(x, y) \in r(x, y)$ such that $p \vdash \forall y \left(\rho(x, y) \to \varphi(y; \tilde{b})\right)$. But then, by definition of $p \mid B$ and the fact that $\rho \in L(A)$ we have $(p \mid B) \vdash \forall y \left(\rho(x, y) \to \varphi(y; b)\right)$, and since $\rho \in r$ we get $(p \mid B) \cup r \vdash \varphi(y; b)$. □

Notation 2.1.18. We adopt from now on the following conventions. The letter A continues to denote a small set. The symbols p, q, possibly with subscripts, denote global invariant types, unless we explicitly write e.g. $p \in S(\emptyset)$ or $q \in S(\mathfrak{U})$, and r stands for a witness (or candidate witness) of domination or equidominance over a base of invariance for both types.

The first use we make of Lemma 2.1.17 is to prove the following statement, which generalises [Las75, Corollaire 11].

Lemma 2.1.19. Suppose $p_0(x) \cup r(x, y) \vdash p_1(y)$, and let $s := r(x, y) \cup \{z = w\}$. Then $(p_0(x) \otimes q(z)) \cup s \vdash p_1(y) \otimes q(w)$. In particular if $p_0 \geq_{\mathrm{D}} p_1$ then $p_0 \otimes q \geq_{\mathrm{D}} p_1 \otimes q$, and the same holds replacing $\geq_{\mathrm{D}}$ with $\equiv_{\mathrm{D}}$.

Proof. Choose any $b \vDash q(w)$. For every $\varphi(y, w; t) \in L(\emptyset)$ and $d \in \mathfrak{U}$ such that $\varphi(y, w; d) \in p_1(y) \otimes q(w)$ we have, by definition of $\otimes$, that $p_1(y) \mid \mathfrak{U}b \vDash \varphi(y, b; d)$. By Lemma 2.1.17 there is some $L(A)$-formula $\rho(x, y) \in r(x, y)$ such that $p_0(x) \mid \mathfrak{U}b \vDash \forall y \left(\rho(x, y) \to \varphi(y, b; d)\right)$, hence $p_0(x) \otimes q(z) \vDash \forall y \left(\rho(x, y) \to \varphi(y, z; d)\right)$. In particular, since $\rho \in r$, we have $(p_0(x) \otimes q(z)) \cup r \vdash \varphi(y, z; d)$. Therefore any completion of $s \cup \left((p_0(x) \otimes q(z)) \restriction A\right) \cup \left((p_1(y) \otimes q(w)) \restriction A\right)$ witnesses that $p_0(x) \otimes q(z) \geq_{\mathrm{D}} p_1(y) \otimes q(w)$.

In the special case where the same r also witnesses $p_1 \geq_{\mathrm{D}} p_0$, for the same s we have that $s \cup \left((p_0(x) \otimes q(z)) \restriction A\right) \cup \left((p_1(y) \otimes q(w)) \restriction A\right)$ witnesses $p_1 \otimes q \geq_{\mathrm{D}} p_0 \otimes q$, and we get $p_1 \otimes q \equiv_{\mathrm{D}} p_0 \otimes q$. □

One may expect a similar result to hold when multiplying a relation of the form $q_0 \geq_{\mathrm{D}} q_1$ by p on the left, and indeed it was claimed (without proof) in [HHM08] that $\equiv_{\mathrm{D}}$ is a congruence with respect to $\otimes$. Unfortunately, this turns out not to be true in general: we will see in Proposition 5.3.15 that it is possible to have $q_0 \equiv_{\mathrm{D}} q_1$ and $p \otimes q_0 \not\geq_{\mathrm{D}} p \otimes q_1$ simultaneously. For the time being, we assume this does not happen as a hypothesis and explore some of its immediate consequences.

Definition 2.1.20. In a fixed theory T, we say that $\otimes$ *respects* (or *is compatible with*, or *preserves*) $\geq_{\mathrm{D}}$ [resp. $\equiv_{\mathrm{D}}$] iff for all global invariant types p, q_0, q_1, if $q_0 \geq_{\mathrm{D}} q_1$ [resp. $q_0 \equiv_{\mathrm{D}} q_1$] then $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ [resp. $p \otimes q_0 \equiv_{\mathrm{D}} p \otimes q_1$].

Corollary 2.1.21.

1. The product $\otimes$ respects the preorder $\geq_{\mathrm{D}}$ if and only if $(S^{\mathrm{inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$ is a preordered semigroup. If this is the case, then $\sim_{\mathrm{D}}$ is a congruence with respect to $\otimes$, and the latter induces on $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$ the structure of a partially ordered semigroup.

2. The product $\otimes$ respects the equivalence relation $\equiv_{\mathrm{D}}$ if and only if $\equiv_{\mathrm{D}}$ is a congruence with respect to $\otimes$.

Proof. Everything follows at once from Lemma 2.1.19. □

The following observation is easy, but useful.

Corollary 2.1.22. Suppose that $q_0 \geq_{\mathrm{D}} q_1$ and, for $i < 2$, we have $p \otimes q_i \sim_{\mathrm{D}} q_i \otimes p$. Then $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$. The same holds replacing $\geq_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$ with $\equiv_{\mathrm{D}}$. In particular, if for all $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ we have $p \otimes q \sim_{\mathrm{D}} q \otimes p$ [resp. $p \otimes q \equiv_{\mathrm{D}} q \otimes p$], then $\otimes$ respects $\geq_{\mathrm{D}}$ [resp. $\equiv_{\mathrm{D}}$].

Proof. By assumption and Lemma 2.1.19 we have

$$p \otimes q_0 \sim_{\mathrm{D}} q_0 \otimes p \geq_{\mathrm{D}} q_1 \otimes p \sim_{\mathrm{D}} p \otimes q_1$$

For equidominance, repeat the proof replacing $\geq_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$ with $\equiv_{\mathrm{D}}$. □

Definition 2.1.23. If $\otimes$ respects $\geq_{\mathrm{D}}$, we shall still denote by $\otimes$ the operation it induces on $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. We call the structure $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$ the *domination monoid* and, if no confusion may arise, we usually denote it simply by $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ and say that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ *is well-defined* to mean that $\otimes$ respects $\geq_{\mathrm{D}}$. Similarly, we call $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ the *equidominance monoid*, we denote it simply by $\overline{\mathrm{Inv}}(\mathfrak{U})$, and we say that $\overline{\mathrm{Inv}}(\mathfrak{U})$ *is well-defined* to mean that $\otimes$ respects $\equiv_{\mathrm{D}}$.

The abuse of notation above should cause no confusion since, as a poset, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$ is *always* well-defined, and similarly for $\overline{\mathrm{Inv}}(\mathfrak{U})$.

Lemma 2.1.24. Suppose that $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ and p is realised. The following are equivalent.

1. The relation $p \equiv_{\mathrm{D}} q$ holds.

2. The relation $p \sim_{\mathrm{D}} q$ holds.

3. The relation $p \geq_{\mathrm{D}} q$ holds.

4. The type q is realised.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are true by definition, even when p is not realised. Let $p = \mathrm{tp}(a/\mathfrak{U})$, where $a \in \mathfrak{U}$.

For $3 \Rightarrow 4$ suppose that $r \in S_{pq}(A)$ is such that $p \cup r \vdash q$. Since $\{x = a\} \vdash p$, we have $\{x = a\} \cup r \vdash q$. But since $\{x = a\} \cup r$ is a small type, it is realised in $\mathfrak{U}$ by some (a, b), and clearly $b \vDash q$.

For $4 \Rightarrow 1$ suppose that for some $b \in \mathfrak{U}$ we have $q = \mathrm{tp}(b/\mathfrak{U})$ and let A be any small set containing a and b. Clearly, $(x = a) \wedge (y = b)$ implies a complete type $r \in S_{xy}(A)$ containing $(p \restriction A) \cup (q \restriction A)$, and since $r(x, y) \vdash p(x) \cup q(y)$ we have that r witnesses $p \equiv_{\mathrm{D}} q$. □

Lemma 2.1.25. Suppose $p_x, q_y \in S^{\mathrm{inv}}(\mathfrak{U})$ and p is realised by $a \in \mathfrak{U}$. Then $\{x = a\} \cup q(y) \vdash p(x) \cup q(y) \vdash p(x) \otimes q(y) = q(y) \otimes p(x)$. Moreover $p \otimes q \equiv_{\mathrm{D}} q$.

Proof. The first part is clear. It follows that, if q is A-invariant and $a \in A$, in order to show that $p(x) \otimes q(y) \equiv_{\mathrm{D}} q(z)$ it suffices to take as r the type $\{x = a\} \cup \{y = z\} \cup (q(y) \restriction A) \cup (q(z) \restriction A)$. □

Notation 2.1.26. When quotienting by $\sim_{\mathrm{D}}$ or $\equiv_{\mathrm{D}}$ we denote by $[\![p]\!]$ the class of p, with the understanding that the equivalence relation we are referring to is clear from context. We write $[\![0]\!]$ for the class of realised types, since it is the class of the unique global 0-type, namely the elementary diagram $\mathrm{ED}(\mathfrak{U})$.

Proposition 2.1.27. Suppose that $\otimes$ respects $\geq_{\mathrm{D}}$ [resp. $\equiv_{\mathrm{D}}$].

1. The semigroup $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ [resp. $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$] has neutral element $[\![0]\!]$.

2. The monoid $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, [\![0]\!])$ [resp. $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes, [\![0]\!])$] is *conical*, i.e. it satisfies the formula $\forall x, y\ (x \otimes y = [\![0]\!]) \to (x = y = [\![0]\!])$. In particular, no element different from $[\![0]\!]$ is invertible.

3. The neutral element $[\![0]\!]$ is also the minimum of $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$.

Proof.

1. Let $p = \operatorname{tp}(a/\mathfrak{U})$ and $q \in S^{\mathrm{inv}}(\mathfrak{U})$, where $a \in \mathfrak{U}$. Apply Lemma 2.1.25 and note that $p \otimes q \equiv_{\mathrm{D}} q$ implies $p \otimes q \sim_{\mathrm{D}} q$.

2. By the previous point, if $[\![p]\!] \otimes [\![q]\!] = [\![0]\!]$ then $p \otimes q$ is realised. In particular p and q are both realised.

3. We have to show that for every $p(x)$ and every realised $q(y)$ we have $p \geq_{\mathrm{D}} q$. If q is realised by $b \in \mathfrak{U}$, it is sufficient to put in r the formula $y = b$. □

2.1.5 Stronger notions

It is possible to define strengthenings of domination and equidominance, for instance by restricting the small type r to be a single formula ρ. These will only be used sparingly, and can be ignored on a first reading.

Definition 2.1.28. If $p(x), q(y) \in S(\mathfrak{U})$, we say that

1. p is greater than q in the *Rudin–Keisler preorder*, or that q *is semi-isolated over* p, and write $p \geq_{\mathrm{RK}} q$, iff there is $\rho(x, y) \in L(\mathfrak{U})$ consistent with $p(x) \cup q(y)$ and such that $p(x) \cup \{\rho(x, y)\} \vdash q(y)$;

2. p, q are *Rudin–Keisler equivalent*, and write $p \sim_{\mathrm{RK}} q$, iff $p \geq_{\mathrm{RK}} q$ and $q \geq_{\mathrm{RK}} p$;

3. p, q are *strongly Rudin–Keisler equivalent*, and write $p \equiv_{\mathrm{RK}} q$, iff there is $\rho(x, y) \in L(\mathfrak{U})$ consistent with $p(x) \cup q(y)$ such that $p(x) \cup \{\rho(x, y)\} \vdash q(y)$ and $q(y) \cup \{\rho(x, y)\} \vdash p(x)$.

In all of the above, "Rudin–Keisler" can be abbreviated to "RK".

These relations were studied for instance in [Tan15]. As they will only be discussed briefly in this thesis (briefly enough that we do not give a name to the analogues of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ and $\overline{\operatorname{Inv}}(\mathfrak{U})$), for the sake of readability I have avoided duplicating every lemma so far. Hence, I hope that the reader can be convinced of, or has the patience to check, the validity of the remark below. Similarly, some (but not all!) of the coming definitions and results also work for $\geq_{\mathrm{RK}}$.

Remark 2.1.29. Every result in Subsection 2.1.3 and Subsection 2.1.4 still holds, with essentially the same proof, when we replace $\geq_{\mathrm{D}}$ with $\geq_{\mathrm{RK}}$, $\sim_{\mathrm{D}}$ with $\sim_{\mathrm{RK}}$, $\equiv_{\mathrm{D}}$ with $\equiv_{\mathrm{RK}}$, and the small type r with a single formula φ consistent with $p \cup q$.

An example of a theory where $\geq_{\mathrm{RK}}$ does not equal $\geq_{\mathrm{D}}$ is for instance DLO; see Remark 4.2.6.[2] There are also stable examples where they differ, the standard one being the theory of a generic equivalence relation E together with a bijection s with no cycles and such that $\forall x\ E(x, s(x))$. If we take as $p(x)$ the type of an element in a new equivalence class, and as $q(y)$ the type of two elements y_0, y_1 in a new equivalence class with $E(y_0, y_1)$ and $y_0 \neq s^n(y_1)$ for every $n \in \mathbb{Z}$, then $p \equiv_{\mathrm{D}} q$ but $p \not\geq_{\mathrm{RK}} q$.

So semi-isolation is indeed stronger than domination. Both relations can be strengthened further by requiring that $p(x) \cup \{\rho(x, y)\}$, or $p(x) \cup r(x, y)$, not only proves $q(y)$, but is a complete type in the variables xy. Equivalently, the strengthening of $\geq_{\mathrm{RK}}$ corresponds to the existence of $a \vDash p$ and $b \vDash q$ such that $\mathrm{tp}_y(b/\mathfrak{U}a)$ is isolated by $\rho(a, y)$. We call the analogous strengthening of $\geq_{\mathrm{D}}$ *small-type isolation*. This corresponds to $\mathrm{tp}_y(b/\mathfrak{U}a)$ being implied by a small $r(a, y)$; in the terminology of [She90, Section IV.2], we would say that it is $\mathrm{F}^{\mathrm{s}}_{\kappa(\mathfrak{U})}$-isolated (or $\mathrm{F}^{\mathrm{t}}_{\kappa(\mathfrak{U})}$-isolated; since $\kappa(\mathfrak{U}) > |T|$ the two notions coincide). We now give examples showing these strengthenings are not vacuous, and the concepts of (small-type) isolation and (small-type) semi-isolation may differ.

If $\rho(x, y)$ witnesses $\mathrm{tp}_x(a/\mathfrak{U}) \geq_{\mathrm{RK}} \mathrm{tp}_y(b/\mathfrak{U})$, this does not mean $\mathrm{tp}_y(b/\mathfrak{U}a)$ is isolated by $\rho(a, y)$. As an example, in the theory DOAG of divisible ordered abelian groups (see Subsection 4.2.2), let a, b be points greater than $\mathfrak{U}$ such that for all $n \in \omega$ we have $n \cdot a < b$, and let $\rho(x, y) \coloneqq x < y$.

It is also possible to construct an example where $p \equiv_{\mathrm{RK}} q$ but, for all $a \vDash p$ and $b \vDash q$, there is no small type $r(a, y)$ isolating $\mathrm{tp}(b/\mathfrak{U}a)$ (above, we could have taken $a = b$ instead, since $a \equiv_{\mathfrak{U}} b$). This can be seen in the theory below, inspired by an example for which I would like to thank Predrag Tanović. To the best of my knowledge, no such example was previously known.

Example 2.1.30. Work in a 2-sorted language, with sorts O ("objects") and D ("digraphs"). Let $L \coloneqq \{E^{(O^2)}, P^{(O)}, R^{(O^2 \times D)}\}$, a relational language with arities indicated as superscripts. Consider the universal axioms below.

[2] Point 2 of Example 2.1.15 is still subject to Remark 2.1.29, though. The particular type p defined there has the property that $p \equiv_{\mathrm{RK}} p^{(2)}$.

1. E is an equivalence relation.

2. $R(x, y, w) \to E(x, y)$.

3. $R(x, y, w) \to \neg R(y, x, w)$.

The collection of finite structures satisfying these axioms is a Fraïssé class, and we take as T the theory of its Fraïssé limit (see for example [Hod93, Theorem 7.1.2]). In a model of T, the sort O carries an equivalence relation with infinitely many classes. On each class a/E the predicate P is infinite and coinfinite, and each point of D induces a random digraph on a/E. Different random digraphs, on the same a/E or on different ones, interact generically with P and between them, but none of them has an edge across different classes. Define $\pi(x) \coloneqq \{\neg E(x, d) \mid d \in \mathfrak{U}\}$, and let $p(x) \coloneqq \pi(x) \cup \{P(x)\}$ and $q(y) \coloneqq \pi(y) \cup \{\neg P(y)\}$. By quantifier elimination and the lack of edges across different equivalence classes, p and q are complete global types, in fact $\emptyset$-invariant ones. The formula $\rho(x, y) \coloneqq E(x, y) \wedge P(x) \wedge \neg P(y)$ shows that $p \equiv_{\mathrm{RK}} q$. Note that the predicate P forbids having $x = y$. By genericity, there is no small A such that for some $r \in S_{pq}(A)$ the partial type $p \cup r$ decides, for all $d \in \mathfrak{U}$, whether $R(x, y, d)$ holds, and similarly for $q \cup r$. Hence, for all $a \vDash p$ and $b \vDash q$, neither $\operatorname{tp}(a/\mathfrak{U}b)$ nor $\operatorname{tp}(b/\mathfrak{U}a)$ is $\mathrm{F}^{\mathrm{s}}_{\kappa(\mathfrak{U})}$-isolated.

If instead of digraphs we use equivalence relations (refining E), the example above still works, and is related to the theory in Subsection 5.3.3.

2.2 Well-definedness

We now investigate sufficient conditions for $\otimes$ to respect $\geq_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$. Some of these conditions are admittedly rather artificial, but we show them to be consequences of other properties easier to test directly, such as stability.

In what follows, types will be usually assumed to have no realised coordinates and no duplicate coordinates, i.e. we will assume, for all $i \neq j < |x|$ and $a \in \mathfrak{U}$, to have $p(x) \vdash (x_i \neq a) \wedge (x_i \neq x_j)$. Up to domination-equivalence, and even equidominance, no generality is lost, as justified by Lemma 2.1.25 and by the fact that, for example, if $p(x_0)$ is any 1-type and $q(y_0, y_1) \vdash p(y_0) \cup \{y_0 = y_1\}$, setting $x_0 = y_0 = y_1$ shows $p \equiv_{\mathrm{D}} q$.

2.2.1 Stationary domination

The most general sufficient condition for well-definedness of $\widetilde{\text{Inv}}(\mathfrak{U})$ that we give is quite technical, and some words to motivate it are in order. Let $q_0(y) \geq_{\mathrm{D}} q_1(z)$ be witnessed by $r(y,z)$. In light of Lemma 2.1.19, a natural candidate for a witness to $p(x) \otimes q_0(y) \geq_{\mathrm{D}} p(w) \otimes q_1(z)$ would be $r(y,z) \cup \{x = w\}$. In a stable T, this will do the trick by commutativity of $\otimes$. Unfortunately, even if $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ does hold, $r \cup \{x = w\}$ may not witness it.

Example 2.2.1. In DLO, if $p(x), q_0(y), q_1(z)$ are all $\operatorname{tp}(+\infty/\mathfrak{U})$, and $r(y,z) \vdash z > y$, then we may have $a, b, c \in \mathfrak{U}_1 \mathrel{^{+}\!\!\succ} \mathfrak{U}$ with $a \vDash p$, $b \vDash q_0$, and $c \vDash q_1$ such that $b < a < c$. In this case, $a \vDash p \mid \mathfrak{U}b$, but $a \nvDash p \mid \mathfrak{U}c$. Therefore, even if r witnesses $q_0 \geq_{\mathrm{D}} q_1$, we have that $r \cup \{x = w\}$ does not witness $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$.

So trying the "obvious thing" does not work. In fact, we will see in Proposition 5.3.15 that sometimes nothing at all works, and $\otimes$ may not respect $\geq_{\mathrm{D}}$. The spirit of what we are about to define is "if you try the second most obvious thing, it works".

Example 2.2.1 does not violate well-definedness of $\widetilde{\text{Inv}}(\mathfrak{U})$ in the case of DLO, since we may use some r' containing $(x > z > y) \wedge (x = w)$ in place of $r \cup \{x = w\}$, and this indeed shows $p(x) \otimes q_0(y) \geq_{\mathrm{D}} p(w) \otimes q_1(z)$. What does r' have that $r \cup \{x = w\}$ does not? If we take a closer look at Example 2.2.1, we discover that

$$(p \mid \mathfrak{U}b) \nvdash (p \mid (\mathfrak{U} \cup r(b, \mathfrak{U}_1)))$$

Therefore, a better candidate for a witness to $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ needs to place x in a position which is generic enough with respect to z. While we are at it, we may allow further changes to r, in order to encompass a larger class of theories where $\otimes$ preserves $\geq_{\mathrm{D}}$. For instance, in the example above we might have taken $r'' \vdash (y = z) \wedge (x = w)$. This is the idea behind the next definition. Or rather, "definitions", since we give four of them at once. The "strict" versions correspond to r' and appear for the first time on these pages, while the non-strict ones allow us to pass to r'' and were introduced in [Men20].

Definition 2.2.2. Let $p(x), q_0(y), q_1(z) \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ and $r \in S_{q_0 q_1}(A)$. Let $\mathfrak{U}_1 \mathrel{^{+}\!\!\succ} \mathfrak{U}$ and $b, c \in \mathfrak{U}_1$ be such that $(b,c) \vDash q_0(y) \cup r(y,z) \cup q_1(z)$, and let $a \vDash p(x) \mid \mathfrak{U}_1$. Define

$$(r[p])(x,y,w,z) := (\operatorname{tp}_{xyz}(abc/A) \cup \{x = w\}) \in S_{(p \otimes q_0)(p \otimes q_1)}(A)$$

Since p is A-invariant, $r[p]$ only depends on r and p, and not on a, b, c.

Definition 2.2.3. We say that T has *strict stationary domination* iff, whenever $p, q_0, q_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ and $q_0 \geq_{\mathrm{D}} q_1$, for all $A \subset^+ \mathfrak{U}$ and $r \in S_{q_0 q_1}(A)$ such that

$$p(x), q_0(y), q_1(z) \text{ are } A\text{-invariant and } q_0 \cup r \vdash q_1 \tag{2.1}$$

the following holds.

$$\text{For all } \mathfrak{U}_1 \mathrel{^+\!\succ} \mathfrak{U}, \text{ all } b, c \in \mathfrak{U}_1 \text{ such that } (b, c) \models q_0 \cup r \text{ and all } a \models p(x) \mid \mathfrak{U}_1, \\ \text{we have } (p(x) \otimes q_0(y)) \cup r[p] \vdash p(w) \otimes q_1(z). \tag{2.2}$$

We say that T has *stationary domination* iff, whenever $p, q_0, q_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ and $q_0 \geq_{\mathrm{D}} q_1$, there are $A \subset^+ \mathfrak{U}$ and $r \in S_{q_0 q_1}(A)$ such that (2.1) and (2.2) hold. We define *(strict) stationary equidominance* similarly, except that we assume that $q_0 \equiv_{\mathrm{D}} q_1$, and r and $r[p]$ are required to be witnesses in both directions.

Proposition 2.2.4. Strict stationary domination implies stationary domination, and similarly for equidominance. If T has stationary domination, then $\otimes$ respects $\geq_{\mathrm{D}}$. If T has stationary equidominance, then $\otimes$ respects $\equiv_{\mathrm{D}}$.

Proof. By carefully reading the definitions. □

I hope that the discussion leading up to Definition 2.2.3 has convinced the reader that the notions defined in the latter are natural, if intricate to state. In the rest of this section, we focus on isolating properties that imply stationary domination or its variants, but are easier to test directly, or have been already tested in the literature in several concrete cases of interest.

Proposition 2.2.5. Let T be stable. Then T has strict stationary domination and strict stationary equidominance. Moreover the monoids $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ and $\overline{\mathrm{Inv}}(\mathfrak{U})$ are commutative.

Proof. Let r witness $q_0(y) \geq_{\mathrm{D}} q_1(z)$. By Lemma 2.1.19 $(q_0(y) \otimes p(x)) \cup r(y, z) \cup \{x = w\} \vdash q_1(z) \otimes p(w)$. As $r \cup \{x = w\} \subseteq r[p]$ and if T is stable then $\otimes$ is commutative (see Example 2.1.7), we have strict stationary domination and commutativity of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. For strict stationary equidominance and commutativity of $\overline{\mathrm{Inv}}(\mathfrak{U})$, argue analogously starting with any r witnessing $q_0(y) \equiv_{\mathrm{D}} q_1(z)$. □

2.2.2 Algebraic domination

The notions in this subsection were defined in [Men20], and are strong forms of stationary domination.

Definition 2.2.6. We say that q_1 *is algebraic over* q_0 [resp. q_0 *and* q_1 *are mutually algebraic*] iff there are $b \vDash q_0$ and $c \vDash q_1$ such that $c \in \operatorname{acl}(\mathfrak{U}b)$ [resp. $c \in \operatorname{acl}(\mathfrak{U}b)$ and $b \in \operatorname{acl}(\mathfrak{U}c)$]. The theory T has *algebraic domination* [resp. algebraic equidominance] iff $q_0 \geq_{\mathrm{D}} q_1$ if and only if q_1 is algebraic over q_0 [resp. q_0 and q_1 are mutually algebraic].

Note that we are only asking that $b \vDash q_0$ and $c \vDash q_1$ *can* be arranged algebraically. We could also ask that *every* arrangement of b and c witnessing domination is algebraic. The reader might take the previous as a definition of *strict algebraic domination*, but from now on we will try to restrict the number of variants of concepts we introduce, even if some of the results would carry through or have stronger statements.

Proposition 2.2.7. Suppose that $q_1(z)$ is algebraic over $q_0(y)$, as witnessed by $b, c \in \mathfrak{U}_1 \mathrel{^+{\succ}} \mathfrak{U}$ with $b \vDash q_0$ and $c \vDash q_1$ algebraic over $\mathfrak{U}b$. Suppose that $\rho(y,z) \in r \in S_{pq}(A)$ is such that $\rho(b,z)$ isolates $\operatorname{tp}(c/\mathfrak{U}b)$. Then for all $p \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ we have $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$, and this is witnessed by the type $r[p]$ in Definition 2.2.2. In particular, algebraic domination [resp. equidominance] implies stationary domination [resp. equidominance].

Proof. We only deal with stationary domination; the proof for stationary equidominance is similar. Let $s \coloneqq \{x = w\} \cup \{\rho(y,z)\}$. Since $s \subseteq r[p]$, it is enough to show that $p(x) \otimes q_0(y) \cup s \vdash p(w) \otimes q_1(z)$. In some $\mathfrak{U}_2 \mathrel{^+{\succ}} \mathfrak{U}_1$, let $a \vDash p \mid \mathfrak{U}_1$ and let $\varphi(w,z) \in p(w) \otimes q_1(z)$. This means that $\varphi(w,z) \in L(\mathfrak{U})$ and $\varphi(w,c) \in \operatorname{tp}(a/\mathfrak{U}c) = p \mid \mathfrak{U}c$.

By hypothesis, there are only finitely many $\tilde{c} \equiv_{\mathfrak{U}b} c$, which must be contained in every model containing $\mathfrak{U}b$ and, by invariance of $p \mid \mathfrak{U}_1$, for all such $\tilde{c} \in \mathfrak{U}_1$ we have $p \mid \mathfrak{U}_1 \vdash \varphi(x, \tilde{c})$. It follows that $\operatorname{tp}_w(a/\mathfrak{U}_2) \vdash \forall z\ \big(\rho(b,z) \to \varphi(w,z)\big)$. As the latter is an $L(\mathfrak{U}b)$-formula, it is contained in $p \mid \mathfrak{U}b$, and it follows that $(p(x) \otimes q_0(y)) \cup s \vdash p(w) \otimes q_1(z)$. □

The proof above actually gives $p \otimes q_0 \geq_{\mathrm{RK}} p \otimes q_1$, and in fact even isolation.

Corollary 2.2.8. Suppose that q_1 is the pushforward $f_* q_0$ of q_0, for some definable function f. Then, for all $p \in S^{\mathrm{inv}}(\mathfrak{U})$, we have $p \otimes q_0 \geq_{\mathrm{RK}} p \otimes q_1$.

Assume moreover that f is a bijection, so $q_1 \equiv_{\mathrm{RK}} q_0$. Then, for all $p \in S^{\mathrm{inv}}(\mathfrak{U})$, we have $p \otimes q_0 \equiv_{\mathrm{RK}} p \otimes q_1$.

Algebraic domination is a pretty strong form of stationary domination, and holds for example in all strongly minimal theories by Proposition 3.2.1. We define a further strengthening, which has a tendency to appear in Fraïssé limits of Fraïssé classes of finite relational structures with free amalgamation.

Definition 2.2.9. Let L_0 be the "empty" language, containing only equality. We say that T has *degenerate domination* iff whenever $p(x) \geq_{\mathrm{D}} q(y)$ there is a small set $r_0(x, y)$ of $L_0(\mathfrak{U})$-formulas consistent with $p(x)$ such that $p(x) \cup r_0(x, y) \vdash q(y)$.

Note that we are not requiring r_0 to contain a restriction of $p \cup q$; in fact, every symbol outside of $L_0(\mathfrak{U})$ is forbidden.

Remark 2.2.10. It is easy to see that there is $r_0(x, y)$ as above if and only if $q(y)$ is included in $p(x)$ up to removing realised and duplicate y_i and renaming each of the remaining ones to a suitable x_j.

Proposition 2.2.11. Suppose T has degenerate domination. Then T has algebraic domination, and in particular $\otimes$ respects $\geq_{\mathrm{D}}$. Moreover for global types p and q the following are equivalent.

1. There is a small set r_0 of $L_0(\mathfrak{U})$-formulas consistent with $p \cup q$ such that $p \cup r_0 \vdash q$ and $q \cup r_0 \vdash p$.

2. $p \equiv_{\mathrm{D}} q$.

3. $p \sim_{\mathrm{D}} q$.

In particular, T has algebraic equidominance and $\otimes$ respects $\equiv_{\mathrm{D}}$.

Proof. By Remark 2.2.10 degenerate domination implies algebraic domination. The implications $1 \Rightarrow 2 \Rightarrow 3$ are trivial and hold in every theory. To prove $3 \Rightarrow 1$ suppose $p(x) \sim_{\mathrm{D}} q(y)$, and let r_1 and r_2 be small sets of $L_0(\mathfrak{U})$-formulas with free variables included in xy and consistent with $p \cup q$ such that $p \cup r_1 \vdash q$ and $q \cup r_2 \vdash p$. It follows easily from Remark 2.2.10 that we may find r_0 satisfying the same restrictions as r_1 and r_2 and such that $p \cup r_0 \vdash q$ and $q \cup r_0 \vdash p$ hold simultaneously. $\square$

2.2.3 Weak binarity

Recall that a theory T is *binary* iff every formula is equivalent modulo T to a Boolean combination of formulas with at most two free variables. Equivalently, for every B and tuples a, b

$$\mathrm{tp}(a/B) \cup \mathrm{tp}(b/B) \cup \mathrm{tp}(ab/\emptyset) \vdash \mathrm{tp}(ab/B)$$

For instance, this is the case whenever T eliminates quantifiers in a binary relational language. We now show that a weaker condition, introduced in [Men20], is already sufficient to imply stationary domination.

Definition 2.2.12. A theory T is *weakly binary* iff whenever $\mathrm{tp}(a/\mathfrak{U})$ and $\mathrm{tp}(b/\mathfrak{U})$ are invariant there is $A \subset^+ \mathfrak{U}$ such that

$$\mathrm{tp}(a/\mathfrak{U}) \cup \mathrm{tp}(b/\mathfrak{U}) \cup \mathrm{tp}(a, b/A) \vdash \mathrm{tp}(a, b/\mathfrak{U}) \tag{2.3}$$

I am grateful to Jan Dobrowolski for pointing out the relationship between binarity and weak binarity, hence implicitly suggesting a name for the latter.

Example 2.2.13. A weakly binary theory which is not binary is the theory of a dense circular order (see Subsection 5.1.4), or any other non-binary theory that becomes binary after naming some constants. A weakly binary theory which does not become binary after adding constants can be obtained by considering the generic countable structure (M, E, R) where E is an equivalence relation with infinitely many classes, on each class $R(x, y, z)$ is a circular order, and $R(x, y, z) \to E(x, y) \wedge E(x, z)$. Another example is the theory of dense meet-trees, by Theorem 5.2.15. The generic 3-hypergraph and the theory ACF_0 of algebraically closed fields of characteristic 0 are not weakly binary.

Remark 2.2.14. If T is weakly binary, p is invariant, and $p \geq_{\mathrm{D}} q$, then q is small-type isolated over p.

Proof. Let $r \in S(A)$ witness domination, so $p \cup r \vdash q$. By definition of weak binarity, up to enlarging A, the partial type $p \cup r \cup q$ is complete, hence so is $p \cup r$, which implies q. □

Lemma 2.2.15. If T is weakly binary and $\mathrm{tp}(a/\mathfrak{U})$, $\mathrm{tp}(b/\mathfrak{U})$ are both invariant, then so is $\mathrm{tp}(a, b/\mathfrak{U})$.

Proof. If (2.3) holds and $\operatorname{tp}(a/\mathfrak{U})$ and $\operatorname{tp}(b/\mathfrak{U})$ are B-invariant then the left-hand side of (2.3) is fixed by $\operatorname{Aut}(\mathfrak{U}/AB)$. As $\operatorname{tp}(a,b/\mathfrak{U})$ is complete we can show, arguing as in the proof of Lemma 2.1.12, that it is AB-invariant. □

Lemma 2.2.16. A theory T is weakly binary if and only if for every $n \geq 2$ we have the following. If $a^0, \dots, a^{n-1}$ are such that for all $i < n$ we have $\operatorname{tp}(a^i/\mathfrak{U}) \in S^{\mathrm{inv}}(\mathfrak{U})$, then there is $A \subset^+ \mathfrak{U}$ such that

$$\Big(\bigcup_{i=0}^{n-1} \operatorname{tp}(a^i/\mathfrak{U})\Big) \cup \operatorname{tp}(a^0, \dots, a^{n-1}/A) \vdash \operatorname{tp}(a^0, \dots, a^{n-1}/\mathfrak{U}) \tag{2.4}$$

Proof. For the nontrivial direction, assume T is weakly binary. For notational simplicity we will only show the case $n = 3$, and leave the easy induction to the reader. Let a, b, c be tuples with invariant global type. By Lemma 2.2.15 $\operatorname{tp}(bc/\mathfrak{U})$ is still invariant, so we can let A witness weak binarity for b, c and for a, bc simultaneously, where bc is considered now as a single tuple. Then $\operatorname{tp}(b/\mathfrak{U}) \cup \operatorname{tp}(c/\mathfrak{U}) \cup \operatorname{tp}(a, b, c/A) \vdash \operatorname{tp}(b, c/\mathfrak{U})$, and by applying weak binarity to a, bc we get

$$\begin{aligned} &\operatorname{tp}(a/\mathfrak{U}) \cup \operatorname{tp}(b/\mathfrak{U}) \cup \operatorname{tp}(c/\mathfrak{U}) \cup \operatorname{tp}(a, b, c/A) \\ &\vdash \operatorname{tp}(a/\mathfrak{U}) \cup \operatorname{tp}(bc/\mathfrak{U}) \cup \operatorname{tp}(a, bc/A) \\ &\vdash \operatorname{tp}(a, bc/\mathfrak{U}) \end{aligned}$$
□

Corollary 2.2.17. Every weakly binary theory has stationary domination and stationary equidominance.

Proof. Let $p(x), q_0(y), q_1(z)$ be A_0-invariant and $r \in S_{q_0 q_1}(A_0)$ be such that $q_0 \cup r \vdash q_1$. We prove stationary domination. For stationary equidominance, start with an r witnessing $q_0 \equiv_{\mathrm{D}} q_1$ and argue analogously.

In some $\mathfrak{U}_1 \mathrel{^+\!\!\succ} \mathfrak{U}$ choose $(b, c) \vDash q_0 \cup r$, then choose $a \vDash p \mid \mathfrak{U}_1$. By the case $n = 3$ of (2.4) there is some $A \subset^+ \mathfrak{U}$, which without loss of generality includes A_0, such that

$$\operatorname{tp}(a/\mathfrak{U}) \cup \operatorname{tp}(b/\mathfrak{U}) \cup \operatorname{tp}(c/\mathfrak{U}) \cup \operatorname{tp}(abc/A) \vdash \operatorname{tp}(abc/\mathfrak{U}) \tag{2.5}$$

After replacing, if necessary,[3] the type $r = \operatorname{tp}(bc/A_0)$ with $\operatorname{tp}(bc/A)$, we obtain

[3] If T is such that in each of these situations we may take $A_0 = A$ in (2.5), then this step is not necessary and T has *strict* stationary domination/equidominance.

that $r \subseteq r[p] = \operatorname{tp}_{xyz}(abc/A) \cup \{x = w\}$, hence $(p \otimes q_0) \cup r[p] \vdash (q_0 \cup r) \vdash q_1 = \operatorname{tp}(c/\mathfrak{U})$. Combining this with (2.5), and observing that $\operatorname{tp}(ab/\mathfrak{U}) = p \otimes q_0$, that $\operatorname{tp}(ac/\mathfrak{U}) = p \otimes q_1$, and that $r[p] \vdash x = w$, we have

$$\begin{aligned}
&\big(p(x) \otimes q_0(y)\big) \cup r[p] \\
&\vdash \big(p(x) \otimes q_0(y)\big) \cup r[p] \cup q_1(z) \cup \{x = w\} \\
&\vdash \operatorname{tp}_x(a/\mathfrak{U}) \cup \operatorname{tp}_y(b/\mathfrak{U}) \cup \operatorname{tp}_z(c/\mathfrak{U}) \cup \operatorname{tp}_{xyz}(abc/A) \cup \{x = w\} \\
&\vdash \operatorname{tp}_{wz}(ac/\mathfrak{U}) = p(w) \otimes q_1(z)
\end{aligned}$$ □

For $k \geq 1$, it is possible to define the notion of a NIP_k theory; see [She14, Definition 5.63], or [CPT19, Definition 2.1]. By [CPT19, Example 2.2.3] binary theories are all NIP_2 and, more generally, all k-ary theories are NIP_k. It is natural to ask whether weak binarity still suffices to imply NIP_2. One can of course modify Definition 2.2.12 to yield a notion of *weakly k-ary* theory for larger k, but since we will not use it beyond this page we leave it to the reader to formulate precisely the definition.[4]

Question 2.2.18. Does weak binarity imply NIP_2? More generally, are all weakly k-ary theories NIP_k?

Question 2.2.19. Is there a theory which does not have stationary domination, but where $\otimes$ respects $\geq_{\mathrm{D}}$ or $\equiv_{\mathrm{D}}$?

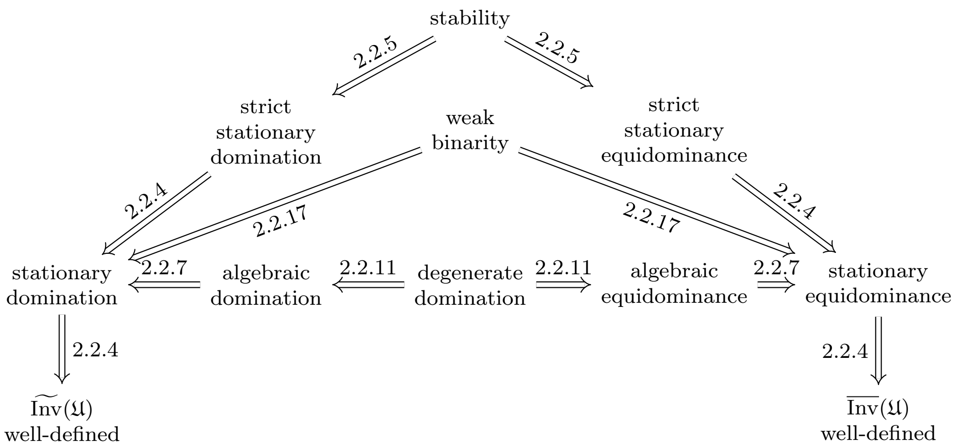


Figure 2.1: the implications shown in Section 2.2.

[4]Hint: T is ternary iff $\operatorname{tp}(ab/B) \cup \operatorname{tp}(bc/B) \cup \operatorname{tp}(ac/B) \cup \operatorname{tp}(abc/\emptyset) \vdash \operatorname{tp}(abc/B)$.

2.2.4 Examples

After all these definitions, I feel that the reader has earned some examples of $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$. These characterisations can be proven with easy ad hoc arguments but, as such computations are made almost immediate by results like Theorem 2.3.16 or Theorem 3.1.24, we postpone proofs.

Example 2.2.20. As shown in detail in Subsection 3.2.1, if T is strongly minimal then $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$ is isomorphic $(\mathbb{N}, +, \geq)$, and generated by the $\sim_{\mathrm{D}}$-class of the unique nonrealised 1-type.

Example 2.2.21. Let T_1 be the theory of the *generic equivalence relation*: an equivalence relation E with infinitely many classes, all infinite. Since T_1 is ω-stable, by Proposition 2.2.5 and Proposition 2.2.4 $\otimes$ respects $\geq_{\mathrm{D}}$. Moreover by [She90, Theorem III.3.12] (see also [Poi00, Theorem 14.2]) for every κ there is a κ-saturated $\mathfrak{U} \vDash T_1$ of size κ. For such $\mathfrak{U}$ we have $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}) \cong \bigoplus_\kappa \mathbb{N}$, where each copy of $\mathbb{N}$ is equipped with the usual $+$ and $\geq$, and $\oplus$ is the direct sum of ordered monoids.

To spell this out and give a little extra information on $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ for T_1, fix a choice of representatives $(b_i \mid 0 < i < \kappa)$ for $\mathfrak{U}/E$ and let $\pi_E \colon \mathfrak{U} \to \mathfrak{U}/E$ be the projection to the quotient. Then an element $[\![p]\!] \in \widetilde{\mathrm{Inv}}(\mathfrak{U})$ corresponds to a κ-sequence $(n_i)_{i<\kappa}$ of natural numbers with finite support where, for every $c \vDash p$, we have $n_0 = |\pi_E c \setminus \pi_E \mathfrak{U}|$ and, for positive i, $n_i = |\{c_j \in c \mid E(c_j, b_i)\} \setminus E(\mathfrak{U}, b_i)|$. In other words, n_0 counts the new equivalence classes represented in p and, when i is positive, n_i counts the number of new points in the equivalence class of b_i. Addition is done componentwise and $(n_i)_{i<\kappa} \leq (m_i)_{i<\kappa}$ iff $\forall i < \kappa \; n_i \leq m_i$. For a proof of this see Proposition 3.2.6.

As we will see in Chapter 3, the fact that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ has the previous forms follows from the stability-theoretic properties of the theories above: Theorem 3.1.24 applies to both and, in the case of Example 2.2.20, Corollary 3.1.30 tells us directly that $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$.

Example 2.2.22. As DLO is binary, $\otimes$ respects $\geq_{\mathrm{D}}$. We have already seen an example of two domination-equivalent types in this theory in Example 2.1.15. A *cut* in a linearly ordered set A is a pair of subsets (L, R) of A such that $A = L \cup R$, $L < R$, and $|L \cap R| \leq 1$. To each $p \in S_1(\mathfrak{U})$ associate a cut (L_p, R_p) in $\mathfrak{U}$ by setting $L_p \coloneqq \{d \in \mathfrak{U} \mid p(x) \vdash x \geq d\}$ and $R_p \coloneqq \{d \in \mathfrak{U} \mid p(x) \vdash x \leq d\}$.

Note that $|L_p \cap R_p| \neq \emptyset$ if and only if p is realised. To describe $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, call a cut in $\mathfrak{U}$ *invariant* iff it has small cofinality on exactly one side, i.e. iff exactly one between the cofinality of L and the coinitiality of R are small. Let $\mathrm{IC}(\mathfrak{U})$ be the set of all such. The domination-equivalence class of a nonrealised invariant type in DLO is determined by the (necessarily invariant) cuts of its 1-subtypes, and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}) \cong (\mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U})), \cup, \supseteq)$. See Subsection 4.2.1 for more details.

2.3 Preservation results

In this section we show that some properties are preserved downwards by domination. These invariants also facilitate computations of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ and $\overline{\mathrm{Inv}}(\mathfrak{U})$ for specific theories; an immediate consequence is for instance Corollary 5.1.13, that such monoids may change when passing to T^{eq} (T must be unstable; see Fact 3.1.9). We also see how the domination monoid of $\mathfrak{U}$ embeds those of certain definable subsets.

2.3.1 Finite satisfiability, definability, generic stability

The notions studied here are well-known properties that an invariant type may or may not have. One of them, *generic stability*, has been defined in different ways across the literature. These definitions are equivalent in the NIP context, but not in general, and the one we use is [ACP12, Definition 1.6]. The reader interested in other definitions, and a comparison between them, is referred to [CG20].

Definition 2.3.1. Let $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A)$. A *Morley sequence* of p over A is an A-indiscernible sequence $(a^i \mid i \in I)$, indexed on a totally ordered set I, such that for every $i_0 < \ldots < i_{n-1}$ we have $\mathrm{tp}(a^{i_{n-1}}, \ldots, a^{i_0}/A) = p^{(n)} \upharpoonright A$ [sic][5].

Definition 2.3.2. Let $M \prec^+ \mathfrak{U}$ and $A \subset^+ \mathfrak{U}$.

1. A partial type π is *finitely satisfiable in M* iff for every finite conjunction $\varphi(x)$ of formulas in π there is $m \in M$ such that $\vDash \varphi(m)$.

2. A global type $p \in S_x(\mathfrak{U})$ is *definable over A* iff it is A-invariant and for every $\psi(x; y) \in L$ the set $(d_p\psi)(y)$ is clopen, i.e. there is $\varphi(y) \in L(A)$ such that $(d_p\psi)(y) = \{q \in S_y(A) \mid \varphi \in q\}$.

[5] For example $(a^1, a^0) \vDash (p(x^1) \otimes p(x^0)) \upharpoonright A$. This awkwardness in notation is an unfortunate consequence of the order in which $\otimes$ is written, i.e. realising the type on the right first. See Subsection 7.1.3.

3. A global type $p \in S_x(\mathfrak{U})$ is *generically stable over* A iff it is A-invariant and for every ordinal $\alpha \geq \omega$, Morley sequence $(a^i \mid i < \alpha)$ of p over A, and $L(\mathfrak{U})$-formula $\varphi(x)$, the set $\{i < \alpha \mid\models \varphi(a^i)\}$ is finite or cofinite.

We say that p is *definable* iff it is definable over A for some small A, and similarly for the other two notions.

It is easy to see that in the definition of generic stability we may replace the arbitrary infinite ordinal α with $\omega + \omega$.

Remark 2.3.3. It is well-known that every partial type which is finitely satisfiable in M extends to a global type still finitely satisfiable in M; see e.g. [Poi00, Lemma 12.10].

If $p \in S(\mathfrak{U})$ is finitely satisfiable in M then p is M-invariant (see [Poi00, Theorem 12.13]; in fact, we almost proved both this statement and the previous remark in the proof of Remark 2.1.3). Moreover all the notions above are monotone; for instance if p is generically stable over A and $A \subseteq B$, then p is generically stable over B: all Morley sequences over B are also Morley sequences over A.

Fact 2.3.4. Suppose that $p \in S^{\mathrm{inv}}(\mathfrak{U}, A)$. If p is definable [resp. generically stable] over some small set, then it is definable [resp. generically stable] over A. If p is finitely satisfiable in some small model, then it is finitely satisfiable in every small model containing A.

Proof. See [HP11, Lemma 3.1] or [Sim15, Lemma 2.18] for definability and finite satisfiability, and [ACP12, Fact 1.9(2)] for generic stability. □

Fact 2.3.5 ([PT11, Proposition 1(ii)])**.** If p is generically stable over a model M, then p is finitely satisfiable in M.

Lemma 2.3.6. Suppose $p \in S_x^{\mathrm{inv}}(\mathfrak{U})$ is finitely satisfiable in M and $r \in S_{xy}(M)$ is consistent with p. Then $p \cup r$ is finitely satisfiable in M.

Proof. Pick any $\varphi(x) \in p$ and $\rho(x, y) \in r$. As $p \cup r$ is consistent, we have $p \vdash \exists y\ (\varphi(x) \wedge \rho(x, y))$, and as p is finitely satisfiable in M there is $m^0 \in M$ such that $\models \exists y\ (\varphi(m^0) \wedge \rho(m^0, y))$. In particular, $\models \exists y\ \rho(m^0, y)$, and since $\rho(m^0, y) \in L(M)$ and M is a model there is $m^1 \in M$ such that $\models \rho(m^0, m^1)$, so $(m^0, m^1) \models \varphi(x) \wedge \rho(x, y)$. □

We can now prove the main result of this section. Point 3 can be seen as a generalisation of [Tan15, Proposition 3.6], which states that generic stability is preserved under *non-weak orthogonality* of *symmetric regular* types; see Definition 2.3.14 for weak orthogonality and Definition 2.3.19 for symmetry and regularity. The missing step to formally call it a generalisation would be to know that, for all regular types p and all invariant types q, the equivalence $p \not\perp^{\mathrm{w}} q \Leftrightarrow p \leq_{\mathrm{D}} q$ held. To the best of my knowledge, this is only known when p is simultaneously *strongly regular* and generically stable (see Definition 2.3.19 and Remark 2.3.20), or under additional assumptions such as stability (see Fact 3.1.21 and Fact 3.1.27), but I am not aware of any counterexamples.

Theorem 2.3.7 ([Men20, Theorem 3.5])**.** Suppose A is a small set such that $p_x, q_y \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ and $r \in S_{pq}(A)$ is such that $p \cup r \vdash q$.

1. If $A = M$ is a model and p is finitely satisfiable in M, then so is q.
2. If p is definable over A, then so is q.
3. If $A = M$ is a model and p is generically stable over M, then so is q.

Proof.

① Let $\psi(y) \in q$, and let by hypothesis and compactness $\varphi(x) \in p$ and $\rho(x, y) \in r$ be such that $\vDash \forall x, y \left((\varphi(x) \wedge \rho(x, y)) \to \psi(y)\right)$. By Lemma 2.3.6 we find $m^0, m^1 \in M$ such that $\vDash \varphi(m^0) \wedge \rho(m^0, m^1)$, and in particular $\vDash \psi(m^1)$.

② We want to show that for every $\psi(y; z) \in L(A)$ the set $d_q\psi \subseteq S_z(A)$ is clopen; it is sufficient to show that $d_q\psi$ is open, as since ψ is arbitrary then the complement $d_q(\neg\psi)$ of $d_q\psi$ will be open as well. Fix d such that $q \vdash \psi(y; d)$; we are going to find a formula $\delta(z) \in \mathrm{tp}(d/A)$ such that every element of $S_z(A)$ satisfying δ lies in $d_q\psi$, proving that $\mathrm{tp}(d/A)$ is in the interior of $d_q\psi$. Let $\rho(x, y) \in r$ be such that $p \vdash \chi(x; d)$, where

$$\chi(x; z) \coloneqq \forall y \left(\rho(x, y) \to \psi(y; z)\right)$$

As $\chi(x; z)$ is an $L(A)$-formula and p is definable over A, the formula $\delta(z) \coloneqq (d_p\chi)(z)$ is as well over A. Suppose $\tilde{d} \in \mathfrak{U}$ is such that $\vDash \delta(\tilde{d})$. Then $p \vdash \chi(x; \tilde{d})$, therefore $p \cup \{\rho\} \vdash \psi(y, \tilde{d})$, so $\psi(y, \tilde{d}) \in q$. As $\delta(z) \in \mathrm{tp}(d/A)$, we are done.

③ Assume that q is not generically stable over M, as witnessed by an $L(M)$-formula $\psi(y; w)$, some $\tilde{d} \in \mathfrak{U}^{|w|}$, an ordinal α and a Morley sequence

$(\tilde{b}^i \mid i < \alpha)$ of q over M such that both $I := \{i < \alpha \mid \models \neg\psi(\tilde{b}^i; \tilde{d})\}$ and $\alpha \setminus I$ are infinite and $\psi(y; \tilde{d}) \in q(y)$. Assume without loss of generality that $\alpha < \kappa(\mathfrak{U})$.

By Fact 2.3.5 and Lemma 2.3.6 $p \cup r$ is finitely satisfiable in M. Since $p \cup r \vdash q$, the partial type $p \cup r \cup q$ is finitely satisfiable in M as well, and by Remark 2.3.3 extends to some $\hat{r} \in S(\mathfrak{U})$ which is, again, finitely satisfiable in M, and in particular M-invariant; take a Morley sequence $((a^i, b^i) \mid i \in I)$ of $\hat{r}$ over M, let $f \in \operatorname{Aut}(\mathfrak{U}/M)$ be such that $f((\tilde{b}^i \mid i \in I)) = (b^i \mid i \in I)$, and set $d := f(\tilde{d})$. Note that p, q, r, and $\psi(y; w)$ are fixed by f.

Now let J be a copy of ω disjoint from I and let $(a^j \mid j \in J)$ realise a Morley sequence of p over $Md\{a^i \mid i \in I\}$. We want to show that the concatenation of $(a^i \mid i \in I)$ with $(a^j \mid j \in J)$ contradicts generic stability of p over M. By construction this is a Morley sequence over M, and if we find $\chi(x; d)$ such that $\models \chi(a^i; d)$ holds for $i \in J$ but for no $i \in I$ then we are done, since I and J are both infinite.

As $\psi(y; d) \in q$ by M-invariance of q, there is by hypothesis $\rho(x, y) \in r$ such that $p(x) \vdash \forall y \left(\rho(x, y) \to \psi(y; d)\right)$. Let $\chi(x; d)$ be the last formula. By hypothesis, for $i \in J$ we have $\models \chi(a^i; d)$. On the other hand, for $i \in I$ we have $(a^i, b^i) \models \rho(a^i, b^i) \land \neg\psi(b^i; d)$, and in particular for all $i \in I$ we have $\models \neg\chi(a^i; d)$. □

Remark 2.3.8. We are assuming that p, q are A-invariant. It is *not* true that if p is finitely satisfiable/definable/generically stable in/over some $B \subseteq A$ then q must as well be such, for the same B. Even when $B = N \prec M = A$ are models, a counterexample can easily be obtained by taking q to be the realised type of a point in $M \setminus N$.

Question 2.3.9. If $p \in S^{\mathrm{inv}}(\mathfrak{U})$ is dominated by some N-invariant type, is p domination-equivalent to some N-invariant type?

By Fact 2.3.4, in the setting of Remark 2.3.8 this would imply that q is domination-equivalent to a type finitely satisfiable/definable/generically stable in/over N. If instead of a model N we have just a set B, the answer is negative. See Counterexample 5.1.11.

Theorem 2.3.7 can be used to produce a variant of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ based on generically stable types which is well-defined in every theory. As we will see in Proposition 5.3.25, generic stability is not preserved under products, so we cannot simply take the quotient of the space of generically stable types. Instead, we use the following standard fact.

Proposition 2.3.10. For all generically stable types p and all invariant types q, we have $p \otimes q = q \otimes p$.

This is [HP11, Proposition 3.5], which holds even without assuming NIP provided the definition of "generically stable" is the one above. For the reader's convenience, we prove it below; see also [Sim15, Proposition 2.33].

Proof. Suppose that $p(x)$ does not commute with $q(y)$, as witnessed by the $L(\mathfrak{U})$-formula $\varphi(x, y) \in p \otimes q$, but it is generically stable over A. In particular, $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A)$. Up to enlarging A, assume $\varphi \in L(A)$ and $q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$. Take a Morley sequence $(a^i \mid i < \omega)$ of p over A, let $b \vDash q \upharpoonright Aa^{<\omega}$, then take a Morley sequence $(a^i \mid \omega \le i < \omega+\omega)$ of p over $Aa^{<\omega}b$. As $\varphi(x, y) \in (p \otimes q) \setminus (q \otimes p)$, we have that $a^i \vDash \varphi(x, b)$ if and only if $i \ge \omega$, contradicting generic stability. □

Definition 2.3.11. Let $\widetilde{\mathrm{Inv}}_{\mathrm{gs}}(\mathfrak{U})$ be the quotient by $\sim_{\mathrm{D}}$ of the space of types which are products of generically stable types.

Corollary 2.3.12. $(\widetilde{\mathrm{Inv}}_{\mathrm{gs}}(\mathfrak{U}), \otimes, \ge_{\mathrm{D}})$ is well-defined and commutative.

Proof. Clearly, if p, q commute with every invariant type, then so does $p \otimes q$. It follows immediately from Lemma 2.1.19 and Proposition 2.3.10 that, when restricting to the set of products of generically stable types, $\otimes$ respects $\ge_{\mathrm{D}}$. As generators of $\widetilde{\mathrm{Inv}}_{\mathrm{gs}}(\mathfrak{U})$ commute, so does every pair of elements from it. □

Remark 2.3.13. By [HP11, Proposition 3.2], in NIP theories a type is generically stable if and only if it is simultaneously definable and finitely satisfiable. As both these properties can be shown to be preserved by products, in NIP theories $\widetilde{\mathrm{Inv}}_{\mathrm{gs}}(\mathfrak{U})$ is indeed the quotient of the space of generically stable types.

The space $\widetilde{\mathrm{Inv}}_{\mathrm{gs}}(\mathfrak{U})$ may be significantly smaller than $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, and even be reduced to a single point; this happens for instance in the theory of the Random Graph, or in DLO.

Since finite satisfiability and definability are preserved under products, one may define analogous quotients, on which well-definedness of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ needs to be proven/assumed: the types in Proposition 5.3.15 are all definable. In the case where $\otimes$ preserves $\ge_{\mathrm{D}}$, these form submonoids of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, which are downward closed by Theorem 2.3.7, and it can be interesting to study e.g. under which conditions every type is domination-equivalent to a product of finitely satisfiable and definable types.

2.3.2 Weak orthogonality

Another property preserved by domination is weak orthogonality to a type. This generalises (by Proposition 3.1.8) a classical result in stability theory; see e.g. [Mak84, Proposition C.13'''(iii)].

Definition 2.3.14. We say that $p \in S_x(\mathfrak{U})$ and $q \in S_y(\mathfrak{U})$ are *weakly orthogonal*, and write $p \perp^{\mathrm{w}} q$, iff $p \cup q$ is a complete global type.

Remark 2.3.15. If p is invariant then $p \perp^{\mathrm{w}} q$ is equivalent to $p \cup q \vdash p \otimes q$, or in other words to the fact that for any $c \vDash q$ we have $p \vdash p \mid \mathfrak{U}c$. Moreover, if p and q are both invariant, since $p \otimes q$ and $q \otimes p$ are both completions of $p \cup q$, then $p \perp^{\mathrm{w}} q$ implies that p and q commute.

In the literature the name *orthogonality* and the symbol $\perp$ are sometimes (e.g. [Sim15, p. 136] or [Tan15, p. 310]) used to refer to the restriction of weak orthogonality to global invariant types. Furthermore, sometimes the name "weak orthogonality" is used to refer to $(p \upharpoonright A) \perp^{\mathrm{w}} (q \upharpoonright A)$, where A is a base for both p and q. We will not adopt neither of these conventions here.

Theorem 2.3.16 ([Men20, Proposition 3.13])**.** Suppose that $p_0, p_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ and $q \in S(\mathfrak{U})$ are such that $p_0 \geq_{\mathrm{D}} p_1$ and $p_0 \perp^{\mathrm{w}} q$. Then $p_1 \perp^{\mathrm{w}} q$.

Proof. Fix $\mathfrak{U}_1 \mathrel{^{+}\!\!\succ} \mathfrak{U}$, work in its elementary diagram and suppose $p_0(x) \cup r(x,y) \vdash p_1(y)$. We have to show that for any $c \in \mathfrak{U}_1$ realising q we have $p_1 \vdash p_1 \mid \mathfrak{U}c$. By hypothesis, $p_0 \vdash p_0 \mid \mathfrak{U}c$, and by Lemma 2.1.17 we have $(p_0 \mid \mathfrak{U}c) \cup r \vdash p_1 \mid \mathfrak{U}c$, therefore $p_0 \cup r \vdash p_1 \mid \mathfrak{U}c$. This means that, for every $\psi(y,z) \in L(\mathfrak{U})$ such that $\psi(y,c) \in p_1 \mid \mathfrak{U}c$, there are $\varphi(x) \in p_0$ and $\rho(x,y) \in r$ such that $\mathfrak{U}_1 \vDash \forall x, y\ \big((\varphi(x) \land \rho(x,y)) \to \psi(y,c)\big)$, therefore

$$\mathfrak{U}_1 \vDash \forall y\ \big(\big(\exists x\ (\varphi(x) \land \rho(x,y))\big) \to \psi(y,c)\big)$$

Note that $p_0(x) \cup r(x,y) \cup p_1(y)$ is consistent, since it is satisfied by any realisation of $p_0(x) \cup r(x,y)$. Therefore, we have $p_1(y) \vdash \exists x\ (\varphi(x) \land \rho(x,y))$, hence by what we have just proved $p_1(y) \vdash \psi(y,c)$. Since $\psi(y,c)$ was an arbitrary formula in $p_1 \mid \mathfrak{U}c$, we have the conclusion. □

As a consequence of this theorem, we obtain the following slight generalisation of [Poi00, Theorem 10.23]. This was implicitly used in [HHM08] in the computation of $\overline{\mathrm{Inv}}(\mathfrak{U})$ in ACVF; see Subsection 5.2.3.

Corollary 2.3.17. Let $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$. If $p \geq_{\mathrm{D}} q$ and $p \perp^{\mathrm{w}} q$, then q is realised.

Proof. From $p \geq_{\mathrm{D}} q$ and $p \perp^{\mathrm{w}} q$ the previous theorem gives $q \perp^{\mathrm{w}} q$. But this can only happen if q is realised, otherwise $q(x) \cup q(y) \cup \{x = y\}$ and $q(x) \cup q(y) \cup \{x \neq y\}$ are both consistent. □

Theorem 2.3.16 allows us to endow $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ with an additional binary relation, induced by $\perp^{\mathrm{w}}$ and denoted by the same symbol. It is natural to ask whether this addition is redundant.

Question 2.3.18. Can $\perp^{\mathrm{w}}$ be defined internally to $\widetilde{\mathrm{Inv}}(\mathfrak{U})$?

The question above can be made precise in a number of ways. For example, we could ask whether there is a formula $\varphi(x, y)$ in the language of posets such that, in all theories T, we have $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}}) \models \varphi(\llbracket p_0 \rrbracket, \llbracket p_1 \rrbracket)$ if and only if $p_0 \perp^{\mathrm{w}} p_1$. Alternatively, restrict to theories T where $\otimes$ preserves $\geq_{\mathrm{D}}$, and allow φ to be a formula in the language of ordered monoids.

Under additional assumptions on T, the question has an affirmative answer. For instance, in thin stable theories (see Definition 3.1.19), by Theorem 3.1.24 we have $\llbracket p_0 \rrbracket \perp^{\mathrm{w}} \llbracket p_1 \rrbracket$ if and only if

$$\forall \llbracket q \rrbracket \left((\llbracket p_0 \rrbracket \geq_{\mathrm{D}} \llbracket q \rrbracket \wedge \llbracket p_1 \rrbracket \geq_{\mathrm{D}} \llbracket q \rrbracket) \rightarrow \llbracket q \rrbracket = \llbracket 0 \rrbracket \right) \tag{2.6}$$

In every theory, if $\llbracket p_0 \rrbracket \perp^{\mathrm{w}} \llbracket p_1 \rrbracket$ holds then (2.6) holds as well by Theorem 2.3.16 and Corollary 2.3.17. On the other hand, the converse fails e.g. in the theory of the Random Graph, where no two nonrealised types are weakly orthogonal. The converse may also fail under NIP; see Counterexample 5.1.14.

2.3.3 Other properties

In this subsection we briefly comment on the interaction of domination and products with other properties that an invariant type may or may not have.

We start by pointing out that a lot of properties are *not* preserved by domination-equivalence, nor by equidominance. For instance, by Remark 3.2.5 there is an ω-stable theory with two equidominant types of different Morley rank. Other properties that are not preserved are for instance stability, being generically NIP, and having a certain dp-rank, as shown respectively by [Usv09, Example 6.14], Counterexample 5.3.22, and Example 4.2.5.

I would now like to draw the reader's attention to certain other properties.

Definition 2.3.19. A global invariant type $p(x) \in S^{\mathrm{inv}}(\mathfrak{U})$ is

1. *regular over* $A \subset^+ \mathfrak{U}$ iff it is A-invariant and for every $B \supseteq A$ and $a \vDash p \restriction A$ either $a \vDash p \mid B$ or $(p \mid B) \vdash (p \mid Ba)$;
2. *regular* iff it is regular over some $A \subset^+ \mathfrak{U}$;
3. *strongly regular* iff there are $A \subset^+ \mathfrak{U}$ and $\varphi(x) \in p \restriction A$ such that p is A-invariant and for every $B \supseteq A$ and $a \vDash \varphi(x)$ either $a \vDash p \mid B$ or $(p \mid B) \vdash (p \mid Ba)$;
4. *symmetric* iff $p(x) \otimes p(y) = p(y) \otimes p(x)$; and
5. *asymmetric* iff it is not symmetric.

Regularity is not preserved by domination: see for instance Remark 3.2.23, or Example 4.2.5. Nevertheless, it is an important property in this context. For example, by a classical result, in every superstable theory every nonrealised global type is domination-equivalent to a regular type if and only if it is $\geq_{\mathrm{D}}$-minimal among the nonrealised types, and every global type is domination-equivalent to a product of regular types.

While we will deal with these matters in a bit more depth in Subsection 3.1.3, what we will *not* see is the real reason why regular types are interesting: generically stable regular types induce pregeometries, allowing to develop a nice dimension theory. We refer the interested reader to e.g. [Bue17, Section 6.3] for the superstable case and [Tan15] for a more general treatment of regularity. Asymmetric regular types induce proper closure operators (not satisfying exchange); see [MT15].

Remark 2.3.20. By [Tan15, Theorem 4.4], if p is strongly regular and generically stable then p is $\geq_{\mathrm{RK}}$-minimal among the nonrealised types, and for all invariant q we have $p \not\perp^{\mathrm{w}} q \iff p \leq_{\mathrm{RK}} q$. An immediate consequence of this result and of Corollary 2.3.17 is that such types are also $\geq_{\mathrm{D}}$-minimal among the nonrealised types.

Weak orthogonality is an equivalence relation when restricted to asymmetric regular types by [MT15, Theorem 3], and when restricted to generically stable regular types by [Tan15, Theorem 1]. In [Tan15, Question 1], it is asked whether $\not\perp^{\mathrm{w}}$ is an equivalence relation on symmetric regular types. A related question is the following.

Question 2.3.21. If p is regular and $q \not\perp^{\mathrm{w}} p$, does $q \geq_{\mathrm{D}} p$?

By [MT15, Theorem 3] symmetry and asymmetry are preserved under non-weak orthogonality of regular types. Asymmetry is not preserved by domination, as realised types are symmetric; a less trivial counterexample can be obtained by considering that if p is symmetric and q is not, then $p \otimes q$ is still asymmetric, but it dominates p. Nevertheless, the question for symmetry remains open. I take the opportunity to also ask about preservation of other properties. Properties with $n \geq 5$ below, for which we refer the reader to [CG20], arise from the fact that in NIP theories generic stability has equivalent definitions that become separated in general. See [CG20] for a study of this phenomenon.

Question 2.3.22. Suppose that $p_1 \leq_{\mathrm{D}} p_0 \in S^{\mathrm{inv}}(\mathfrak{U})$. If p_0 has property n, does p_1 have property n as well, for the properties listed below?[6]

1. Symmetry.
2. Commutativity with a fixed invariant type q.
3. Distality (see [Sim13] or [Sim15, Chapter 9]).
4. Generic simplicity, in the sense of [Sim17].
5. Finite approximability.
6. Stationarity.
7. Weak stationarity.

We conclude with some remarks about Question 2.3.22. Firstly, note that by Proposition 2.3.10 generically stable types are symmetric. Furthermore, by [MT15, Theorem 5.9], symmetric types cannot dominate *regular* asymmetric ones.

For the first three properties, the question is answered in the positive in distal theories: trivially for 3, while preservation of 1 and 2 is a corollary of Theorem 2.3.16 and the fact that, in distal theories, $p \perp^{\mathrm{w}} q \iff p \otimes q = q \otimes p$.

Regarding generic simplicity, there is some heuristic evidence that the property might be interesting in this context: [Sim17, Lemma 3.6] says that it is preserved under pushforwards, and [Sim17, Lemma 3.7] implies that it is preserved by products.

[6] See [CG20] for the definitions of property n for $n \geq 5$.

2.3.4 Stable embeddedness

Naively, it could be expected that if $\mathfrak{U}'$ is an expansion of $\mathfrak{U}$ by new relations, then there is an embedding of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ into $\widetilde{\mathrm{Inv}}(\mathfrak{U}')$, or a surjective map in the other direction. This is not the case, as can be seen by expanding a pure set $\mathfrak{U}$ with a dense linear order $\mathfrak{U}' = (\mathfrak{U}, <)$. Similar problems arise with interpretations.

Remark 2.3.23. If T_0 is the theory of a pure set, and T_1 is the theory of the generic equivalence relation (Example 2.2.21), then there is a natural 2-dimensional interpretation of T_1 in T_0: if $M_0 \vDash T_0$, a model of T_1 can be defined inside M_0^2 by setting $(x_0, x_1)E(y_0, y_1) \coloneqq x_0 = y_0$. The structure induced on the domain of the interpretation is, anyway, not just a generic equivalence relation, but a generic equivalence relation *with a bijection between classes*, and this prevents $\widetilde{\mathrm{Inv}}(\mathfrak{U}_1) \cong \bigoplus_\kappa \mathbb{N}$ from embedding into $\widetilde{\mathrm{Inv}}(\mathfrak{U}_0) \cong \mathbb{N}$. The extra structure makes complete types in T_1 become partial after being interpreted in T_0, and is also responsible for more domination relations.

In fact, the situation is even worse, and the reduct of an invariant type may not be invariant: in DLO together with a convex bounded predicate with no supremum, the type just right of the predicate is $\emptyset$-definable, but its DLO reduct is not invariant. See [RS17b] for a study of these phenomena.

Nonetheless, if $\mathfrak{U}$ has for example multiple sorts, call them H_i, it is sometimes possible (and convenient) to compute each $\widetilde{\mathrm{Inv}}(H_i(\mathfrak{U}))$ and argue that it embeds in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. As the reader might expect after the previous discussion, this can be done if $H_i(\mathfrak{U})$ does not inherit extra structure from $\mathfrak{U}$; this was implicitly used in [HHM08]. This subsection deals with such matters in detail.

Definition 2.3.24. Let $\pi(x)$ be a partial type over $\emptyset$ with $|x| = 1$, and let $P = \pi(\mathfrak{U})$. View P as a structure in its own right as follows. For every $\emptyset$-definable m-ary relation R of $\mathfrak{U}$, let $R^P \coloneqq R(\mathfrak{U}) \cap P^m$.

We say that P is *stably embedded* iff whenever $D \subseteq \mathfrak{U}^m$ is definable, then $D \cap P^m$ is definable with parameters from P.

Let M_0, M_1 be structures in possibly different languages, such that M_1 is defined inside M_0, with domain the $\emptyset$-type-definable set P. We say that M_1 is *canonically embedded* in M_0 iff the structure induced on P in the sense above is exactly M_1. We call M_1 *fully embedded* in M_0 iff M_1 is canonically embedded in M_0 with domain a $\emptyset$-type-definable set which is stably embedded in M.

If $P = \pi(\mathfrak{U})$ is viewed as a structure in its own right as specified above, then it is canonically embedded by definition, and it is stably embedded if and only if it is fully embedded, if and only if the definable sets it inherits from $\mathfrak{U}$ are exactly the same as the sets it defines as a structure in its own right. We refer the reader to the appendix to [CH99] and to [Sim15, Section 3.1] for a more thorough treatment of stable embeddedness. The only thing we will need here is the following fact.

Fact 2.3.25. If P is stably embedded, $M \prec^+ \mathfrak{U}$ is $|T|^+$-saturated, and $d, d' \in P$, then $d \equiv_M d' \iff d \equiv_{M \cap P} d'$.

Proof sketch. Suppose first that $\pi(x)$ is a single formula $\varphi(x)$, and moreover that stable embeddedness is *uniform*, i.e. that, for every formula $\psi(x, y)$ over $\emptyset$ there is $\theta_\psi(x, y)$ over $\emptyset$ such that

$$\vDash \forall y\ \exists z \left[\Big(\bigwedge_{i<|z|} \varphi(z_i)\Big) \wedge \forall w \Big(\Big(\bigwedge_{i<|w|} \varphi(w_i)\Big) \to \big(\psi(w, y) \leftrightarrow \theta_\psi(w, z)\big)\Big)\right]$$

If this is the case, then stable embeddedness of π does not depend on the model on which we are working, i.e. $\pi(M)$ is stably embedded in M, and the result follows. In the general case π is not implied by a single formula, and θ may depend not just on $\psi(w, y)$, but on the parameters to be plugged in y. In this case, $|T|^+$-saturation of M is needed, and details are left to the reader. □

If P is stably embedded, by definition every element of $S_n(P)$ implies a complete type in $S_{x^0,\ldots,x^{n-1}}(\mathfrak{U})$.

Definition 2.3.26. If P is stable embedded and $p \in S_n(P)$, let $\iota(p)$ be the unique element of $S_{x^0,\ldots,x^{n-1}}(\mathfrak{U})$ implied by p.

Lemma 2.3.27. Let P be stably embedded, say in the sort of x. The map ι is an embedding of the space $S_n(P)$ of types over P concentrating on $\pi^n := \bigwedge_{i<n} \pi(x^i)$ into $S_{x^0,\ldots,x^{n-1}}(\mathfrak{U})$, and an homeomorphism onto $S_{\pi^n}(\mathfrak{U})$.

Proof. The map ι is clearly injective, and is continuous by compactness. Since it is from a compact space to a Hausdorff one, it is closed. To conclude, note that its image is included in $S_{\pi^n}(\mathfrak{U})$, and that by stable embeddedness every element of $S_{\pi^n}(\mathfrak{U})$ is implied by its restriction to P. □

Therefore, we may freely confuse $S_n(P)$ with $S_{\pi^n}(\mathfrak{U})$.

Lemma 2.3.28. Let P be stably embedded and $p \in S_n(P)$. Then p is invariant if and only if $\iota(p)$ is.

Proof. Suppose p is A-invariant, for some small $A \subseteq P$. So p is a fixed point of the action of $\operatorname{Aut}(P/A)$. Since π is over $\emptyset$, it is fixed setwise by every element of $\operatorname{Aut}(\mathfrak{U})$. It follows that p is fixed by every element of $\operatorname{Aut}(\mathfrak{U}/A)$ and, since it implies $\iota(p)$, which is complete, so is the latter.

Suppose that p is not A-invariant for any small $A \subseteq P$, but $\iota(p)$ is M-invariant for some $M \prec^+ \mathfrak{U}$, where we may assume that M is $|T|^+$-saturated. Let $A \coloneqq M \cap P$, and let $\varphi(y; d) \triangle \varphi(y; d')$ be a formula in $p(y)$ with $d \equiv_A d'$. Then this formula is still in $\iota(p)$, and by Fact 2.3.25 $d \equiv_M d'$, hence we have reached a contradiction. □

Lemma 2.3.29. Let P be stably embedded. Then $p \geq_{\mathrm{D}} q$ in $S(P)$ if and only if $\iota(p) \geq_{\mathrm{D}} \iota(q)$ in $S(\mathfrak{U})$. Similarly for $\equiv_{\mathrm{D}}$.

Proof. Left to right is obvious, since $q \vdash \iota(q)$. In the other direction, given a small type r witnessing $\iota(p) \geq_{\mathrm{D}} \iota(q)$, we may replace r with a small type over a subset of P since, by definition, every formula concentrating on a cartesian power of π is equivalent to one over P. □

So if P is stably embedded and $p \in S(P)$, then $[\![p]\!]_P \subseteq [\![p]\!]_{\mathfrak{U}}$. Note that the inclusion may be strict, as p may be domination-equivalent to the type of a tuple with some coordinates not satisfying π.

Lemma 2.3.30. The map ι is a $\otimes$-homomorphism.

Proof. The restriction of $\iota(p) \otimes \iota(q)$ is $p \otimes q$, which implies $\iota(p \otimes q)$. □

Proposition 2.3.31. Let P be stably embedded. The map ι induces an embedding of $\widetilde{\operatorname{Inv}}(P)$ into $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ as posets and of $\overline{\operatorname{Inv}}(P)$ into $\overline{\operatorname{Inv}}(\mathfrak{U})$ as sets. This embedding is also a $\perp^{\mathrm{w}}$-homomorphism, a $\not\perp^{\mathrm{w}}$-homomorphism, and, if $\otimes$ respects $\geq_{\mathrm{D}}$ [resp. $\equiv_{\mathrm{D}}$], an embedding of monoids.

Proof. The only statements we have not proven yet are those regarding $\perp^{\mathrm{w}}$. If $p_0 \perp^{\mathrm{w}} p_1$ then $p_0 \cup p_1$ is complete, and $\iota(p_0 \cup p_1)$ is implied by $p_0 \cup p_1$. But each p_i is contained in $\iota(p_i)$. Conversely, if $\iota(p_0) \cup \iota(p_1)$ is complete, it follows easily from stable embeddedness that so is $p_0 \cup p_1$. □

Remark 2.3.32. Suppose that $\mathfrak{U}$ is the disjoint union of the structures in $(\mathfrak{U}_i \mid i \in I)$, where each $\mathfrak{U}_i$ has its own sorts and there are no relations among points coming from different $\mathfrak{U}_i$. Then each $\mathfrak{U}_i$ is fully embedded in $\mathfrak{U}$, and clearly we have $\widetilde{\operatorname{Inv}}(\mathfrak{U}) = \bigoplus_{i\in I} \widetilde{\operatorname{Inv}}(\mathfrak{U}_i)$, with a technical caveat. In order for this to behave as expected, we really need to work in multi-sorted first-order logic, as opposed to identifying sorts with unary predicates P_i. If we use predicates, and I is infinite, then $\mathfrak{U}$ is not even ω-saturated anymore, since it does not realise the 1-type $p(x) := \{\neg P_i(x) \mid i \in I\}$. If we replace it with a saturated structure we obtain $\widetilde{\operatorname{Inv}}(\mathfrak{U}) = (\bigoplus_{i\in I} \widetilde{\operatorname{Inv}}(\mathfrak{U}_i)) \oplus \mathbb{N}$, a generator of the extra copy of $\mathbb{N}$ being $[\![p]\!]$.

Remark 2.3.33. Fix $\mathfrak{U}$, and consider $\mathfrak{U}^{\mathrm{eq}}$ as a structure in multi-sorted logic, lest we incur the problems explained in Remark 2.3.32. By [TZ12, Proposition 8.4.5] $\mathfrak{U}$ is fully embedded in $\mathfrak{U}^{\mathrm{eq}}$. By the results in this subsection, this induces an embedding $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \hookrightarrow \widetilde{\operatorname{Inv}}(\mathfrak{U}^{\mathrm{eq}})$. This embedding *need not be surjective* in general (Corollary 5.1.13), but it is if T is stable (Fact 3.1.9), or if T eliminates imaginaries, since in that case every tuple from $\mathfrak{U}^{\mathrm{eq}}$ is interdefinable with a tuple from $\mathfrak{U}$.

2.4 Other viewpoints

The final part of this chapter is dedicated to alternative points of view: we first provide an equivalent definition of domination, and then we take a step back and "undo" the quotient $\widetilde{\operatorname{Inv}}(\mathfrak{U})$.

With the exception of Lemma 2.4.4, the material in this section will not be used later on, and can be skipped. While it appears at this point of this thesis, because it works in full generality and does not use anything we have not proven yet, the ideas come from the stable case, and there are some immediate consequences when stability is assumed (see Remark 2.4.11). The reader unfamiliar with stability theory might want to come back to this section after reading Subsection 3.1.1.

2.4.1 A rephrasing

The goal of this subsection is to rephrase domination in terms of an independence relation. This independence relation is quaternary, rather than ternary as usual, and is only defined on certain tuples. We do not define it directly, but merely describe the domination relation it induces.

Definition 2.4.1. Write $a \triangleright^{\mathrm{i}}_{M,N} b$ iff $M \prec^+ N$, $\mathrm{tp}(a/N)$ is M-invariant, and for every $C \supseteq N$ if $\mathrm{tp}(a/C)$ is M-invariant, then $\mathrm{tp}(b/C)$ is M-invariant.

Note that in particular $\mathrm{tp}(b/N)$ must be M-invariant.

Lemma 2.4.2. Suppose that $a \triangleright^{\mathrm{i}}_{M,N} b$ and $b' \equiv_{Na} b$. Then $a \triangleright^{\mathrm{i}}_{M,N} b'$.

Proof. Let $C \supseteq N$ be such that $\mathrm{tp}(a/C)$ is M-invariant but $\mathrm{tp}(b'/C)$ is not, as witnessed by $b' \vDash \varphi(y;d) \triangle \varphi(y;\tilde{d})$, say, with $d \equiv_M \tilde{d}$ and $\varphi(y;w) \in L(M)$. Let $\mathfrak{U}_1 \mathrel{{}^{+}{\supset}} Cabb'$. By hypothesis there is $f \in \mathrm{Aut}(\mathfrak{U}_1/Na)$ such that $f(b') = b$. Then $\mathrm{tp}(a/f(C))$ is M-invariant, $f(C) \supseteq f(N) = N$, and clearly $f(d) \equiv_M f(d')$, but $b = f(b') \vDash \varphi(y; f(d)) \triangle \varphi(y; f(\tilde{d}))$. This contradicts $a \triangleright^{\mathrm{i}}_{M,N} b$ □

Proposition 2.4.3. Let $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $M \prec^+ N \prec^+ \mathfrak{U}$. The following are equivalent.

1. There is $r \in S_{pq}(N)$ such that $p \cup r \vdash q$.

2. There are $a \vDash p$ and $b \vDash q$ such that $a \triangleright^{\mathrm{i}}_{M,N} b$.

Proof. For $1 \Rightarrow 2$, let $ab \vDash p \cup r$. In particular, by assumption, we also have $b \vDash q$. Suppose that $C \supseteq N$ and $\mathrm{tp}(a/C)$ is M-invariant. Since $C \supseteq N$ and $a \vDash p \restriction N$, we must have $\mathrm{tp}(a/C) = p \mid C$. By Lemma 2.1.17 we have $(p \mid \mathfrak{U}C) \cup r \vdash (q \mid \mathfrak{U}C)$, and by Lemma 2.1.16 $(p \mid C) \cup r \vdash (q \mid C)$ holds too. Since $ab \vDash r$, we obtain $b \vDash q \mid C$, and the latter is an M-invariant type.[7]

For $2 \Rightarrow 1$, let $r := \mathrm{tp}(ab/N)$. Suppose that $a'b' \vDash p \cup r$, and assume up to an isomorphism in $\mathrm{Aut}(\mathfrak{U}_1/\mathfrak{U})$ that $a' = a$. We have to show that $b' \vDash q$. As $b' \vDash q \restriction N$ and $M \prec^+ N$, $(q \restriction N)$ has a unique M-invariant extension to every parameter set so, if we show that $\mathrm{tp}(b'/\mathfrak{U})$ is M-invariant, then it must coincide with q. But $b \equiv_{Na} b'$, so by Lemma 2.4.2 we have $a \triangleright^{\mathrm{i}}_{M,N} b'$, and since $\mathrm{tp}(a/\mathfrak{U}) = p \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ we are done. □

If the reader is accustomed to stability theory, or took the author's advice and is reading this section after Subsection 3.1.1, I suggest to compare the proof above with that of Proposition 3.1.8. The idea of proof is the same,

[7] Even if this proof uses $p \mid \mathfrak{U}C$, this does not necessarily mean that a satisfies it.

except the model N here is compensating for lack of stationarity by fixing defining schemes for M-invariant types.

The following lemma is related to this order of ideas, and will be found useful in Chapter 6. In a sense, it says that we can avoid mentioning N if we are allowed to fix defining schemes.

Lemma 2.4.4. Let $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$. Suppose that there are $a, b \in \mathfrak{U}$ with $a \vDash p \restriction M$, $b \vDash q \restriction M$, and

$$\forall C\ M \subseteq C \subseteq \mathfrak{U} \implies \big((a \vDash p \restriction C) \implies (b \vDash q \restriction C)\big) \tag{2.7}$$

Let $r \coloneqq \mathrm{tp}(ab/M)$. Then $p \cup r \vdash q$.

Proof. We first show that if $a', b' \in \mathfrak{U}$ are such that $a'b' \equiv_M ab$ then a' and b' still satisfy (2.7). Suppose this is not the case. Let $a'b' \vDash \mathrm{tp}(ab/M)$ and let $C' \supseteq M$ be such that $a' \vDash p \restriction C'$ but $b' \nvDash q \restriction C'$. Let $f \in \mathrm{Aut}(\mathfrak{U}/M)$ be such that $f(a'b') = ab$ and let $C \coloneqq f(C')$. Then $a \vDash (f \cdot p) \restriction C$ and $b \nvDash (f \cdot q) \restriction C$. Now note that, since p, q are M-invariant, they are fixed by f.

By saturation of $\mathfrak{U}$, there is N such that $M \prec^+ N \prec^+ \mathfrak{U}$ and $a \vDash p \restriction N$. Then, by what we proved above, $(p \restriction N) \cup r \vdash (q \restriction N)$, and we conclude by using Lemma 2.1.17, with N taking the role of $\mathfrak{U}$. □

2.4.2 Functoriality

In this subsection we take a step back and look at domination and product of invariant types *before* quotienting by domination-equivalence. In order to keep track of the different ways in which domination can be witnessed, we encode this information in the morphisms of a category. I would like to thank Gabriele Lobbia for the useful discussions around this topic.

In what follows, types are not assigned predetermined variables. For example, if x and y are tuples of variables with $|x| = |y|$, we do not distinguish between $p(x)$ and $p(y)$. When we write e.g. $p(x)$, we implicitly assume that x is a tuple of variables compatible with p. As usual, if we plug variables in several different types simultaneously, we use pairwise distinct ones.

Definition 2.4.5. The *domination category* $\mathrm{Inv}(\mathfrak{U})$ is defined as follows.

1. Objects of $\mathrm{Inv}(\mathfrak{U})$ are invariant types $p \in S^{\mathrm{inv}}(\mathfrak{U})$.

2. If $r(x, y)$ is a small partial type consistent with $p(x) \cup q(y)$ and such that $p \cup r \vdash q$, then the deductive closure of $p \cup r$ is a morphism from p to q. We abuse the notation[8] and denote this by $r\colon p \to q$. All morphisms are obtained this way.

3. The identity $\mathrm{id}_p\colon p \to p$ is the deductive closure of $p(x) \cup \{x = y\}$.

4. If $r_0\colon p_0 \to p_1$ and $r_1\colon p_1 \to p_2$, the composition $r_1 \circ r_0$ is the deductive closure of $p_0(x) \cup r(x, z)$, where r is the partial type

$$r(x, z) \coloneqq \exists y\ (r_0(x, y) \cup r_1(y, z))$$

Proposition 2.4.6. Composition is well-defined, and $\mathrm{Inv}(\mathfrak{U})$ is a category.

Proof. Composition is well-defined because $r_1 \circ r_0$ actually witnesses $p_0 \geq_{\mathrm{D}} p_2$ by Lemma 2.1.10, and if $p_i \cup r_i$ has the same deductive closure as $p_i \cup r'_i$ for $i < 2$, then $p_0 \cup r$ has the same deductive closure as $p_0 \cup r'$, where r' is defined analogously from r'_0 and r'_1. Verifying that composition is associative boils down to introducing two quantifiers in two different orders, but of course for every partial type $\Phi(x, y, z, w)$ the partial types $\exists y\ \exists z\ \Phi(x, y, z, w)$ and $\exists z\ \exists y\ \Phi(x, y, z, w)$ are the same. Finally, we need to check that the identities are indeed identities with respect to composition, but this is obvious. □

The reason we take a deductive closure when defining morphisms is because otherwise the identities id_p are not uniquely defined, for inconspicuous reasons: if $r(x, y)$ and $r'(x, y)$ are distinct, both consistent with $p(x)$, and both contain $x = y$, then they would be two different identity maps.

Note that, contrary to our usual conventions, we are allowing r not to be complete, for the following reason. If we insist that a morphism $p \to q$ should be a complete $r \in S_{pq}(A)$, then in order to define the composition of $r_0(x, y) \in S_{p_0p_1}(A_0)$ with $r_1(y, z) \in S_{p_1p_2}(A_1)$ we need to choose a way to extend $\exists y\ (r_0(x, y) \cup r_1(y, z))$ to a complete type in $S_{p_0p_2}(A_0A_1)$. This might get in the way of associativity, or yield several versions of $\mathrm{Inv}(\mathfrak{U})$, each with its own composition map.

Lemma 2.4.7. If $p_0 \cup r$ and $p_0 \cup r'$ have the same deductive closure, then so do $p_0(x^0) \otimes q(y^0) \cup r(x^0, x^1) \cup \{y^0 = y^1\}$ and $p_0(x^0) \otimes q(y^0) \cup r'(x^0, x^1) \cup \{y^0 = y^1\}$.

[8] Such an r is not necessarily unique: for instance there may be a restriction r' of r such that $(p \cup r') \vdash (p \cup r)$.

Proof. If the partial type on the left, say, proves $\varphi(x^0, x^1, y^0)$, then for b realising the restriction of q to the parameters in r and φ together with a base of p, we have $p(x^0) \cup r(x^0, x^1) \vdash \varphi(x^0, x^1, b)$. By assumption, $p \cup r$ and $p \cup r'$ have the same deductive closure, so $p(x^0) \cup r'(x^0, x^1) \vdash \varphi(x^0, x^1, b)$, but then $p(x^0) \otimes q(y^0) \cup r'(x^0, x^1) \vdash \varphi(x^0, x^1, y^0)$, as desired. □

Definition 2.4.8. Let $q \in S^{\mathrm{inv}}(\mathfrak{U})$. We define a map $- \otimes q \colon \mathrm{Inv}(\mathfrak{U}) \to \mathrm{Inv}(\mathfrak{U})$ as follows. On objects of $\mathrm{Inv}(\mathfrak{U})$, set $(- \otimes q)(p) = p \otimes q$; on morphisms, let $(- \otimes q)(r_0 \colon p_0 \to p_1) \colon p_0 \otimes q \to p_1 \otimes q$ be the deductive closure of $p_0(x^0) \otimes q(y^0) \cup r(x^0, x^1) \cup \{y^0 = y^1\}$.

This is well-defined: by Lemma 2.1.19 we really get a witness of $p_0 \otimes q \geq_{\mathrm{D}} p_1 \otimes q$, and by Lemma 2.4.7 this does not depend on r, but only on the deductive closure of $p \cup r$.

Proposition 2.4.9. For all $q \in S^{\mathrm{inv}}(\mathfrak{U})$, the map $- \otimes q$ is a functor.

Proof. We need to show that composition and identities are preserved. For identities, $\mathrm{id}_{p \otimes q}$ is the deductive closure of $p(x^0) \otimes q(y^0) \cup p(x^1) \otimes q(y^1) \cup \{x^0 = x^1 \wedge y^0 = y^1\}$. By definition, this is the same as $(- \otimes q)\mathrm{id}_p$. For composition, let $r_0 \colon p_0 \to p_1$ and $r_1 \colon p_1 \to p_2$. Then $\big((- \otimes q)r_1\big) \circ \big((- \otimes q)r_0\big)$ is the deductive closure of

$$p_0(x) \otimes q(w^0) \cup \Big(\exists y, w^1 \; \big(r_0(x, y) \cup \{w^0 = w^1\} \cup r_1(y, z) \cup \{w^1 = w^2\}\big)\Big)$$

while $(- \otimes q)(r_1 \circ r_0)$ is the deductive closure of

$$p_0(x) \otimes q(w^0) \cup \Big(\exists y \; \big(r_0(x, y) \cup r_1(y, z)\big)\Big) \cup \{w^0 = w^2\}$$

and these are clearly the same. □

Remark 2.4.10. If q is the unique 0-type, i.e. the elementary diagram $\mathrm{ED}(\mathfrak{U})$, then $p \otimes q$ is not merely equidominant to p, but we actually have the equality $p \otimes q = p$.

Remark 2.4.11. Suppose that T is stable, so $\otimes$ is commutative, as we saw in Example 2.1.7. Then we can define a functor $(p \otimes -)$ by simply setting it equal to $(- \otimes p)$. It follows from the previous remark that, for T stable, $\mathrm{Inv}(\mathfrak{U})$ together with the associative bifunctor $- \otimes -$ and the object $\mathrm{ED}(\mathfrak{U})$ is a *strict symmetric monoidal category*.

We refer the reader to [ML13, Section VII.1] for monoidal categories. Since $\otimes$ may not be commutative, $(\mathrm{Inv}(\mathfrak{U}), \otimes, \mathrm{ED}(\mathfrak{U}))$ is not strict symmetric monoidal in general, but if we drop strictness, and possibly symmetry, this might hold in a class of theories larger than the stable ones. I expect stationary domination (or rather, its strict version) to play a role here, since trying to define $p \otimes -$ in the obvious way presents us with obstructions to functoriality even in a NIP theory: as we saw in Example 2.2.1, the "obvious" attempt at a definition of $p \otimes -$ on morphisms does not necessarily yield a morphism.

While these investigations are left to future work, observe immediately that, as we will see in Proposition 5.3.15, it is possible for $p \otimes q_0$ not to dominate $p \otimes q_1$ even if $q_0 \geq_{\mathrm{D}} q_1$. Therefore, the map $p \otimes -$, defined on objects, cannot in general be extended to a functor: the set of morphisms $p \otimes q_0 \to p \otimes q_1$ can be empty even if the set of morphisms $q_0 \to q_1$ is not.

Chapter 3

The stable case

In almost all doctoral theses I have read, the introduction is followed by a chapter containing a survey of existing results. Apart from the nonstandard collocation, this is that chapter: what follows is not new, except possibly in presentation, but it provides part of the motivation for the rest of this work. In the first section, we will review the classical stability-theoretic theorems about domination, and point out some facts that are relevant for our purposes. The second section is instead dedicated to the domination monoids of some concrete stable theories.

Until the end of the chapter, T will be stable unless otherwise stated, but we may repeat this for emphasis. For the sake of concision, we will assume some knowledge of stability theory from the reader. For example, we will not define what a stable theory is (but see Fact 3.1.1). Forking will follow a similar fate, and we will take various properties of the latter for granted. Similarly, as the material here is well-established, we will omit several proofs.

In this chapter, when pointing to a result, I have tried to use the source which stated it in the form which was most convenient for our purposes, hence the reader is cautioned against taking these citations to imply attribution. As most of these results are due to Shelah, the primary reference is [She90]; other standard references are e.g. [Bal88, Bue17, Pil96, Poi00]. Note that, in [Bue17], some results are only stated for theories with regular $\kappa(T)$; the reason for this is that [Bue17] defines an *a-model* to be a strongly $\kappa(T)$-saturated model, as opposed to a strongly $\kappa_{\mathrm{r}}(T)$-saturated one.

3.1 Domination in stable theories

In stable theories $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined by Proposition 2.2.5. Moreover, every global type is invariant, and we may simply write $S(\mathfrak{U})$ instead of $S^{\mathrm{inv}}(\mathfrak{U})$. In fact, when working over models, we may forget about invariance, and replace it by nonforking. This is a consequence of the facts below, which will be tacitly used throughout this chapter.

Fact 3.1.1. The following are equivalent.

1. T is stable.
2. Every global type is definable.
3. Every global type is invariant.

Fact 3.1.2. If T is stable, $M \vDash T$, and $p \in S(\mathfrak{U})$, the following are equivalent.

1. p is M-invariant.
2. p is M-definable.
3. p is finitely satisfiable in M.
4. p does not fork over M.

Fact 3.1.3. If T is stable, every $p \in S(M)$ is *stationary*, i.e. has a unique extension to $\mathfrak{U}$ that does not fork over M.

Fact 3.1.4. If T is stable, and $a^i \vDash p_i \in S(\mathfrak{U})$ for $i < 2$, then $a^0 a^1 \vDash p_0 \otimes p_1$ if and only if $a^0 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathfrak{U}} a^1$. In particular, $\otimes$ is commutative, i.e. for all $p, q \in S(\mathfrak{U})$ we have $p(x) \otimes q(y) = q(y) \otimes p(x)$.

3.1.1 Domination via forking

In the following definition A is allowed to be a large set, e.g. $A = \mathfrak{U}$.

Definition 3.1.5. We say that *a weakly dominates b over A* iff for all d we have $a \mathop{\smile\!\!\!\!\!\!\mid}_A d \Longrightarrow b \mathop{\smile\!\!\!\!\!\!\mid}_A d$. We say that *$a$ dominates b over A*, written $a \rhd_A b$, iff for every $B \supseteq A$ if $ab \mathop{\smile\!\!\!\!\!\!\mid}_A B$ then a weakly dominates b over B.

Fact 3.1.6 ([Pil96, Lemma 1.4.3.4] and [Poi00, Lemma 19.18])**.** Suppose $A \subseteq B$ and $ab \mathop{\smile\!\!\!\!\!\!\mid}_A B$. Then $a \rhd_B b$ if and only if $a \rhd_A b$. Moreover, over a $|T|^+$-saturated model domination and weak domination are equivalent.

Definition 3.1.7. For stationary $p, q \in S(A)$ we write $p \triangleright q$ iff there are $a \models p$ and $b \models q$ such that $a \triangleright_A b$. Write $p \bowtie q$ iff $p \triangleright q \triangleright p$. Write $p \doteq q$ iff there are $a \models p$ and $b \models q$ such that $a \triangleright_A b \triangleright_A a$.

Proposition 3.1.8. Suppose that T is stable and p, q are global types. Then the following hold.

1. $p \geq_{\mathrm{D}} q$ if and only if $p \triangleright q$.
2. $p \equiv_{\mathrm{D}} q$ if and only if $p \doteq q$.
3. If $p \geq_{\mathrm{D}} q$ then this is witnessed by some $r \in S_{pq}(M)$ with $|M| \leq |T|$.

We provide a proof of this, essentially that in [Poi00, Theorem 19.27], as it motivates Subsection 2.4.1.

Proof. Assume that $r \in S_{pq}(M)$ is such that $p(x) \cup r(x, y) \vdash q(y)$. By enlarging M, we may assume that p and q do not fork over M, and that M is $|T|^+$-saturated, so weak domination and domination for tuples coincide by Fact 3.1.6. We begin by proving that $(p \upharpoonright M) \triangleright (q \upharpoonright M)$; more specifically that, if $(a, b) \models r$, then $a \triangleright_M b$. Assume that $b \not\perp_M d$, as witnessed by $\psi(y, d)$. As in a stable theory types over models are stationary and p does not fork over M, to show that $a \not\perp_M d$ it is sufficient to prove that a does not satisfy $p \upharpoonright Md$, the unique nonforking extension of $p \upharpoonright M$ to Md. We have $q \vdash \neg\psi(y, d)$, as q does not fork over M. By Lemma 2.1.16 $(p \upharpoonright Md) \cup r \vdash (q \upharpoonright Md)$, and this completes the proof that $(p \upharpoonright M) \triangleright (q \upharpoonright M)$. If now ab satisfies the unique nonforking extension of r to $\mathfrak{U}$, then $a \models p$, $b \models q$, and $ab \perp_M \mathfrak{U}$. By Fact 3.1.6, we have $a \triangleright_{\mathfrak{U}} b$, and therefore $p \triangleright q$, thereby proving left to right in 1.

Now suppose that $a, b \in \mathfrak{U}_1 \mathrel{^{+}\!\succ} \mathfrak{U}$ are such that $a \models p$, $b \models q$, and $a \triangleright_{\mathfrak{U}} b$. By local character of forking in stable theories, there is a small M, in fact of size at most $|T|$, such that $ab \perp_M \mathfrak{U}$. Let $r := \mathrm{tp}(ab/M)$. By Fact 3.1.6 we have $a \triangleright_M b$. Modulo an element of $\mathrm{Aut}(\mathfrak{U}_1/\mathfrak{U})$, to prove that $p \cup r \vdash q$ it is sufficient to show that if $b' \equiv_{Ma} b$ then $b' \models q$, and since q is the unique nonforking extension of $q \upharpoonright M$, it is sufficient to show that $b' \perp_M \mathfrak{U}$. Whether $a \triangleright_M b$ or not only depends on[1] $\mathrm{tp}(b/Ma)$: if $f \in \mathrm{Aut}(\mathfrak{U}_1/Ma)$ is such that $f(b') = b$, then if $a \perp_M d$ but $b' \not\perp_M d$ we have $a \perp_M f(d)$ and $b \not\perp_M f(d)$. Therefore we have $a \triangleright_M b'$, and since $a \perp_M \mathfrak{U}$ we get $b' \perp_M \mathfrak{U}$.

[1] See also Lemma 2.4.2.

This proves 1, and combining the proofs of the two implications gives 3. For left to right in 2, note that if additionally $q \cup r \vdash p$ then $a \triangleleft_{\mathfrak{U}} b$. In the other direction, if additionally $a \triangleleft_{\mathfrak{U}} b$ then, with the same r, we have $q \cup r \vdash p$. □

More conceptual proofs of 1 and 3 can be obtained from the classical results that $p \triangleright q$ if and only if q is realised in the *a-prime* model containing a realisation of p, and that a-prime models are *a-atomic* (see [Pil96, Lemma 1.4.2.4], or [Bue17, Proposition 5.6.3]). We will not define what these notions starting in "a-" mean here, and refer the reader to the literature. Note that a consequence of this equivalence is that in a stable theory *semi-a-isolation* (i.e. $\geq_{\mathrm{D}}$ by point 3 of the previous proposition) is the same as *a-isolation*: if $p \cup r \vdash q$ then r can be chosen such that $p \cup r$ is complete, despite r being small.

I would like to thank Anand Pillay for pointing out to me the following fact. Recall that for us $\mathfrak{U}^{\mathrm{eq}}$ is a structure in multi-sorted logic. See Remark 2.3.32 and Remark 2.3.33.

Fact 3.1.9. If T is stable, the embedding $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ is surjective.

Proof sketch. Even without assuming stability, every type $q \in S(\mathfrak{U}^{\mathrm{eq}})$ is dominated by a type p with all variables in the home sort via a suitable tuple of projection maps. Suppose now that T is stable and let M be $|T|^+$-saturated and such that p and q do not fork over M. It is enough to show that there is a (possibly forking) extension of $p \upharpoonright M$ which is equidominant with q, since such an extension has all variables in the home sort. This is essentially (the proof of) [Poi00, Lemma 19.21], and the argument goes as follows. Fix $a \vDash p \upharpoonright M$ and $b \vDash q \upharpoonright M$ such that $a \triangleright_M b$. If there is no $d^0 \in \mathfrak{U}$ such that $b \mathop{\smile\!\!\!\!\!\mid}_M d^0$ but $a \not\mathop{\smile\!\!\!\!\!\mid}_M d^0$ we are done. Otherwise, let M_1 be a small $|T|^+$-saturated model containing Md^0 and such that $ab \mathop{\smile\!\!\!\!\!\mid}_{Md^0} M^1$, and note that by assumption and transitivity of forking $b \mathop{\smile\!\!\!\!\!\mid}_M M_1$. If there is no $d^1 \in \mathfrak{U}$ such that $b \mathop{\smile\!\!\!\!\!\mid}_{M_1} d^1$ but $a \not\mathop{\smile\!\!\!\!\!\mid}_{M_1} d^1$ then $\mathrm{tp}(a/M_1)$ is the required forking extension of $p \upharpoonright M$. Otherwise, iterate, and at limit steps take a $|T|^+$-saturated model containing all the M_i considered so far. By stability, this process must stop at an ordinal smaller than $\kappa(T)$.[2] □

[2]This proof of course assumes that $\kappa(\mathfrak{U})$ is large enough.

3.1.2 Orthogonality

Definition 3.1.10. Two types $p_0, p_1 \in S(A)$ are *almost orthogonal* iff for every $a^i \vDash p_i$ we have $a^0 \mathop{\smile\!\!\!\!\!\!\,|}_A a^1$.

Note that if one of the p_i is stationary, e.g. if A is a model, then almost orthogonality is the same as weak orthogonality: some tuples $a^i \vDash p_i$ such that $a^0 \mathop{\smile\!\!\!\!\!\!\,|}_A a^1$ always exist, and if the type of one of them is stationary then the type of the pair is uniquely determined.

Definition 3.1.11. If $p_i \in S(A_i)$ for $i < 2$, we say that p_0 and p_1 are *orthogonal*, denoted by $p_0 \perp p_1$, iff for every $B \supseteq A_0A_1$, and nonforking extensions q_i of p_i to B, the types q_0 and q_1 are almost orthogonal.

Fact 3.1.12 ([Poi00, Lemma 19.14.])**.** If M is $|T|^+$-saturated and $p_i \in S(M)$, then $p_0 \perp^{\mathrm{w}} p_1 \iff p_0 \perp p_1$.

Note that orthogonality makes sense even for types over different domains. The proof of the following fact illustrates why it is useful to be able to preserve weak orthogonality when passing to nonforking extensions.

Fact 3.1.13 ([Bue17, Lemma 5.6.1])**.** If T is stable, then $p \perp q_0 \otimes q_1$ if and only if $p \perp q_0$ and $p \perp q_1$.

Proof. Left to right follows from Theorem 2.3.16, and holds in every theory if we replace $\perp$ by $\perp^{\mathrm{w}}$. Right to left, let $a \vDash p$ and $(b^0, b^1) \vDash q_0 \otimes q_1$. Since $p \perp^{\mathrm{w}} q_1$, we have $a \vDash p \mid \mathfrak{U}b^1$. Since $p \perp q_0$, we have $(p \mid \mathfrak{U}b^1) \perp^{\mathrm{w}} (q_0 \mid \mathfrak{U}b^1)$. As $b^0 \vDash q_0 \mid \mathfrak{U}b^1$, we are done. □

In stable theories, there is another nice property relating $\geq_{\mathrm{D}}$, $\otimes$, and $\perp^{\mathrm{w}}$. We include a proof, borrowed from [Her91, Proposition 4.12], for the reader's convenience. I do not know if this still holds in an arbitrary theory.

Fact 3.1.14. If T is stable, $p_0 \otimes p_1 \geq_{\mathrm{D}} q$, and $p_1 \perp q$, then $p_0 \geq_{\mathrm{D}} q$.

Proof. The reader acquainted with the theory of a-models will know that $p \geq_{\mathrm{D}} q$ if and only if q is realised in the a-prime model $\mathfrak{U}[a]$, where $a \vDash p$. Let $(a^0, a^1) \vDash p_0 \otimes p_1$, and set $\mathfrak{M} := \mathfrak{U}[a^1]$. By assumption, there is $b \in \mathfrak{M}[a^0] \cong \mathfrak{U}[a^0a^1]$ such that $b \vDash q$. Since $p_1 \perp q$, in fact we have $b \vDash q \mid \mathfrak{M}$. Since $b \in \mathfrak{M}[a^0]$, this means that $q \mid \mathfrak{M}$ is realised in the a-prime model over a realisation of $p_0 \mid \mathfrak{M}$, and therefore $(p_0 \mid \mathfrak{M}) \geq_{\mathrm{D}} (q \mid \mathfrak{M})$. By Proposition 3.1.8 and Fact 3.1.6,[3] this implies $p_0 \geq_{\mathrm{D}} q$. □

Question 3.1.15. Is it true that, in an arbitrary T, for all invariant p_0, p_1, q, if $p_1 \perp^{\mathrm{w}} q$ and $p_0 \otimes p_1 \geq_{\mathrm{D}} q$ then $p_0 \geq_{\mathrm{D}} q$? What if instead $p_1 \otimes p_0 \geq_{\mathrm{D}} q$? What if we also assume $p_0 \perp^{\mathrm{w}} p_1$? What if we also assume NIP?

3.1.3 Thin theories

In stable theories, domination-equivalence comes with a cardinal invariant, called *weight*. If this is finite on every type (this is the case in all superstable theories), then the theory is called *thin*, and $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ obeys a strong structure theorem, which reduces the study of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ to the identification of nice representatives for the domination-equivalence classes that generate it.

Definition 3.1.16. If $p \in S(A)$, let w_p be the set of all cardinals μ such that there are $a \vDash p$ and $D = \{d_i \mid i < \mu\}$ of size μ such that $d_i \mathop{\smile\!\!\!\!\!\!|}_A D \setminus \{d_i\}$ and for every $i < \mu$ we have $a \not\mathop{\smile\!\!\!\!\!\!|}_A d_i$. The *preweight* $\mathrm{pwt}(p)$ of p is defined as follows. Let $\kappa = \sup w_p$. If $\kappa = \max w_p$, then $\mathrm{pwt}(p) := \kappa$. If w_p has no maximum, write $\mathrm{pwt}(p) := \kappa^-$.

By forking calculus, if $\mu \in w_p$ then $\mu < \kappa(T) \leq |T|^+$, so $\sup w_p$ exists.

Fact 3.1.17 ([Bue17, Lemma 5.6.4(ii)]). If $p \in S(M)$, where M is $|T|^+$-saturated, then $\mathrm{pwt}(p \mid \mathfrak{U}) = \mathrm{pwt}(p)$.

This allows us to take suprema across nonforking extensions, and give the following definition.

Definition 3.1.18. The *weight* $\mathrm{wt}(p)$ of p is defined as follows. Let $W_p := \{\mathrm{pwt}(q) \mid q \text{ nonforking extension of } p\}$. Let $\kappa = \sup W_p$. If $\kappa = \max W_p$, then $\mathrm{wt}(p) := \kappa$. If W_p does not have a maximum, we write $\mathrm{wt}(p) := \kappa^-$. If $\mathrm{wt}(p) = \omega^-$, we say that p has *rudimentarily finite* weight.

By Fact 3.1.17, over sufficiently saturated models preweight equals weight.

[3] Or, to be precise, their versions for a-models, as opposed to $|T|^+$-saturated ones. See [Bue17, Proposition 5.6.3, Proposition 5.6.4].

Definition 3.1.19. A stable theory T is *thin* iff every complete[4] type has finite weight.

A *dense forking chain* is a set of types $\{p_i \mid i \in \mathbb{Q}\}$ such that if $i < j$ then p_j is a forking extension of p_i.

Fact 3.1.20 ([Pil96, Corollary 1.4.5.8, Lemma 4.3.7, Proposition 4.3.10])**.** If T is superstable, has no dense forking chains, or is such that every complete type has rudimentarily finite weight, then T is thin.

Fact 3.1.21 ([Bue17, Corollary 5.6.5], [Kim14, Remark 4.4.7(4)])**.** If $p \not\perp^{\mathrm{w}} q$ and $\mathrm{wt}(q) = 1$, then $p \geq_{\mathrm{D}} q$. If p and q have both weight 1 then the following are equivalent.

1. $p \not\perp q$.
2. $p \sim_{\mathrm{D}} q$.
3. $p \equiv_{\mathrm{D}} q$.

Fact 3.1.22 ([Bue17, Lemma 5.6.4 (iv) and Proposition 5.6.5 (ii)])**.** If $p \geq_{\mathrm{D}} q$ then $\mathrm{wt}(p) \geq \mathrm{wt}(q)$. Moreover, $\mathrm{wt}(p \otimes q) = \mathrm{wt}(p) + \mathrm{wt}(q)$.

Lemma 3.1.23. If p has weight $\mathrm{wt}(p) = 1$, then the monoid generated by $[\![p]\!]$ in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is isomorphic to $\mathbb{N}$.

Proof. Since weight is additive over $\otimes$ we have $\mathrm{wt}(p^{(n)}) = n$ and we conclude by Fact 3.1.22 that the map $n \mapsto [\![p^{(n)}]\!]$ is an isomorphism between $\mathbb{N}$ and the monoid generated by $[\![p]\!]$. □

Although not stated as below, the following result is in [She90] for superstable theories, and was generalised to thin theories by Pillay using [Hyt95], according to [Bue17, p. 290].

Theorem 3.1.24. If T is thin, then there are a cardinal κ, possibly depending on $\mathfrak{U}$, and an isomorphism $f\colon \widetilde{\mathrm{Inv}}(\mathfrak{U}) \to \bigoplus_\kappa \mathbb{N}$. Moreover, $p \perp q$ if and only if $f(p)$ and $f(q)$ have disjoint supports.

[4]There is a related notion of *strong* theory, in which every partial type π has finite weight (or generalisations thereof), defined by taking a further supremum over the realisations of π. This is not the same as thinness since, for example, for every n there might be a 1-type p_n with $\mathrm{wt}(p_n) = n$. In this case the weight of the partial type $x = x$ will be rudimentarily finite, but not finite.

Proof. Let $([\![p_i]\!] \mid i < \kappa)$ be an enumeration without repetitions of the $\sim_{\mathrm{D}}$-classes of types of weight 1. By Fact 3.1.21, the p_i are pairwise orthogonal. Define $f([\![p_i]\!])$ to be the characteristic function of $\{i\}$, then extend f to classes of products of weight-one types by sending $[\![p \otimes q]\!]$ to $f([\![p]\!]) + f([\![q]\!])$ and $[\![0]\!]$ to the function which is constantly 0.

Suppose that p is domination-equivalent to both $q_{0,0} \otimes \ldots \otimes q_{0,m}$ and $q_{1,0} \otimes \ldots \otimes q_{1,n}$, where all the $q_{0,i}$ and $q_{1,j}$ have weight 1. Fact 3.1.21 implies that if $q_{1,0}$, say, is not dominated by $q_{0,0} \otimes \ldots \otimes q_{0,m}$, then it is orthogonal to it, hence to $q_{1,0} \otimes \ldots \otimes q_{1,n}$ by Theorem 2.3.16. By Fact 3.1.13, we have $q_{1,0} \perp q_{1,0}$, and by Corollary 2.3.17 $q_{1,0}$ is realised, contradicting that it has weight 1. It follows that we must have $\{q_{0,i} \mid i \leq m\} = \{q_{1,j} \mid j \leq n\}$. Moreover, if there are exactly k of the $q_{0,i}$ such that $q_{0,i} \in [\![p_0]\!]$, say, then there are exactly k of the $q_{1,j}$ such that $q_{0,j} \in [\![p_0]\!]$: otherwise, by discarding all factors not in $[\![p_0]\!]$ and using again Fact 3.1.13, together with Fact 3.1.14 and Fact 3.1.22, $p_i^{(k)}$ is dominated by one between $q_{0,0} \otimes \ldots \otimes q_{0,m}$ and $q_{1,0} \otimes \ldots \otimes q_{1,n}$, but not by the other, hence these two products cannot be domination-equivalent. This tells us that f is well-defined, i.e. does not depend on the decomposition as product of weight-one types, nor on the representatives that we choose for each domination-equivalence class, and that f is injective.

By [Pil96, Proposition 4.3.10] in a thin theory every type is domination-equivalent to a finite product of weight-one types, so f is defined on the whole of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. By Lemma 3.1.23 if $\mathrm{wt}(p) = 1$ then the monoid generated by $[\![p]\!]$ is isomorphic to $\mathbb{N}$ and this easily entails that f is surjective. It is also clear that f is an isomorphism of ordered monoids.

The "moreover" part follows from the facts that two types of weight 1 are either weakly orthogonal or domination-equivalent (Fact 3.1.21), and that $p \perp q_0 \otimes q_1$ if and only if $p \perp q_0$ and $p \perp q_1$ (Fact 3.1.13). □

Remark 3.1.25. Weight, which is preserved by domination-equivalence by Fact 3.1.22, can, in the thin case, be read off $f(\widetilde{\mathrm{Inv}}(\mathfrak{U}))$ by taking "norms". Specifically, if $f([\![p]\!]) = (n_i)_{i<\kappa}$, then $\mathrm{wt}(p) = \sum_{i<\kappa} n_i$ (recall that every $(n_i)_{i<\kappa} \in \bigoplus_\kappa \mathbb{N}$ has finite support).

Proposition 3.1.26. If T is thin, then $\equiv_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$ coincide.

Proof. By [Kim14, Theorem 4.4.10] every type is in fact *equidominant* with a finite product of types of weight 1. The conclusion then follows from Fact 3.1.21 and the fact that, as T is stable, $\otimes$ respects both $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$. □

In a stable theory, a type p is regular (Definition 2.3.19) if and only if it is orthogonal to all of its forking extensions. It is well-known that in superstable theories every type is equidominant with a product of regular types. This follows from what we have seen so far and the fact below.

Fact 3.1.27 ([Bue17, Proposition 6.3.1 and Corollary 6.3.4])**.** It T is stable, every regular type has weight 1. If T is superstable, every type of weight 1 is domination-equivalent to a regular type.

3.1.4 Unidimensionality

Definition 3.1.28. A stable theory is *unidimensional* iff whenever $p \perp q$ at least one between p and q is algebraic.

If T is totally transcendental, then by [Bue17, Proposition 7.1.1] unidimensionality is the same as categoricity in every cardinality strictly larger than $|T|$. Example 3.2.21 shows that unidimensional theories may still fail to be totally transcendental. By a classical result, the situation cannot be much worse.

Theorem 3.1.29 ([Hru90, Theorem 4])**.** Let T be a stable unidimensional theory. Then T is superstable.

Corollary 3.1.30. Let T be stable. Then T is unidimensional if and only if $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}) \cong (\mathbb{N}, +, \geq)$. Moreover, in this case $\widetilde{\mathrm{Inv}}(\mathfrak{U}) = \overline{\mathrm{Inv}}(\mathfrak{U})$.

Proof. Superstable theories are thin by Fact 3.1.20. If T is unidimensional, by Theorem 3.1.29 we have the hypothesis of Theorem 3.1.24, and the conclusion then follows easily from unidimensionality. In the other direction, every pair of types is $\geq_{\mathrm{D}}$-comparable, but if $p \perp^{\mathrm{w}} q$ and $p \geq_{\mathrm{D}} q$ then q is realised by Corollary 2.3.17. The "moreover" part follows by Proposition 3.1.26. □

Note that the hypothesis that T is stable is necessary: in the Random Graph if $p \perp^{\mathrm{w}} q$ then one between p and q must be realised, but $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is not commutative by Corollary 5.3.2.

Corollary 3.1.30 should be compared with [Las75, Proposition 5], which tells us, under the assumption that T is countable and ω-stable, that T is $\aleph_1$-categorical if and only if $(S(\mathfrak{U})/\sim_{\mathrm{RK}}, \otimes, \geq_{\mathrm{RK}}) \cong (\mathbb{N}, \otimes, \geq)$.

Proposition 3.1.31. If T is stable then $\mathbb{N}$ embeds in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$.

Proof. Recall that we only consider theories with infinite models. By [Poi00, Lemma 13.3 and p. 336] in every stable theory there is always a type p of U-rank 1, and in particular of weight $\mathrm{wt}(p) = 1$ (see [Poi00, before Theorem 19.9]). The conclusion follows from Lemma 3.1.23. □

Therefore, unidimensional theories are the stable ones where $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is reduced to the bare minimum. The existence of the embedding above also has the following consequence. Note that $(\mathbb{N}, \geq)$ contains an infinite chain, hence is unstable.

Corollary 3.1.32. If T is stable, then $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ and $\overline{\mathrm{Inv}}(\mathfrak{U})$ are not inverse semigroups, and are not type-definable in $\mathfrak{U}$.

Proof. By [Hal18, Proposition 3.1.12], a semigroup S type-definable in a stable structure is always *strongly-π-regular*, i.e. for every $p \in S$ there is n such that p^n is contained in a subgroup of S. Let p have weight 1, and assume that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is type-definable. Let $[\![p]\!]^n$ be contained in a subgroup of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ and let $[\![q]\!]$ be an inverse in the sense of this subgroup. Then $n = \mathrm{wt}(p^{(n)}) = \mathrm{wt}(p^{(n)} \otimes q \otimes p^{(n)}) \geq 2n$, a contradiction. The first statement is proven analogously by assuming that there is q such that $p \otimes q \otimes p = p$. □

3.2 Examples

Here we describe what $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ looks like in some specific stable theories. Sometimes, I have taken the liberty of adopting a hands-on approach, and some arguments might be shortened. Keep in mind that, when we say "put in $r \in S_{pq}(A)$ the formula φ", we are implicitly assuming that A is big enough that $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$.

3.2.1 Strongly minimal theories

Strongly minimal theories are notoriously $\aleph_1$-categorical, hence unidimensional, and by Corollary 3.1.30 this is equivalent to $\overline{\mathrm{Inv}}(\mathfrak{U}) = \widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$. We prove it (kind of) directly for illustrative purposes.

Proposition 3.2.1. If T is strongly minimal, then $\overline{\mathrm{Inv}}(\mathfrak{U}) = \widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$. Moreover, T has algebraic domination and algebraic equidominance.

Proof. If $p(x) \in S_1(\mathfrak{U})$ is the generic type and $k < \ell \in \mathbb{N}$, then obviously $p^{(\ell)} \geq_{\mathrm{D}} p^{(k)}$. Moreover $p^{(k)} \geq_{\mathrm{D}} p^{(k+1)}$ cannot hold by Fact 3.1.22 and the fact that $\mathrm{wt}(p) = 1$. We now show that, if q has Morley rank n, then $q \sim_{\mathrm{D}} p^{(n)}$, and q is algebraic over p, in the sense of Definition 2.2.6. By permuting the variables we may assume that $q(x_0, \ldots, x_m)$ proves that $x_0, \ldots, x_{n-1}$ are independent and $x_n, \ldots, x_{m-1} \in \mathrm{acl}(\mathfrak{U}x_0, \ldots, x_{n-1})$. To show equidominance with $p^{(n)}(y)$ It is then sufficient to put in $r(x, y)$ the formulas $x_i = y_i$ for $i < n$. Algebraic equidominance follows easily, and the proof of algebraic domination is similar. □

We will see in Remark 3.2.5 that characterising domination-equivalence classes via Morley rank is misleading: what really characterises them is weight, but the two happen to coincide in this case.

3.2.2 Cross-cutting equivalence relations

Definition 3.2.2. If κ is a cardinal, denote with T_κ the theory of κ *cross-cutting* equivalence relations $\{E_i \mid i < \kappa\}$, asserting that each E_i is an equivalence relation with infinitely many classes, and for each $i_0 < \ldots < i_n < \kappa$ and $y_0, \ldots, y_n$ there are infinitely many x such that $\bigwedge_{j \leq n} E_{i_j}(x, y_{i_j})$ holds. For $\kappa = 1$, we call T_1 the theory of the *generic equivalence relation*.[5]

A standard back-and-forth argument shows that each T_κ is complete and eliminates quantifiers. This in turns allows us to characterise types: a nonrealised type $p(x)$ over A is completely determined once we specify

- for each $i < \kappa$, $j < |x|$, and $a \in A$, whether $E_i(x_j, a)$ holds or not, and
- for those $i < \kappa$ and $j_0 < j_1 < |x|$ such that x_{j_0} and x_{j_1} are not E_i-related to any element of A, whether $E_i(x_{j_0}, x_{j_1})$ holds or not.

It is then easy to count types and see that, for all infinite A, we have $|S_1(A)| = |A|^\kappa$. Therefore, if κ is finite, T_κ is ω-stable, and in particular superstable, while for infinite κ we are in the presence of a strictly stable theory. Forking is easily characterised: $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ if and only if $a \cap b \subseteq C$ and, for all i, if π_i denotes the projection to the quotient by E_i, then $\pi_i a \cap \pi_i b \subseteq \pi_i C$. In other words,

[5] For finite κ, it is possible to show that T_κ is the Fraïssé limit of the class of finite structures equipped with κ equivalence relations.

forking means that we are either becoming realised, or that we are falling in an equivalence class that was not determined before.

We now study domination, starting from T_1.

Definition 3.2.3. In T_1, for every $a \in \mathfrak{U}$, we define $p_a(x) \coloneqq \{E_0(x,a) \wedge x \neq d \mid d \in \mathfrak{U}\}$ to be the type of a new element in the class of a. We define $p_{\mathrm{g}}(x) \coloneqq \{\neg E_0(x,d) \mid\in \mathfrak{U}\}$ to be the *generic* type, i.e. the type an element in a new equivalence class.

Note that if $\models \neg E_0(a,b)$ then $p_a \perp^{\mathrm{w}} p_b$ and $p_a \perp^{\mathrm{w}} p_{\mathrm{g}}$. Moreover, all p_a and p_{g} have weight one, as follows easily from the characterisation of forking. Therefore, their domination-equivalence classes are among the generators of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, which by Theorem 3.1.24 is of the form $\bigoplus_\lambda \mathbb{N}$, for some cardinal λ. In fact, these are the only generators.

Proposition 3.2.4. In T_1, every global type is domination-equivalent to a product of types of the form p_{g} or p_a.

Proof. Let $p(x) \in S(\mathfrak{U})$, with $|x| = n$, and assume without loss of generality that for all $i < j < n$ and $c \in \mathfrak{U}$ we have $p \vdash x_i \neq x_j$ and $p \vdash x_i \neq c$. Up to a permutation of the variables, which can be performed with no problems inside r, we may assume that there are $a_1, \ldots, a_m \in \mathfrak{U}$ and $0 = k_0 < k_1 < \ldots < k_m = \ell_0 < \ldots < \ell_s = n \in \mathbb{N}$ such that p says the following.

- For each $i \in \{1, \ldots, m\}$, the elements $x_{k_{i-1}}, \ldots, x_{k_i - 1}$ are all in the same class as a_i.
- For each $i \in \{1, \ldots, s\}$, the elements $x_{\ell_{i-1}}, \ldots, x_{\ell_i - 1}$ are all in the same class, which is different from all the ones represented in $\mathfrak{U}$ and from the ones of x_j for $j < \ell_{i-1}$.

Now observe that whenever y is in a new equivalence class and $E_0(y, z_i)$ holds for all $i < |z|$, then the type of yz is determined by the information above. In other words, $p_{\mathrm{g}}(y) \cup \{E_0(y,z_i) \mid i < |z|\} \cup \{z_i \neq z_j \mid i < j < |z|\}$ is complete. Therefore, by introducing in r formulas such as $E_0(x_{\ell_{i-1}}, x_{\ell_{i-1}+1})$, we may furthermore assume that for all $i \in \{1, \ldots, s\}$ we have $\ell_i - \ell_{i-1} = 1$. Hence $p \sim_{\mathrm{D}} p_{\mathrm{g}}^{(s)} \otimes \bigotimes_{i=1}^m p_{a_i}^{(k_i - k_{i-1})}$. □

Remark 3.2.5. U-rank and Morley rank, which in this theory coincide, are invisible in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, as there is an automorphism of $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$ which swaps an arbitrary $[\![p_a]\!]$ with $[\![p_{\mathrm{g}}]\!]$. In fact, by passing to T_1^{eq}, we also see that U-rank is not preserved by domination: p_{g} has U-rank 2 but is domination-equivalent to a type of U-rank 1, namely the generic type in the sort $\mathfrak{U}/E$. This is not a coincidence, as every type of weight 1 in a superstable T is equivalent in T^{eq} to one of U-rank of the form ω^α; see e.g. [Poi00, Corollary 19.25].

Before moving to larger κ we observe that, in T_1, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ has size $|\mathfrak{U}/E|$, but T_1 can be interpreted in the strongly minimal theory of a pure set, which has domination monoid $\mathbb{N}$. We redirect the puzzled reader to Remark 2.3.23.

Denote by $\prod_\kappa^{\mathrm{b}} \mathbb{N}$ the monoid of bounded κ-sequences of natural numbers: the set of functions $f\colon \kappa \to \omega$ such that $\exists n \in \omega\ \forall \alpha < \kappa\ f(\alpha) < n$, equipped with pointwise sum.

Proposition 3.2.6. In T_κ we have $\overline{\mathrm{Inv}}(\mathfrak{U}) = \widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \left(\bigoplus_\lambda \mathbb{N}\right) \oplus \left(\prod_\kappa^{\mathrm{b}} \mathbb{N}\right)$, where λ is the number of possible sets of the form $\bigcap_{i<\kappa} X_i$, with each X_i an E_i-class. The order is the product order, i.e.[6]

$$(a_i)_{i<\lambda+\kappa} \le (b_i)_{i<\lambda+\kappa} \iff \forall i < \lambda+\kappa\ a_i \le b_i$$

and weak orthogonality corresponds to having disjoint supports.

Proof. For every $\alpha \in 2^\kappa$ and $a \in \mathfrak{U}$, define $p_{\alpha,a} \in S_1(\mathfrak{U})$ as

$$p_{\alpha,a}(x) := \{E_i(x,a) \wedge x \neq d \mid \alpha(i) = 0, d \in \mathfrak{U}\} \cup \{\neg E_i(x,d) \mid \alpha(i) = 1, d \in \mathfrak{U}\}$$

If for some i we have $\alpha(i) \neq 0$, then the equidominance class of $p_{\alpha,a}$ is determined regardless of a: the partial type $p_{\alpha,a}(x) \cup \{E_i(x,y) \mid \alpha(i) = 1\}$ forces y to be nonrealised, and to obtain $p_{\alpha,a} \equiv_{\mathrm{D}} p_{\alpha,b}$ it is enough to add formulas specifying to which E_j-classes x and j belong for $\alpha(j) = 0$. These formulas are automatically contained in r as soon as its domain is large enough.

Suppose now that $\alpha \neq \beta$ are both not constantly 0, say $\alpha(i) = 1 \neq \beta(i)$, and let $\chi_{\{i\}}$ be the characteristic function of $\{i\}$. Clearly, for every $a \in \mathfrak{U}$, we have $p_{\alpha,a} \geq_{\mathrm{D}} p_{\chi_{\{i\}},a}$ and $p_{\beta,a} \perp^{\mathrm{w}} p_{\chi_{\{i\}},a}$. Hence by Theorem 2.3.16 $p_{\beta,a} \not\geq_{\mathrm{D}} p_{\alpha,a}$, otherwise $p_{\beta,a}$ both dominates and is orthogonal to $p_{\chi_{\{i\}},a}$. By Corollary 2.3.17 this implies that $p_{\chi_{\{i\}},a}$ is realised, a contradiction.

If α_0 is constantly 0, then $p_{\alpha_0,a} = p_{\alpha_0,b} \iff \forall i < \kappa\ \mathfrak{U} \vDash E_i(a,b)$, and it is easy to see that each $p_{\alpha_0,a}$ is weakly orthogonal to every other 1-type.

[6] Here $\lambda + \kappa$ is the ordinal sum. Note that by saturation of $\mathfrak{U}$ we must have $\lambda > \kappa$.

With an argument almost identical to the proof of Proposition 3.2.4, we can show that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ equals $\overline{\mathrm{Inv}}(\mathfrak{U})$, is generated by types of the form $p_{\alpha,a}$, and is the direct sum of one copy of $\mathbb{N}$ for each set of the form $\bigcap_{i<\kappa} X_i$, where each X_i is an E_i-class, plus a direct summand of the form $\prod_\kappa^{\mathrm{b}} \mathbb{N}$. The reason only bounded κ-sequences are allowed is that a 1-type can contribute at most one new equivalence class to every E_i. □

In these theories we can already observe a number of phenomena.

Remark 3.2.7. Fix $a \in \mathfrak{U} \models T_2$ and, for $i < 2$, define $p_i(x) \coloneqq \{\neg E_i(x, d) \mid d \in \mathfrak{U}\} \cup \{E_{1-i}(x, a)\}$. Then neither of the p_i is domination-equivalent to any $\emptyset$-invariant type, but $p_0 \otimes p_1$, which clearly dominates both, is domination-equivalent to the $\emptyset$-invariant type $q(x) \coloneqq \{\neg E_i(x, d) \mid i < 2, d \in \mathfrak{U}\}$. Therefore, the analogue of Question 2.3.9 where instead of a model N we have just a set B has a negative answer. Note that if we pass to T_2^{eq} then both p_i are domination-equivalent to the generic type of $\mathfrak{U}/E_i$, which *is* $\emptyset$-invariant.

Counterexample 5.1.11 shows that passing to T^{eq} does not always help.

The same example as in Remark 3.2.7 shows that, even if $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is generated by the classes of nonrealised 1-types, there may be redundant generators, since $p_0 \otimes p_1 \sim_{\mathrm{D}} q$. In fact, if κ is infinite, *every* set of generators must contain redundant ones, and we have an example of a (necessarily strictly) stable theory where $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ cannot be generated by pairwise orthogonal classes. This follows easily from the next observation.

Proposition 3.2.8. If $\kappa \geq \omega$, the commutative monoid $\prod_\kappa^{\mathrm{b}} \mathbb{N}$ is not free.

Proof. Suppose that it is freely generated (as a commutative monoid) by a set $\mathcal{G}$. Since all elements of $\mathbb{N}$ are non-negative, there is at least one infinite $A \subseteq \kappa$ such that its characteristic function χ_A is in $\mathcal{G}$. Partition $A = B \sqcup C$, with both B and C infinite. By writing χ_B and χ_C as a sum of elements of $\mathcal{G}$, we can write $B = \bigsqcup_{i<n} B_i$ and $C = \bigsqcup_{j<m} C_j$, where the χ_{B_i} and χ_{C_j} are in $\mathcal{G}$. But then $\chi_A = \sum_{i<n} \chi_{B_i} + \sum_{j<m} \chi_{C_j}$, contradicting freeness. □

3.2.3 Modules

Theories of modules are always stable, and have been extensively studied. The main reference for this subsection is [Pre88]. Basic facts can also be found in a variety of textbooks, e.g. [Poi00, Section 6.5]. It turns out that, in the theory of a module, understanding domination boils down to understanding the poset of its pp-definable subgroups. I would like to express my gratitude to Mike Prest for the useful discussions around domination in modules.

Definition 3.2.9. Let Λ be a ring, and let L be the language $\{+, 0, -\} \cup \{\lambda \cdot - \mid \lambda \in \Lambda\}$. Identify a Λ-module with an L-structure by interpreting $+$, 0, $-$ in the natural way, and $\lambda \cdot -$ as scalar multiplication by λ.

Definition 3.2.10. A *pp-formula*[7] is a formula of the form $\exists y \bigwedge_{i<n} \varphi_i(x, y)$, where each $\varphi_i(x, y)$ has the form $\sum_{j<|x|} \lambda_j \cdot x_j = \sum_{k<|y|} \lambda'_k \cdot y_k$.

Fact 3.2.11. If M is a module, in $\mathrm{Th}(M)$ every complete type over M only depends on the set of pp-formulas with parameters from M that it implies.

Fact 3.2.12 ([Pre88, Corollary 2.2 (ii)])**.** If a subgroup of a module is definable by a pp-formula, then it is definable by a pp-formula over $\emptyset$.

This allows the following characterisation of 1-types.

Definition 3.2.13. A *filter of subgroups* of a module M is a filter on the poset of its ($\emptyset$-)pp-definable subgroups closed under taking finite-index pp-definable subgroups. More precisely, it is a family $\mathcal{F}$ of pp-definable subgroups of M such that

1. $M \in \mathcal{F}$,
2. $\mathcal{F}$ is closed under intersections, and
3. if $G \in \mathcal{F}$ and $[G : G \cap H]$ is finite, then $H \in \mathcal{F}$.

Note that the last clause implies, in particular, upward closure.

Fact 3.2.14 ([Poi00, Section 6.5])**.** In a theory of modules, a type $p(x) \in S_1(M)$ over a model M is determined by a filter $\mathcal{F}_p$ of subgroups and, for every $G \in \mathcal{F}_p$, an $a_G \in M$ such that $p(x) \vdash (x - a_G) \in G$ and the overall choice of cosets is consistent. For pp-definable subgroups not in $\mathcal{F}_p$, the type $p(x)$ says that x is in a new coset.

Note that the trivial filter, the one containing (0), corresponds to realised types. The reason why we require the third condition in Definition 3.2.13

[7] The abbreviation "pp" stands for "positive primitive".

is that, if $\varphi(x)$ defines a subgroup G, and $\psi(x)$ defines $H < G$ with $G = \bigsqcup_{i<n} a_i H$, where $a_i \in M$, say, then $\vDash \forall x\ \big(\varphi(x) \leftrightarrow \bigvee_{i<n} \psi_i(x - a_i)\big)$, hence if $p(x) \in S_1(M)$ and $p(x) \vdash \varphi(x)$ there must be $i < n$ such that $p(x) \vdash \psi(x - a_i)$.

For types over arbitrary sets A, we need to take into account the fact that there may be pp-definable subgroups $H \leq G$ with $[G : H]$ finite, but A might not contain parameters to represent all cosets of H in G.

Definition 3.2.15. If $p \in S_1(A)$, define $\mathcal{F}_p$ to be the intersection

$$\mathcal{F}_p := \bigcap \{\mathcal{F}_q \mid p \subseteq q \in S_1(M), A \subseteq M \vDash T\}$$

The behaviour of forking is easily understood by using the fundamental order, or via dividing. Note that, if $p \subseteq q$ are 1-types, then q must specify cosets modulo at least as many subgroups as p, hence $\mathcal{F}_p \subseteq \mathcal{F}_q$.

Fact 3.2.16 ([Pre88, Theorem 5.3])**.** If $p \in S_1(M)$ and $p \subseteq q$, then q is a nonforking extension of p if and only if $\mathcal{F}_p = \mathcal{F}_q$.

Remark 3.2.17. Global 1-types with the same filter of subgroups $\mathcal{F}$ are domination-equivalent: just put in r, for every $G \notin \mathcal{F}$, the formula $x - y \in G$. By Example 3.2.22, the converse does not hold.

Proposition 3.2.18. In every module $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is generated by the equivalence classes of 1-types.

Proof. Consider $\operatorname{tp}_{xy}(a, b/\mathfrak{U})$, where $|a| = 1$. By induction, it is sufficient to show that it is equivalent to $p \otimes \operatorname{tp}(b/\mathfrak{U})$, for a suitable type p. Let $q_0(x) := \operatorname{tp}(a/\mathfrak{U})$ and $q_1(x) := \operatorname{tp}(a/\mathfrak{U}b)$, and let $\mathcal{F}_0$ and $\mathcal{F}_1$ be the respective filters of subgroups. If $\mathcal{F}_0 = \mathcal{F}_1$; then q_1 is the nonforking extension of q_0, and we may take $p = q_0$. Otherwise, we have $\mathcal{F}_0 \subsetneq \mathcal{F}_1$, i.e. q_1 is specifying cosets modulo more subgroups than q_0. In this case, let $p(w)$ be any 1-type over $\mathfrak{U}$ with filter of subgroups $\mathcal{F}_1$ and put in r formulas $(x - w) \in G$ for subgroups $G \notin \mathcal{F}_1$. □

By Fact 3.2.12 every 1-type $p(x)$ is domination-equivalent to a type that does not fork over $\emptyset$; namely, to the type $q(y)$ such that, for all $G \in \mathcal{F}_p$, we have $q(y) \vdash y \in G$, and for all $G \notin \mathcal{F}_p$ and $a \in \mathfrak{U}$ we have $q(y) \vdash (y - a) \notin G$. This has the following consequence.

Fact 3.2.19 ([Pre88, Corollary 6.21])**.** In every theory of a module, every type is domination-equivalent to a type that does not fork over $\emptyset$.

We conclude by looking at some $\mathbb{Z}$-modules, i.e. abelian groups. We will use repeatedly (and tacitly) the following classical result by Szmielew.

Fact 3.2.20 ([Hod93, Theorem A.2.2])**.** Let M be an abelian group and R an $\emptyset$-definable n-ary relation on M. Then R is a Boolean combination of formulas of the form $t(x) = 0$ or $q^m \mid t(x)$, where t is a term, q a prime, $m \in \mathbb{N}$, and $k \mid t(x)$ is a shorthand for $\exists y\ \sum_{i<k} y = t(x)$.

Hence, a 1-type $p(x)$ over $\emptyset$ is determined by the order of x and the powers of primes that divide it, in the sense above. This allows us to characterise the pp-definable subgroups of a given abelian group.

In what follows, q denotes a prime number, (k) the principal ideal that k generates in the ring of integers, $\mathbb{Z}_k$ the quotient $\mathbb{Z}/(k)$, regarded as an abelian group, $\mathbb{Z}_{q^\infty}$ the Prüfer group $\varinjlim \mathbb{Z}_{q^n}$, and $\mathbb{Z}_{(q)}$ the localisation of the integers on the prime ideal (q), again regarded as an abelian group.

Example 3.2.21. In the theory of the abelian group $\mathbb{Z}$ we have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$.

Proof. The only pp-definable subgroup of $\mathbb{Z}$ with infinite index is (0), hence we only have two filters of subgroups: the trivial one corresponds to realised types, and the other one corresponds to the type $p(x)$ of an element $x \notin \mathfrak{U}$ divisible by every integer. It follows that $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$, i.e. $\mathrm{Th}(\mathbb{Z})$ is unidimensional, by Corollary 3.1.30. □

In fact, this is a special case of a more general result. It is easy to see that every 1-type in $\mathrm{Th}(\mathbb{Z})$ has U-rank 1, and if a theory of modules has U-rank 1 then it is unidimensional by [Pre88, Corollary 7.8]. Note that, since for every prime q the pp-definable subgroups $q^m\mathfrak{U}$ form an infinite descending chain, this theory is not ω-stable, hence not $\aleph_1$-categorical.

Example 3.2.22. In the theory of the abelian group $\mathbb{Z}_{q^n}^{\aleph_0}$ we have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$.

Proof. In this case there is no proper pp-definable subgroup of finite index. There are $n+1$ pp-definable subgroups, linearly ordered by inclusion: for each $m \le n$, we have a subgroup G_m given by the elements of order at most q^m. In particular, there is no infinite descending chain of definable subgroups, and T is ω-stable. Filters coincide with upward-closed subsets, which in this case are just cones above points; for $m > 0$, let p_m be the type of an element of G_m in a new coset modulo G_{m-1}. Then $U(p_m) = m$ hence, if $n \ge 2$, there are types of U-rank higher than 1. Anyway, we still have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$ as, for instance, it is possible to show that $p_1 \sim_{\mathrm{D}} p_2$ by using multiplication by q. To see this, suppose that $a = q \cdot b$. If $b \vDash p_2$ then b must have order q^2, hence a must have order at most q. Moreover, $a \notin \mathfrak{U}$, because otherwise there is $d \in \mathfrak{U}$ such that $a = q \cdot d$, but then $q \cdot (b - d) = 0$, against the fact that the coset of b modulo G_1 is new. Conversely, suppose that $a \vDash p_1$. We see immediately from $a = q \cdot b$ that b has order at most q^2. If there was $d \in \mathfrak{U}$ such that $b - d$ had order at most q, then $0 = q \cdot (b - d) = a - q \cdot d$, contradicting that $a \notin \mathfrak{U}$. In conclusion, this theory too is unidimensional, hence $\aleph_1$-categorical, by ω-stability. □

Remark 3.2.23. In the example above, p_m is regular if and only if $m = 1$: types of U-rank 1 are always regular, but if $m > 1$ then $p_m \sim_{\mathrm{D}} p_{m-1}$, and in particular $p_m \not\perp^{\mathrm{w}} p_{m-1}$, even if $[\![p_{m-1}]\!]$ contains a forking extension of a restriction of p_m to an invariance base.

Example 3.2.24. In the theory of $\mathbb{Z}_{q^\infty}^{\aleph_0}$ we have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \mathbb{N}$.

Proof. The pp-definable subgroups form a chain of order type $\omega + 1$, and the situation is partially reminiscent of the one above: all filters corresponding to cones on an element of $\omega \setminus \{0\}$ give rise to equivalent types. Let p_∞ be the global 1-type corresponding to the filter containing only G. Since p_∞ has U-rank ω, it is regular by superstability and [Bue17, Lemma 6.3.3]. Let p be any nonrealised type with a different filter of subgroups. Then p is domination-equivalent to a forking extension of a restriction of p_∞ by Fact 3.2.16 and Remark 3.2.17. Since regular types are orthogonal to their forking extensions we have $p \perp p_\infty$. This exhausts the possibilities for the domination-equivalence class of a nonrealised 1-type, and therefore $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \mathbb{N}$. □

Example 3.2.25. In the theory of $\mathbb{Z}_{(q)}^{\aleph_0}$ we have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \mathbb{N}$.

Proof. The poset of pp-definable subgroups is a chain of order type $(\omega + 1)^*$, i.e. $\omega + 1$ with the reverse order, this time accounting for divisibility. There is a filter $\mathcal{F}_n$ corresponding to elements divisible by q^n but in a new coset modulo $q^{n+1}\mathfrak{U}$, a filter $\mathcal{F}_\omega$ corresponding to elements not in $\mathfrak{U}$ but divisible by all q^n, and the trivial filter $\mathcal{F}_{\omega+1}$ corresponding to realised types. By using multiplication by q, we see that types corresponding to $\mathcal{F}_n$ are domination-equivalent to every type corresponding to $\mathcal{F}_{n+1}$, for all $n \in \omega$.

Suppose now that, for instance, p_0 is the nonforking extension of the type over $\emptyset$ asserting that x is not divisible by q (so, a type corresponding to $\mathcal{F}_0$), and p_1 is a type of U-rank 1, e.g. the nonforking extension of the type over $\emptyset$ corresponding to $\mathcal{F}_\omega$. We check that $p_0 \perp^{\mathrm{w}} p_1$: since in this group no nonzero element has finite order, we only need to check divisibility conditions on terms $t(x, y)$, which are of the form $n \cdot x + m \cdot y + a$ for some $m, n \in \mathbb{Z}$ and $a \in \mathfrak{U}$, but these are clearly decided by $p_0(x) \cup p_1(y)$: for every $k \in \omega$ we have that y is divisible by q^k, hence $n \cdot x + m \cdot y + a$ is divisible by q^k if and only if $n \cdot x + a$ is. It follows that $p_0 \perp^{\mathrm{w}} p_1$, and therefore $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \mathbb{N}$. □

Note that this theory is not superstable, since there is an infinite descending sequence of definable subgroups, each of infinite index in the previous one. This also follows from the fact that, in $[\![p_0]\!]$, there are no regular types: if $p(x) \sim_{\mathrm{D}} p_0$ says that $q^n \mid x - a$ but for no $b \in \mathfrak{U}$ we have $q^{n+1} \mid x - b$ then, as we said above, p is domination-equivalent to the pushforward $(q \cdot -)_* p$, but the latter is domination-equivalent to a forking extension of a suitable restriction of p, since it specifies cosets for more subgroups.

Example 3.2.26. In the theory of $(\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(3)})^{\aleph_0}$ we have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}$.

Proof. By the Chinese Remainder Theorem and lack of torsion, a pp-definable subgroup different from (0) is determined by specifying the highest powers of 2 and 3 that divide all of its elements. Therefore, the options for proper filters of subgroups are given by tuples $(a, b) \in ((\omega + 1)^*)^2$, where for instance $[\![p_{(m,n)}]\!]$ specifies cosets for $2^m\mathfrak{U}$ and $3^n\mathfrak{U}$, but not for $2^{m+1}\mathfrak{U}$ and $3^{n+1}\mathfrak{U}$, and $[\![p_{(\omega,n)}]\!]$ specifies cosets for all $2^m\mathfrak{U}$, and for $3^n\mathfrak{U}$, but not for $3^{n+1}\mathfrak{U}$. By multiplying by a suitable power of 3 we get that for all $a, b \in \omega$ we have $[\![p_{(\omega,a)}]\!] = [\![p_{(\omega,b)}]\!]$, and similarly all types corresponding to a filter of the form (a, ω) are in the same class. By arguing as in the previous example, we see that $p_{(0,\omega)} \perp^{\mathrm{w}} p_{(\omega,0)}$, and that both are orthogonal to $p_{(\omega,\omega)}$.

We now prove that $p_{(0,0)}(x) \sim_{\mathrm{D}} p_{(0,\omega)}(y) \otimes p_{(\omega,0)}(z)$. Let r be a small type containing $(2 \mid x - y) \wedge (3 \mid x - z)$. It is immediate that $p_{(0,\omega)}(y) \otimes p_{(\omega,0)}(z) \cup r \vdash p_{(0,0)}(x)$. For the other direction, use the same r, and recall that the domain of r is assumed to be large enough to contain a base of $p_{(0,\omega)}$ and $p_{(\omega,0)}$; in particular, r specifies the $3^k\mathfrak{U}$-cosets of y and the $2^k\mathfrak{U}$-cosets of z for all k.

It follows that $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}^3$, with generators $[\![p_{(0,\omega)}]\!]$, $[\![p_{(\omega,0)}]\!]$, and $[\![p_{(\omega,\omega)}]\!]$. □

The argument above can be generalised to show that, if $q_0, \ldots, q_{s-1}$ are pairwise distinct primes, and we consider the theory of $\left(\bigoplus_{i<s} \mathbb{Z}_{(q_i)}\right)^{\aleph_0}$, then

$\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \mathbb{N}^{s+1}$. One generator corresponds to the type over $\emptyset$ of a nonzero element divisible by every power of every q_i, while each of the remaining generators corresponds to the type over $\emptyset$ of an element not divisible by a certain q_i, but divisible by q_j^m for all $j \neq i$ and $m \in \omega$.

Example 3.2.27. In the theory of $\mathbb{Z}^{\aleph_0}$ we have $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \prod^{\mathrm{b}}_{\aleph_0} \mathbb{N}$.

Proof. The situation is similar to the previous example, except now there are infinitely many pairwise orthogonal types, and $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \prod^{\mathrm{b}}_{\aleph_0} \mathbb{N}$, where the sequence $(1, 0, 0, 0, \dots)$ corresponds to an element divisible by every power of every prime and, say, $(0, 1, 0, 0, 0, \dots)$ corresponds to an element in a new coset modulo $2\mathfrak{U}$ but divisible by every power of every odd prime. The proof is similar to the case above and we omit it but, for the benefit of the unconvinced reader, we show that, if $p_0(x)$ is the type of an element divisible by every prime q, but in a new coset of every $q^2\mathfrak{U}$, and $p_1(y)$ is the type of an element in a new coset of every $q\mathfrak{U}$, then $p_0 \sim_{\mathrm{D}} p_1$ (both correspond to the sequence $(0, 1, 1, 1, \dots)$). Since no integer a is divisible by all primes, we cannot obtain this through a formula of the form $x = a \cdot y$, and clearly we cannot simultaneously use, say, $x = 2 \cdot y$ and $x = 3 \cdot y$. Instead of gluing x to a multiple of y, we content ourselves with gluing cosets, and take as r some small type containing all the formulas $q^2 \mid (x - q \cdot y)$, as q ranges among the primes. □

We invite the reader to compare these examples with the theories in Subsection 3.2.2, and note that being in the same coset of a definable subgroup is a definable equivalence relation. Anyway (see Remark 2.3.23), $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is not well-behaved with respect to reducts; due to definable bijections between cosets induced by translations, $\sim_{\mathrm{D}}$-classes are quite large.

3.2.4 The cycle-free pairing function

In our last stable example we will not reach a full computation of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$, yet we will still be able to point out some qualitative differences with all of the other domination monoids in this chapter. References for this subsection are [BP88, Bel99, Bel89], all of which build on results originally proven in [Mal71]. My understanding of this theory has benefited from several conversations with John Howe, whom I would like to thank.

Definition 3.2.28. Let L be the functional language $\{f, g, d\}$, with f binary and g, d unary. The theory T of the *cycle-free pairing function*, or of *locally free magmas with no indecomposables*, is axiomatised as follows.

1. The function f is a bijection.[8]

2. For all x we have $f(g(x), d(x)) = x$.

3. If $|x| = 1$ and $t(x, y)$ is some $\{f\}$-term containing the variable x, but not equal to the sole x, then $\forall x, y\ t(x, y) \neq x$ ("there are no cycles").

Fact 3.2.29 ([Bel99, Proposition 5.4, Theorem 5.1])**.** The theory of the cycle-free pairing function is complete and eliminates quantifiers in L.

Given an element $a \in M \models T$, we can think of $g(a)$ and $d(a)$ as, respectively, the "left part" and "right part"[9] of a. Taking further left and right parts generates a tree, and the "no cycles" axiom says that no element of M appears twice on the same branch of one of these trees. This allows us to describe a 1-type over A by specifying which points of the full binary tree are identified with each other or with an element of $\operatorname{dcl}(A)$. We now spell this out a bit more rigorously, and examine an example of 1-type.

Definition 3.2.30. Let $\mathcal{T}$ be the set of $\{g, d\}$-terms where no variable different from x appears. Consider $\mathcal{T}$ as a full binary tree via the following recursively defined bijection $\tau \colon 2^{<\omega} \to \mathcal{T}$. If ε is the empty string (the root of $2^{<\omega}$), then $\tau(\varepsilon) \coloneqq x$. Otherwise, let $\tau(\sigma \frown 0) \coloneqq g(\tau(\sigma))$, and let $\tau(\sigma \frown 1) \coloneqq d(\tau(\sigma))$.

The first few levels of this tree are depicted in Figure 3.1.

By quantifier elimination and the axioms of T, a 1-type $p(x)$ is determined by a consistent choice of which $\{g, d\}$-terms in $\mathcal{T}$ to realise in $\operatorname{dcl} A$, and which nonrealised $\{g, d\}$-terms in $\mathcal{T}$ to set equal to each other. By the "no cycles" axiom of T, whenever t_0 and t_1 lie in the same branch, $p(x) \vdash t_0(x) \neq t_1(x)$.

From this description of types, an easy type-counting argument, for which we refer the reader to [Bel99, Theorem 7.6], yields that T is strictly stable. Unsuperstability will nonetheless follow from Remark 3.2.32.

[8] "There are no indecomposables" is the statement that f is surjective. One can relax this condition to mere injectivity, but then in order to specify a completion one needs to say how many elements are *indecomposable*, i.e. lie outside of the image of f. See [Bel99, Theorem 5.4]

[9] The notation is borrowed from [BP88], which is written in French.

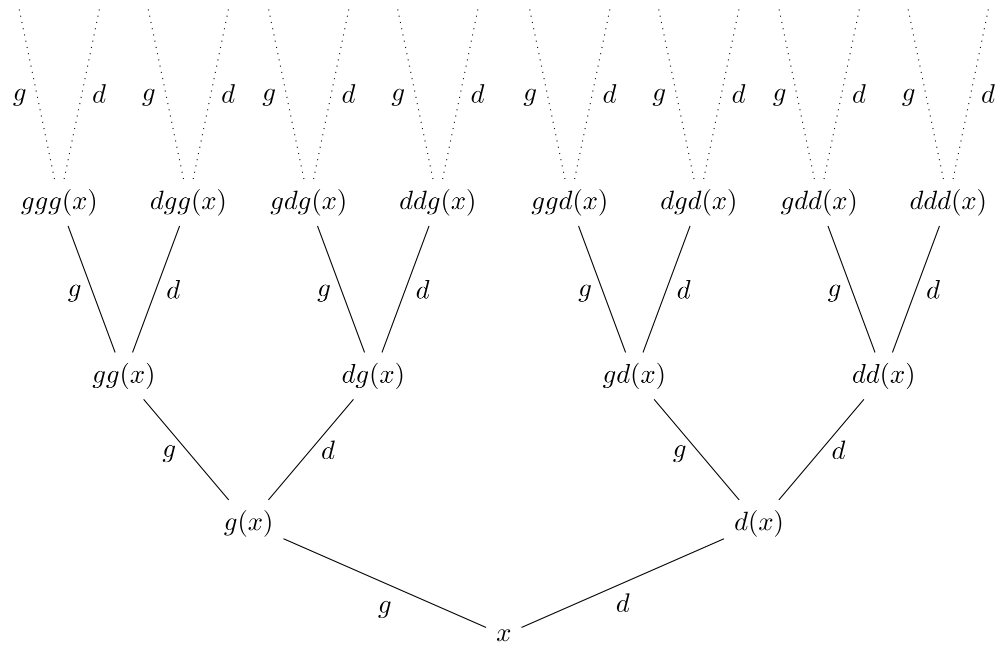


Figure 3.1: first levels of $\mathcal{T}$ in the theory of the cycle-free pairing function.

Example 3.2.31. Let $A = \operatorname{dcl}(A) \neq \emptyset$ and fix $a \in A$. Define $p(x) \in S_1(A)$ by declaring that, for all $b \in A$ and all distinct $\{g, d\}$-terms $t_0(y)$ and $t_1(y)$,

$$\begin{aligned} p(x) \vdash \big(d(x) = a\big) \wedge \big(gg(x) = dg(x)\big) \wedge \big(dgg(x) = g(a)\big) \\ \wedge \big(t_0(ggg(x)) \neq b\big) \wedge \big(t_0(ggg(x)) \neq t_1(ggg(x))\big) \end{aligned} \tag{3.1}$$

This type is depicted in Figure 3.2. Saying that $d(x) = a$ corresponds to realising the right son of the root of $\mathcal{T}$. Clearly, this determines the entire right half of $\mathcal{T} \setminus \{x\}$, i.e. the value of every $\{g, d\}$-term of the form $t(d(x))$; depending on the value of a, this may yield further identifications in the subtree above $d(x)$. Similar consequences follow from $dg(x) = gg(x)$, and from $dgg(x) = d(a)$. Since the last two conjuncts of (3.1) hold for all $b \in A$ and distinct terms t_0, t_1, no other identification is made; for instance, $g(x) = gg(x)$ is inconsistent because the two are on the same branch, and if $gg(x)$ was realised in $\operatorname{dcl}(A)$, then so would be $ggg(x)$.

Remark 3.2.32. In T, for every $n \in \omega \setminus \{0\}$ and every $p(x) \in S_n(\mathfrak{U})$ there is $q(y) \in S_1(\mathfrak{U})$ such that $p \equiv_{\mathrm{RK}} q$, hence $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is equal to the collection of the

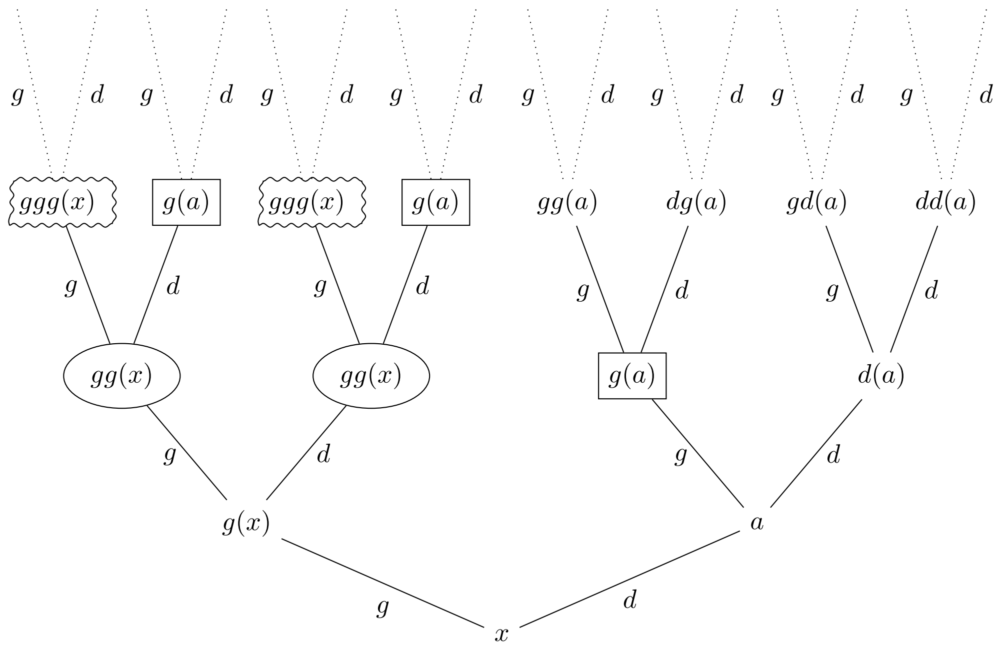


Figure 3.2: the 1-type $p(x)$ in Example 3.2.31. Points enclosed by the same shape are equal according to $p(x)$. Note how this induces equalities between corresponding descendants.

classes of 1-types. If $n \geq 2$, use one of the equivalent formulas below.

$$x_{n-1} = d^{n-1}(y) \wedge \bigwedge_{i<n-1} x_i = g(d^i(y))$$

$$y = f(x_0, f(x_1, f(x_2, f(\dots, f(x_{n-2}, x_{n-1} \underbrace{) \dots)))}_{n-1 \text{ parentheses}}$$

In particular, if p is the "generic" 1-type, not realising any $\{g, d\}$-term nor identifying any pair of them, then $p(x) \equiv_{\mathrm{RK}} p(y) \otimes p(z)$ can be seen by using the formula $x = f(y, z)$.

Hence this is a stable theory where $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ contains nonzero idempotents. This is not possible in a thin theory (so in particular T is not superstable) because a nonrealised idempotent type must have infinite weight. The converse of this, i.e. that every type with infinite weight is idempotent, is not true, as can be shown in the theory $T_{\aleph_0}$ from Subsection 3.2.2. The point is that an idempotent type has infinite weight *with respect to itself*; no type in $T_{\aleph_0}$ has this property.

Remark 3.2.33. It can be shown that $q \supseteq p$ is a nonforking extension of p if and only if the only points of $\mathcal{T}$ that q proves to be realised are those already realised according to p. In other words, q is a forking extension of p if and only if q realises more points of $\mathcal{T}$ than p does. It follows that a set $\{a_i \mid i \in I\}$ is independent over A if and only if for every $i \neq j$ we have that $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_j$. A deeper study is available in [BP88, Bel99, Bel89].

If instead of a binary function we look at a ternary one with similar properties, we find an example of a stable theory in which domination-equivalence and equidominance differ. This is another behaviour which is forbidden in a thin theory, by Proposition 3.1.26.

Example 3.2.34 (essentially [Wag00, Example 5.2.9])**.** Let T be the theory of a cycle-free bijection $f\colon M^3 \to M$. Analogous results hold ([Bel99, Bel89] work in this case too, and even in more general situations), except now trees are ternary, i.e. $\mathcal{T} \cong 3^{<\omega}$. Let $p(x)$ be the "generic" 1-type, neither realising nor identifying anything. Then $p \sim_{\mathrm{D}} p^{(2)}$, but $p \not\equiv_{\mathrm{D}} p^{(2)}$

Proof. Clearly $p^{(2)} \geq_{\mathrm{D}} p$, and $p(x) \geq_{\mathrm{D}} p(y) \otimes p(z)$ can be witnessed by the formula $\exists w\ x = f(y, z, w)$. Hence $p \sim_{\mathrm{D}} p^{(2)}$, and we need to show $p \not\equiv_{\mathrm{D}} p^{(2)}$.

By Proposition 3.1.8 we may argue via forking (see Remark 3.2.33), so suppose that $a \vDash p$ and $(b, c) \vDash p^{(2)}$ witness equidominance in the sense of Definition 3.1.5. In other words, we are assuming towards a contradiction that, for every e, we have $bc \mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} e \iff a \mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} e$. In particular, from $bc \not\mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} bc$, we get $a \not\mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} bc$, hence we must be able to find a as a $\{f\}$-term involving one of b, c, or the other way around.

If for some e we have $a = f(b, c, e)$, then $bc \mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} e$, because p says that the three elements immediately above the root in his tree are independent, but clearly $a \not\mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} e$. One can use a similar argument if $a = t(b, c, h)$ (even if it mentions just one of b, c) for t a term in $\{f\}$ and h a tuple: take as e a suitable point not in the tree generated by b, nor in the tree generated by c, but still in the tree generated by a: such an e exists by definition of p.

So a is one of the elements in the tree of either b or c, say $b = t(a, e)$, for e some tuple and t some $\{f\}$-term. For simplicity, suppose that $b = f(a, e_0, e_1)$. Then $a \mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} e$ but $bc \not\mathop{\smile\!\!\!\!\!\!|}_{\mathfrak{U}} e$. □

Note that anyway, analogously to Remark 3.2.32, in this theory $p \equiv_{\mathrm{D}} p^{(3)}$.

Chapter 4

The o-minimal case

In this chapter T is an o-minimal theory. O-minimality was introduced in [PS86]; another standard reference is the book [vdD98]. We will assume that the reader is acquainted with results such as the Monotonicity Theorem, dcl-independence and its properties, and definable choice for o-minimal expansions of ordered groups and its consequences.

The main contributions of this chapter are two, one for each section. The first consists of Theorem 4.1.27, which reduces the problem of studying $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in a given o-minimal theory to completing two tasks: showing that each invariant type is domination-equivalent to a product of invariant 1-types, and identifying a maximal set of pairwise weakly orthogonal invariant 1-types. The general strategy is inspired by [HHM08], but the present approach is more general, and allows for a uniform treatment of different o-minimal theories.

In the second section we apply these techniques to specific theories. In particular, we fill a gap in the computation of $\overline{\mathrm{Inv}}(\mathfrak{U})$ in DOAG from [HHM08] and, more interestingly, we compute $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in the theory RCF of real closed fields. Besides the intrinsic interest, we will see in Subsection 5.2.3 that, using [EHM19], this also settles the study of the domination monoid for the weakly o-minimal theory RCVF of real closed valued fields.

For some results, if $M_0 \prec^+ \mathfrak{U}$, we want to be able to take some kind of completion M of M_0, whose size can be bounded by $\beth_n(|M|)$ for some $n \in \omega$, and then find N, N_1 such that $M \prec^+ N \prec^+ N_1 \prec^+ \mathfrak{U}$. The technical reasons why we are able to do this are in Subsection 2.1.1.

4.1 Reducing to generation by 1-types

In this section T denotes an arbitrary o-minimal theory. Some of the technical facts we will use are standard, follow from lemmas scattered across the literature, or are variations thereof. In order to be as self-contained as possible, we include proofs.

4.1.1 Preliminaries

Definition 4.1.1. If $p(x) \in S_1(\mathfrak{U})$, recall the definitions of L_p and R_p from Example 2.2.22. If the cofinality of L_p is small we will say that p has *small cofinality on the left*, and if the coinitiality of R_p is small that p has *small cofinality on the right*.

From now on, we will freely use the following.

Remark 4.1.2. In every o-minimal theory, the following hold.

1. Every 1-type $p(x) \in S_1(A)$ is determined by a cut in $\operatorname{dcl}(A)$, since it is enough to specify to which A-definable sets x belongs, and by o-minimality these are unions of points of $\operatorname{dcl}(A)$ and intervals with extremes in $\operatorname{dcl}(A) \cup \{\pm\infty\}$.
2. By saturation of $\mathfrak{U}$, a global 1-type has small cofinality simultaneously on the left and on the right if and only if it is realised, and it has neither if and only if it is not invariant. Moreover, $p \in S_1(\mathfrak{U})$ is M-invariant if and only if M contains a set cofinal in L_p, or a set coinitial in R_p.

Remark 4.1.3. If $p \in S_1^{\text{inv}}(\mathfrak{U})$ is nonrealised and L_p has a maximum, R_p has a minimum, or one of them is empty, then p is definable, while if none of the previous three hold then p is finitely satisfiable.

So every 1-type is definable or finitely satisfiable. By [Sim14a], this is true for types of dp-rank 1 in every NIP theory, and in particular for 1-types in *dp-minimal* theories, which include for example ACVF (see Subsection 5.2.3). Note that the two options are not in general mutually exclusive, e.g. every global type in a stable theory is simultaneously definable and finitely satisfiable by Fact 3.1.2. This cannot happen in an o-minimal theory, except in the case of realised types, by [Sim13, Proposition 2.28]. See [MS94, Theorem 2.1] for a characterisation of definable types in o-minimal theories.

The following lemma and corollary are essentially [PP07, Lemma 6.1].

Lemma 4.1.4. Let $p \in S_1(A)$ and f be A-definable. If $f_*p = p$, then f is strictly increasing on p, i.e. $p(x) \cup p(y) \cup \{x < y\} \vdash f(x) < f(y)$.

Proof. By the Monotonicity Theorem, f is either strictly increasing, strictly decreasing, or constant on p. If f is constant on p then p is realised in $\operatorname{dcl}(A)$, and $f_*p = p$ yields $p(x) \vdash f(x) = x$. Hence, we only need to exclude that f is strictly decreasing on p. Assume towards a contradiction this is the case.

Let $a \vDash p$, and suppose $f(a) \le a$. By assumption $f^{-1}(a)$ satisfies p, which proves $f(x) \le x$, and so $a = f(f^{-1}(a)) \le f^{-1}(a)$. Since f is strictly decreasing on p if and only if f^{-1} is, by replacing f with f^{-1} we may assume $f(a) \ge a$.

Since f^{-1} is strictly decreasing on p, from $f(a) \ge a$ we get $a \le f^{-1}(a)$. But, similarly to what we did in the previous paragraph, $f^{-1}(a) \vDash p(x) \vdash f(x) \ge x$, so $a = f(f^{-1}(a)) \ge f^{-1}(a) \ge a$. But then $p(x) \vdash f(x) = x$, contradicting that f is decreasing on p. □

Corollary 4.1.5. Suppose that $a, b \vDash p \in S_1(A)$. Then either $p(\operatorname{dcl}(Aa))$ and $p(\operatorname{dcl}(Ab))$ are cofinal and coinitial in each other, or one of them lies entirely to the left of the other.

Proof. If none lies entirely to the left of the other, we can find without loss of generality $a^0 \le b^0 \le a^1$, where $b^0 \in p(\operatorname{dcl}(Ab))$ and $a^i \in p(\operatorname{dcl}(Aa))$. If p is realised in $\operatorname{dcl}(A)$ we are done, so assume both a^i are in $\operatorname{dcl}(Aa) \setminus \operatorname{dcl}(A)$. By exchange, there is an A-definable f such that $f(a^0) = a^1$, which is increasing by Lemma 4.1.4, so $f(b^0) \ge f(a^0) = a^1$. Since this argument works with arbitrarily large $a^1 \in p(\operatorname{dcl}(Aa))$, this proves cofinality of $p(\operatorname{dcl}(Ab))$ in $p(\operatorname{dcl}(Aa))$. For coinitiality, argue symmetrically. Since $b_0 \le a_1 \le f(b_0)$, the same argument yields cofinality and coinitiality of $p(\operatorname{dcl}(Aa))$ in $p(\operatorname{dcl}(Ab))$. □

Lemma 4.1.6. Let $M \prec^+ N \prec^+ \mathfrak{U}$. Suppose $p \in S_n^{\mathrm{inv}}(\mathfrak{U}, M)$ and b is a tuple such that $\operatorname{tp}(b/\mathfrak{U})$ is M-invariant. If p is realised in $\operatorname{dcl}(\mathfrak{U}b)$, then it is realised in $\operatorname{dcl}(Nb)$ as well.

Proof. Suppose that for some M-definable function $f(y, u)$ and $d \in \mathfrak{U}$ we have $f(b, d) \vDash p$. Let $\tilde{d} \in N$ be such that $\tilde{d} \equiv_M d$. Let $\varphi(z; w) \in L(M)$ and $e \in \mathfrak{U}$ be such that $\varphi(z; e) \in p$. We want to show that $f(b, \tilde{d}) \vDash \varphi(z; e)$. Let $h \in \operatorname{Aut}(\mathfrak{U}/M)$ be such that $h(d) = \tilde{d}$. By M-invariance, $\varphi(z; h^{-1}(e)) \in p$. Therefore $f(b, d) \vDash \varphi(z; h^{-1}(e))$, hence $b \vDash \varphi(f(y, d); h^{-1}(e))$. By applying h and using that $\operatorname{tp}(b/\mathfrak{U})$ is M-invariant, it follows that $b \vDash \varphi(f(y, \tilde{d}); e)$, hence $f(b, \tilde{d}) \vDash \varphi(z, e)$. □

This can be improved for points that are actually named. We will use the following lemma tacitly to assume independence of a tuple without changing the invariance base. The notation $p(x,y) \vdash x \in \operatorname{dcl}(Ay)$ means "there is an A-definable function f such that $p \vdash x = f(y)$".

Lemma 4.1.7. Let $p(x) \in S_n^{\mathrm{inv}}(\mathfrak{U}, M)$. If $p(x) \vdash x_0 \in \operatorname{dcl}(\mathfrak{U}x_1, \ldots, x_{n-1})$, then $p(x) \vdash x_0 \in \operatorname{dcl}(Mx_1, \ldots, x_{n-1})$.

Proof. Let $p(x) \vdash x_0 = f(x_1, \ldots, x_{n-1}, d)$, where $f(x_1, \ldots, x_{n-1}, w)$ is M-definable. Up to changing f, we may assume that d is M-independent. If $|d| = 0$ we are done. Inductively, assume that the conclusion holds for $|d| \le k$, let $|d| = k+1$, and let $\tilde{d}$ be d with d_k replaced by some different $\tilde{d}_k \equiv_{Md_{<k}} d_k$. Let $b \vDash p$; by M-invariance of p we have $\tilde{d}_k \equiv_{Mbd_{<k}} d_k$. Again by M-invariance, $p(x) \vdash x_0 = f(x_1, \ldots, x_{n-1}, \tilde{d})$, therefore $f(b_1, \ldots, b_{n-1}, \tilde{d}) = b_0 = f(b_1, \ldots, b_{n-1}, d)$. Since $\tilde{d}_k \equiv_{Mbd_{<k}} d_k$, by the Monotonicity Theorem there is a $Mb_{>0}d_{<k}$-definable set, say defined by $\varphi(b_{>0}, d_{<k}, w_k)$, which contains d_k, hence also $\tilde{d}_k$, and where the function $f(b_{>0}, d_{<k}, w_k)$ is constant in the last coordinate. Therefore

$$p(x) \vdash (\exists w_k\ \varphi(x_{>0}, d_{<k}, w_k)) \land (\forall w_k\ \varphi(x_{>0}, d_{<k}, w_k) \to x_0 = f(x_{>0}, d_{<k}, w_k))$$

It follows that $p(x) \vdash x_0 \in \operatorname{dcl}(Mx_1, \ldots, x_{n-1}, d_0, \ldots, d_{k-1})$, and we conclude by applying the induction hypothesis. □

4.1.2 The Idempotency Lemma

This subsection is dedicated to the proof of this section's main lemma, namely the Idempotency Lemma 4.1.9. As its name suggests, its principal consequence is that every 1-type is idempotent modulo equidominance. Nevertheless, this lemma will also find some technical use in certain proofs. A precursor of this result, dealing with definable types only, is [Sta08, Claim 2.4], itself using [Tre04, Lemma].

Notation 4.1.8. For sets X, R, let $X_{<R} \coloneqq \{x \in X \mid \forall r \in R\ x < r\}$.

Lemma 4.1.9 (Idempotency Lemma)**.** Let $M \prec^+ N \preceq \mathfrak{U}$. For all $p(x) \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$ and $b^0 \vDash p$ the set $p(\operatorname{dcl}(Nb^0))$ is cofinal and coinitial in $p(\operatorname{dcl}(\mathfrak{U}b^0))$.

Proof. Without loss of generality, p is not realised. We deal with the case where p has small cofinality on the right, the other case being symmetrical.

The bulk of this proof consists in showing that $p(\mathrm{dcl}(Nb^0))$ is cofinal in $p(\mathfrak{U}b^0)$. Let $R \subseteq M$ be coinitial in R_p.

Assume towards a contradiction that there are an M-definable function $f(t,w)$ and a tuple $d \in \mathfrak{U}$ such that $p(\mathrm{dcl}(Nb^0)) < f(b^0,d) < R$. Note that $p(x) \vdash$ "$f(x,d) \models p$", and in particular by Lemma 4.1.4 the function $f(t,d)$ is strictly increasing on p. Moreover, up to changing $f(t,w)$, we may assume that d is M-independent, hence so is $b^0 d$. For $i \geq 0$, define inductively $b^{i+1} := f(b^i,d)$. The core of the proof consists in justifying the claim below.

Claim. For every $\ell \in \omega$, we have $b^{\ell+1} \models p \mid \mathrm{dcl}(Nb^0 \dots b^\ell)$.

Note that, by Remark 4.1.2 and the definitions of invariant extension and $\otimes$, the Claim is equivalent to saying that $p(\mathrm{dcl}(Nb^0 \dots b^\ell)) < b^{\ell+1} < R$, or that $(b^0, \dots, b^{\ell+1}) \models p^{(\ell+2)} \upharpoonright N$.

Proof of Claim. We argue by induction, the case $\ell = 0$ holding by assumption. Assume that the Claim holds for $\ell - 1$, i.e. $\mathrm{dcl}(Nb^0 \dots b^{\ell-1})_{<R} < b^\ell < R$. Since b^1 satisfies p as well, if we apply the inductive hypothesis starting with b^1 instead of b^0, we obtain that $\mathrm{dcl}(Nb^1 \dots b^\ell)_{<R} < b^{\ell+1} < R$. What we need to show is that $\mathrm{dcl}(Nb^0 \dots b^\ell)_{<R} < b^{\ell+1} < R$. Let $h(u_0, \dots, u_\ell, v)$ be an M-definable function, let $n \in N$ be a without loss of generality M-independent tuple (up to changing h), and suppose that $h(b^0, \dots, b^\ell, n) \models p$. In particular, $h(b^0, \dots, b^\ell, n) < R$, and we need to show that $b^{\ell+1} > h(b^0, \dots, b^\ell, n)$.

Let Y be the set of realisations of $\mathrm{tp}(b^0/Mb^1 \dots b^\ell n)$ in a larger monster model. Since $b^0 \dots b^\ell n$ is M-independent, by the Monotonicity Theorem $h(u_0, b^1, \dots, b^\ell, n)$ is either strictly increasing, strictly decreasing, or constant in u_0 on Y. In the last case, $h(b^0, \dots, b^\ell, n) \in \mathrm{dcl}(Nb^1 \dots b^\ell)_{<R}$, so $b^{\ell+1} > h(b^0, \dots, b^\ell, n)$ holds by inductive hypothesis.

Suppose now that $h(u_0, b^1, \dots, b^\ell, n)$ is strictly decreasing in u_0 on Y. Let $b^{-1} \in N$ be such that $b^{-1} \models p \upharpoonright Mn$. By associativity of $\otimes$ and inductive hypothesis $(b^1, \dots, b^\ell) \models p^{(\ell)} \mid \mathrm{dcl}(Nb^0)$, hence $(b^{-1}, b^1, \dots, b^\ell) \equiv_{Mn} (b^0, \dots, b^\ell)$ because, using the inductive hypothesis again, both tuples have type $p^{(\ell+1)} \upharpoonright Mn$. This implies that $h(b^{-1}, b^1, \dots, b^\ell, n) < R$, and that $b^{-1} \in Y$. Since $p \vdash$ "$x > (p \upharpoonright Mn)(\mathfrak{U})$", we have $b^0 > b^{-1}$, and we get

$$h(b^0, \dots, b^\ell, n) < h(b^{-1}, b^1, \dots, b^\ell, n) \in \mathrm{dcl}(Nb^1 \dots b^\ell)_{<R} \tag{4.1}$$

and it follows that $b^{\ell+1} > h(b^0, \dots, b^\ell, n)$.

If instead $h(u_0, b^1, \dots, b^\ell, n)$ is strictly increasing in u_0 on Y, let $\tilde{d} \in N$ be such that $\tilde{d} \equiv_{Mn} d$. Let $b^\varepsilon := f(b^0, \tilde{d})$. Since p is M-invariant, from $p \vdash$ "$f(x,d) \vDash p \restriction Mn$" we obtain $b^\varepsilon \vDash p \restriction Mn$, and as $b^\varepsilon \in \operatorname{dcl}(Nb^0)_{<R}$ we have $b^1 > b^\varepsilon$. Since p is invariant, from $p \vdash f(x,d) > x$ we obtain $p \vdash f(x, \tilde{d}) > x$, hence we have $b^0 < b^\varepsilon < b^1$. In particular both b^0, b^ε satisfy p. It follows that $(f(t,\tilde{d}))_* p = p$, and by Lemma 4.1.4 $f(t,\tilde{d})$ is strictly increasing on p; let $g(t,\tilde{d})$ be its inverse. As $g(t,\tilde{d})$ must also be strictly increasing, we have that $b^{1-\varepsilon} := g(b^1, \tilde{d}) > g(b^\varepsilon, \tilde{d}) = b^0$. Since p is M-invariant and proves that $g(x,\tilde{d})$ is the inverse of $f(x,\tilde{d})$, it also proves that $g(x,d)$ is the inverse of $f(x,d)$. Using invariance of p one more time we obtain $(g(b^1,\tilde{d}), b^1) \equiv_{Mn} (g(b^1,d), b^1)$, or in other words $(b^{1-\varepsilon}, b^1) \equiv_{Mn} (b^0, b^1)$. Moreover, by inductive hypothesis $(b^2, \dots, b^\ell) \vDash p^{(\ell-1)} \mid Nb^0b^1$, and since $b^{1-\varepsilon} \in \operatorname{dcl}(Nb^1)_{<R}$ and $\otimes$ is associative, $(b^{1-\varepsilon}, b^1, \dots, b^\ell) \vDash p^{(\ell+1)} \restriction Mn$. Again by inductive hypothesis, $(b^0, b^1, \dots, b^\ell) \vDash p^{(\ell+1)} \restriction Mn$ as well, therefore $(b^{1-\varepsilon}, b^1, \dots, b^\ell) \equiv_{Mn} (b^0, b^1, \dots, b^\ell)$. This implies that $h(b^{1-\varepsilon}, b^1, \dots, b^\ell, n) < R$, and that $b^{1-\varepsilon} \in Y$. Since $b^0 < b^{1-\varepsilon}$ we have

$$h(b^0, \dots, b^\ell, n) < h(b^{1-\varepsilon}, b^1, \dots, b^\ell, n) \in \operatorname{dcl}(Nb^1 \dots b^\ell)_{<R}$$

So $b^{\ell+1} > h(b^0, \dots, b^\ell, n)$, and we are done. □ CLAIM

Let $\ell := |d|$. By definition, we have $b^1, \dots, b^\ell \in \operatorname{dcl}(Mdb^0)$. Moreover, since by the Claim $(b^0, b^1, \dots, b^\ell) \vDash p^{(\ell+1)} \restriction M$, the b^i form an M-independent tuple, hence by exchange $d \in \operatorname{dcl}(Mb^0 \dots b^\ell)^{|d|}$. But then we have $b^{\ell+1} = f(b^\ell, d) \in \operatorname{dcl}(Mb^0 \dots b^\ell)$, in contradiction with $b^{\ell+1} \vDash p \mid \operatorname{dcl}(Nb^0 \dots b^\ell)$. This completes the proof that $p(\operatorname{dcl}(Nb^0))$ is cofinal in $p(\operatorname{dcl}(\mathfrak{U}b^0))$.

Finally, if for some other M-definable function $f(t,w)$ the point $b^1 := f(b^0, d) \vDash p$ witnesses that $p(Nb^0)$ is not coinitial in $p(\operatorname{dcl}(\mathfrak{U}b^0))$, i.e. $b^1 < p(Nb^0)$, then by Corollary 4.1.5 $p(\operatorname{dcl}(Nb^1)) < p(\operatorname{dcl}(Nb^0))$. Once again by Lemma 4.1.4, the function $f(t,d)$ is strictly increasing on p, hence it has an inverse, but then we obtain $b^0 = f(t,d)^{-1}(b^1) > p(\operatorname{dcl}(Nb^1))$, which contradicts cofinality of $p(\operatorname{dcl}(Nb^1))$ in $p(\operatorname{dcl}(\mathfrak{U}b^1))$. □

Corollary 4.1.10. In every o-minimal theory, every 1-type is idempotent modulo domination-equivalence, and even modulo equidominance.

Proof. Consider $p(y^1) \otimes p(y^0)$, where p is M-invariant and without loss of generality nonrealised, say with small cofinality on the right, so $p^{(2)} \vdash y^1 > y^0$.

Let $R \subseteq M$ be coinitial in R_p and fix N such that $M \prec^+ N \prec^+ \mathfrak{U}$. By the Idempotency Lemma 4.1.9, any $r(x,y) \in S_{p,p^{(2)}}(N)$ extending $\{x = y_0\} \cup$ "$\operatorname{dcl}_{<R}(Ny^0) < y^1 < R$" witnesses equidominance. □

4.1.3 Weak orthogonality

In this subsection we summarise some things about weak orthogonality which we will need later. The first lemma is a standard o-minimal fact, while for distality we refer the reader to [Sim13] or [Sim15, Chapter 9].

Lemma 4.1.11. If T is o-minimal and $p, q \in S_1(M)$ are nonrealised, the following are equivalent.

1. $p \not\perp^{\mathrm{w}} q$.
2. There is an M-definable function f such that $q = f_* p$.

Moreover, every f as in 2 must be a bijection on p.

Proof. If 2 holds, since q is not realised, the formulas $y = f(x)$ and $y \neq f(x)$ witness that $p \cup q$ has more than one completion, hence $2 \Rightarrow 1$. To prove $1 \Rightarrow 2$, let $a \vDash p$ and $b \vDash q$. If there is no such f then $q(M) = q(\operatorname{dcl}(Ma))$, hence the cut of b in M determines the cut of b in $\operatorname{dcl}(Ma)$, hence $p \cup q$ is complete by Remark 4.1.2. For the "moreover" part note that, since q is nonrealised, f must be a bijection on p by the Monotonicity Theorem. □

Lemma 4.1.12. Let T be o-minimal and $M \prec^+ N \prec^+ \mathfrak{U}$. For every pair of nonrealised $p, q \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$ the following are equivalent.

1. $p \not\perp^{\mathrm{w}} q$.
2. $p \equiv_{\mathrm{D}} q$.
3. $p \sim_{\mathrm{D}} q$.
4. There is a $\mathfrak{U}$-definable function f such that $q = f_* p$.
5. There is an N-definable bijection f on p such that $q = f_* p$.

Proof. As remarked in Example 2.1.15, point 3, we have $5 \Rightarrow 2$, while $2 \Rightarrow 3$ is trivial. Corollary 2.3.17 implies $3 \Rightarrow 1$, and Lemma 4.1.11 implies $1 \Leftrightarrow 4$. The implication $5 \Rightarrow 4$ is also trivial; to see that $4 \Rightarrow 5$, use Lemma 4.1.6 to show that f may be chosen N-definable, and observe that f must be a bijection on p by Lemma 4.1.11. □

Definition 4.1.13. A theory is *distal* iff it is NIP and whenever $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ we have $p \otimes q = q \otimes p \iff p \perp^{\mathrm{w}} q$.

Note that right to left holds in every theory by Remark 2.3.15. The above is not the original definition, but is equivalent to it by [Sim13, Lemma 2.18].

Fact 4.1.14 ([Sim13, Corollary 2.30])**.** O-minimal theories are distal.

Proof sketch. In order for $p(x)$ and $q(y)$ not to be weakly orthogonal, there must be $a \vDash p$, $b \vDash q$, and a nonrealised cut in $\mathfrak{U}$ filled by both an element of $\operatorname{dcl}(\mathfrak{U}a)$, say $f(a)$, and an element of $\operatorname{dcl}(\mathfrak{U}b)$, say $g(b)$, where $f(x)$ and $g(y)$ are definable functions. This cut must be invariant by Lemma 2.1.12. If it has small cofinality on the right, then $p(x) \otimes q(y) \vdash f(x) > g(y)$, while $q(y) \otimes p(x) \vdash f(x) < g(y)$. If it has small cofinality on the left, then $p(x) \otimes q(y) \vdash f(x) < g(y)$, while $q(y) \otimes p(x) \vdash f(x) > g(y)$ □

Lemma 4.1.15. Let T be distal. If $q_0 \perp^{\mathrm{w}} p$ and $q_1 \perp^{\mathrm{w}} p$, then $q_0 \otimes q_1 \perp^{\mathrm{w}} p$.

Proof. If both q_i commute with p then $q_0 \otimes q_1$ commutes with p. □

Corollary 4.1.16. If T is distal, $p \perp^{\mathrm{w}} q$, and $n, m \in \omega$, then $p^{(n)} \perp^{\mathrm{w}} q^{(m)}$.

Proof. By induction on n, obtain $p^{(n)} \perp^{\mathrm{w}} q$. Conclude by induction on m. □

Question 4.1.17. Let T be arbitrary, and suppose that $p_0 \perp^{\mathrm{w}} q$ and $p_1 \perp^{\mathrm{w}} q$. Is it true that $p_0 \otimes p_1 \perp^{\mathrm{w}} q$? What if we also assume NIP?

We saw that the answer is positive in distal theories. By Fact 3.1.12 and Fact 3.1.13, this is also the case in stable ones. We need one more fact about NIP theories; under distality, it also follows from [Wal19, Proposition 3.25].

Fact 4.1.18 ([Sim14b, Corollary 4.7])**.** Let T be NIP and $\{p_i \mid i \in I\}$ be a family of types $p_i \in S^{\mathrm{inv}}(\mathfrak{U})$ such that if $i \neq j$ then $p_i \perp^{\mathrm{w}} p_j$. Then $\bigcup_{i \in I} p_i(x^i)$ is complete.

4.1.4 The reduction

We can now move to the final steps in proving the main theorem of this section, i.e. the characterisation of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ assuming the statement below.

Assumption 4.1.19. Every invariant type is equidominant to a product of invariant 1-types.

Remark 4.1.20. Assumption 4.1.19 has a weaker variant, replacing $\equiv_D$ with $\sim_D$. The proofs that follow also show that the weaker assumption is enough to prove weaker versions of the results, where all mentions of $\equiv_D$ are replaced by $\sim_D$. I have not made this explicit for readability.

Remark 4.1.21. By Lemma 4.1.12 and the fact that realised types are weakly orthogonal to every type, for a sequence $(q_i \mid i \in I)$ of nonrealised invariant 1-types the following are equivalent.

1. The sequence $(q_i \mid i \in I)$ is a a maximal sequence of pairwise weakly orthogonal invariant 1-types.
2. The sequence $(q_i \mid i \in I)$ is a sequence of representatives for the $\equiv_D$-classes of nonrealised invariant 1-types.
3. The sequence $(q_i \mid i \in I)$ is a sequence of representatives for the $\sim_D$-classes of nonrealised invariant 1-types.

Definition 4.1.22. Fix a maximal sequence $(q_i \mid i \in I)$ of pairwise weakly orthogonal invariant 1-types. For $p \in S^{\text{inv}}(\mathfrak{U})$, define

$$I_p := \{i \in I \mid p \geq_D q_i\}$$

Proposition 4.1.23. Let T be o-minimal satisfying Assumption 4.1.19, and let $p \in S^{\text{inv}}(\mathfrak{U})$. Then I_p is finite and, if $p' \in S^{\text{inv}}(\mathfrak{U})$, the following hold.

1. The following are equivalent. (a) $p \equiv_D p'$. (b) $p \sim_D p'$. (c) p and p' dominate the same 1-types. (d) $I_p = I_{p'}$.
2. The following are equivalent. (a) $p \geq_D p'$. (b) For every $q \in S_1^{\text{inv}}(\mathfrak{U})$, if $p' \geq_D q$ then $p \geq_D q$. (c) $I_p \supseteq I_{p'}$.

Proof. In what follows, we will freely use Lemma 4.1.15 and that two types commute if and only if they are weakly orthogonal. Let $p \in S^{\text{inv}}(\mathfrak{U})$ be nonrealised. By Assumption 4.1.19, we can write $p \equiv_D p_0 \otimes \ldots \otimes p_m$, where the p_j are nonrealised invariant 1-types. By Lemma 4.1.12, Lemma 4.1.15, and Theorem 2.3.16, p is orthogonal to every $\equiv_D$-class which is not one of the $[\![p_i]\!]$; therefore the set I_p must be finite. Moreover, since different q_i are orthogonal, they commute, hence products of the form $q_{i_0} \otimes \ldots \otimes q_{i_n}$ with pairwise distinct indices do not depend on the indexing. Suppose that $I_p = \{q_{i_0}, \ldots, q_{i_n}\}$ has size $n+1$; we prove that $p \equiv_D q_{i_0} \otimes \ldots \otimes q_{i_n}$ by induction on m. From this, it follows easily that (1d) $\Rightarrow$ (1a) and that (2c) $\Rightarrow$ (2a). Since, in both points of the conclusion, each property trivially implies the one on its right, this suffices.

If $m = 0$, then $p_0 \equiv_{\mathrm{D}} q_{i_0}$, because otherwise by Lemma 4.1.12 $p = p_0 \perp^{\mathrm{w}} q_{i_0}$ and $p \geq_{\mathrm{D}} q_{i_0}$, so q_{i_0} is realised by Corollary 2.3.17, which is absurd.

If $m > 0$, let us focus on p_m. By distality and Lemma 4.1.12, 1-types that do not commute with p_m commute with every type that commutes with p_m. Therefore, by swapping some types in $p_0 \otimes \ldots \otimes p_{m-1}$, we may assume assume that, for some $k < m$, no pair of types from $p_{k+1} \ldots, p_m$ commutes, but that each of $p_{k+1} \ldots, p_m$ commutes with each p_j for $j \leq k$, and by inductive hypothesis $p_0 \otimes \ldots \otimes p_k \equiv_{\mathrm{D}} q_{j_0} \otimes \ldots \otimes q_{j_\ell}$. By Lemma 2.1.19,

$$p = p_0 \otimes \ldots \otimes p_k \otimes p_{k+1} \otimes \ldots \otimes p_m \equiv_{\mathrm{D}} q_{j_0} \otimes \ldots \otimes q_{j_\ell} \otimes p_{k+1} \otimes \ldots \otimes p_m$$

Note that $p_0 \otimes \ldots \otimes p_k \perp^{\mathrm{w}} p_{k+1} \otimes \ldots \otimes p_m$, hence $q_{j_0} \otimes \ldots \otimes q_{j_\ell} \perp^{\mathrm{w}} p_{k+1} \otimes \ldots \otimes p_m$ by Theorem 2.3.16. By maximality of $(q_i \mid i \in I)$ there is $\bar{\imath} \in I$ such that $q_{\bar{\imath}} \equiv_{\mathrm{D}} p_m$. Since for all $j \geq k+1$ we have $p_m \not\perp^{\mathrm{w}} p_j$, by Lemma 4.1.12 and Corollary 2.2.8 we obtain $p_{k+1} \otimes \ldots \otimes p_m \equiv_{\mathrm{D}} q_{\bar{\imath}}^{(m-k)}$, and by Corollary 4.1.10 $q_{\bar{\imath}}^{(m-k)} \equiv_{\mathrm{D}} q_{\bar{\imath}}$. Moreover $p_{k+1} \otimes \ldots \otimes p_m$ is weakly orthogonal to, hence commutes with, $q_{j_0} \otimes \ldots \otimes q_{j_\ell}$. Again by Lemma 2.1.19,

$$\begin{aligned} p &\equiv_{\mathrm{D}} q_{j_0} \otimes \ldots \otimes q_{j_\ell} \otimes p_{k+1} \otimes \ldots \otimes p_m \\ &= p_{k+1} \otimes \ldots \otimes p_m \otimes q_{j_0} \otimes \ldots \otimes q_{j_\ell} \equiv_{\mathrm{D}} q_{\bar{\imath}} \otimes q_{j_0} \otimes \ldots \otimes q_{j_\ell} \end{aligned}$$

To conclude, we need to show that the inclusion $I_p \supseteq \{\bar{\imath}, j_0, \ldots, j_\ell\}$ (a corollary of what we just proved) cannot be strict. If it is, as witnessed by j, then $p \geq_{\mathrm{D}} q_j$ but $q_j \perp^{\mathrm{w}} q_{\bar{\imath}}$ and $q_j \perp^{\mathrm{w}} q_{j_\alpha}$ for $\alpha \leq \ell$. By Lemma 4.1.15 then

$$p \geq_{\mathrm{D}} q_j \perp^{\mathrm{w}} q_{\bar{\imath}} \otimes q_{j_0} \otimes \ldots \otimes q_{j_\ell} \equiv_{\mathrm{D}} p$$

hence q_j is realised by Corollary 2.3.17, which is absurd. □

Corollary 4.1.24. Let T be o-minimal satisfying Assumption 4.1.19. For all $p_0, p_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ we have $I_{p_0 \otimes p_1} = I_{p_0} \cup I_{p_1}$.

Proof. Clearly, $I_{p_0 \otimes p_1} \supseteq I_{p_0} \cup I_{p_1}$. If the inclusion is strict, there is a nonrealised $q \in S_1^{\mathrm{inv}}(\mathfrak{U})$ such that $p_0 \otimes p_1 \geq_{\mathrm{D}} q$, but for $i < 2$ we have $p_i \perp^{\mathrm{w}} q$. By Lemma 4.1.15, $p_0 \otimes p_1 \perp^{\mathrm{w}} q$, so q is realised by Corollary 2.3.17, a contradiction. □

With similar arguments, one shows the corollary below.

Corollary 4.1.25. Let T be o-minimal satisfying Assumption 4.1.19. If $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$, then $p \perp^{\mathrm{w}} q$ if and only if p and q dominate no common nonrealised 1-type. Moreover, if $q \in S_1^{\mathrm{inv}}(\mathfrak{U})$, then $p \geq_{\mathrm{D}} q \iff p \not\perp^{\mathrm{w}} q$.

Corollary 4.1.26. Let T be o-minimal satisfying Assumption 4.1.19. Then $\otimes$ respects $\geq_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$.

Proof. Suppose $q_0 \geq_{\mathrm{D}} q_1$. By Proposition 4.1.23, this means that $I_{q_0} \supseteq I_{q_1}$. We want to show that, for all invariant p, we have $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$, i.e. $I_{p\otimes q_0} \supseteq I_{p\otimes q_1}$. Similarly, if we start with $q_0 \equiv_{\mathrm{D}} q_1$ then $I_{q_0} = I_{q_1}$, and we want to show that $I_{p\otimes q_0} = I_{p\otimes q_1}$. Both follow at once from Corollary 4.1.24. □

After recalling Remark 4.1.20, we can state the main result of this section.

Theorem 4.1.27. Let T be an o-minimal theory and assume that every invariant type is domination-equivalent to a product of 1-types. Then $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined, and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}, \perp^{\mathrm{w}}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup, \supseteq, D)$, where X is any maximal set of pairwise weakly orthogonal invariant 1-types and $D(x, y)$ holds iff $x \cap y = \emptyset$. Moreover, if every invariant type is equidominant to a product of 1-types, then $\equiv_{\mathrm{D}}$ is the same as $\sim_{\mathrm{D}}$, hence $\overline{\mathrm{Inv}}(\mathfrak{U}) = \widetilde{\mathrm{Inv}}(\mathfrak{U})$.

Proof. By the previous results, $[\![p]\!] \mapsto I_p$ is the required isomorphism. □

4.1.5 Reducing further

In the next section we will be concerned with the study of some specific o-minimal theories. Given T, because of Theorem 4.1.27, we are interested in showing that T satisfies Assumption 4.1.19, and in giving a nice description of a maximal family of pairwise weakly orthogonal invariant 1-types. Before undertaking this task, we isolate a property implying Assumption 4.1.19, and prove other results that help to show that a given T satisfies Assumption 4.1.19.

Assumption 4.1.28. Denote by $\mathcal{F}_T^{m,1}$ the set of functions $\emptyset$-definable in T with domain a definable subset of $\mathfrak{U}^m$ and codomain[1] $\mathfrak{U}^1$. Suppose that c is a $\mathfrak{U}$-independent tuple and let $p = \mathrm{tp}_x(c/\mathfrak{U})$. Then $\pi(x) \vdash p(x)$, where

$$\pi(x) \coloneqq \bigcup_{f \in \mathcal{F}_T^{|x|,1}} \mathrm{tp}_{w_f}(f(c)/\mathfrak{U}) \cup \left\{ w_f = f(x) \;\middle|\; f \in \mathcal{F}_T^{|x|,1} \right\}$$

[1] I.e. they are single definable functions, not tuples thereof: they output a single element.

Note that if this assumption is satisfied, and c is not $\mathfrak{U}$-independent, a similar statement still holds, by working with a basis c' of c over $\mathfrak{U}$ and then adding to $\pi(x)$ the formulas isolating $\operatorname{tp}(c/\mathfrak{U}c')$.

Lemma 4.1.29. Let $p \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$, let $M \prec^+ N \prec^+ \mathfrak{U}$, and let $b \vDash p^{(n+1)}$. If p has small cofinality on the right [resp. left] then $p(\operatorname{dcl}(Nb_n))$ is cofinal [resp. coinitial] in $p(\mathfrak{U}b)$.

Proof. The case where p is realised is trivial, so assume p is not. Assume further that p has small cofinality on the right (the other case is symmetrical) and let $R \subseteq M$ be coinitial in R_p. Let $f(t_0, \ldots, t_n, w)$ be an M-definable function such that $p(\operatorname{dcl}(Nb_n)) < f(b_0, \ldots, b_n, d) < R$. Let $\hat{b} \in \mathfrak{U}$ be such that $\hat{b} \vDash p^{(n)} \upharpoonright Nd$. Since $b_n \vDash p \mid \mathfrak{U}b_0, \ldots, b_{n-1}$ we have $\hat{b}b_n \equiv_{Nd} b$, hence $p(\operatorname{dcl}(Nb_n)) < f(\hat{b}, b_n, d) < R$. This violates the Idempotency Lemma 4.1.9. □

Corollary 4.1.30. Let $p \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$ and $b \vDash p^{(n)}$. Suppose that $c \vDash p$. If $c > p(\operatorname{dcl}(\mathfrak{U}b))$ or $c < p(\operatorname{dcl}(\mathfrak{U}b))$ then $(b, c) \vDash p^{(n+1)}$ or $(c, b) \vDash p^{(n+1)}$.

Proof. As usual, assume that p has small cofinality on the right. If $c > p(\operatorname{dcl}(\mathfrak{U}b))$, then $(c, b) \vDash p \otimes p^{(n)}$ by definition. If $c < p(\operatorname{dcl}(\mathfrak{U}b))$, in particular $c < p(\operatorname{dcl}(\mathfrak{U}b_0))$. By Corollary 4.1.5, we have $p(\operatorname{dcl}(\mathfrak{U}c)) < b_0$, hence $b_0 \vDash p \mid \mathfrak{U}c$. Since $b_1 > p(\operatorname{dcl}(\mathfrak{U}b_0)) \supseteq p(\operatorname{dcl}(Nb_0))$, it follows from Lemma 4.1.29 that $b_1 > p(\operatorname{dcl}(\mathfrak{U}cb_0))$, hence $b_1 \vDash p \mid \mathfrak{U}cb_0$. We conclude by induction. □

Proposition 4.1.31. Let T be o-minimal. Let $p(x) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, let $c \vDash p$ and assume that c is $\mathfrak{U}$-independent. The following facts hold.

1. There is a tuple $b \in \operatorname{dcl}(\mathfrak{U}c)$ of maximal length among those satisfying a product of nonrealised invariant 1-types.

2. Let b be as above, and let $q := \operatorname{tp}(b/\mathfrak{U}) = q_0 \otimes \ldots \otimes q_n$, where the q_i are invariant 1-types. Up to replacing each q_i with another type $\tilde{q}_i$ in definable bijection with it, we may assume that for $i, j \leq n$ either $q_i \perp^{\mathrm{w}} q_j$ or $q_i = q_j$. Moreover $\tilde{q}_0 \otimes \ldots \otimes \tilde{q}_n \equiv_{\mathrm{D}} q_0 \otimes \ldots \otimes q_n$.

Let b, q be as above, and let M, N, N_1 be such that each q_i is M-invariant and $M_0 \preceq M \prec^+ N \prec^+ N_1 \prec^+ \mathfrak{U}$.

3. Up to replacing b with another $\tilde{b} \vDash q$, we may assume $b \in \operatorname{dcl}(Nc)$.

4. Let b, q be as in the previous points, let $r := \operatorname{tp}_{xy}(cb/N_1)$, and let $\mathcal{F}^{m,1}_{T(M)}$ be the set of functions which are M-definable in T and have domain a definable subset of $\mathfrak{U}^m$ and codomain $\mathfrak{U}^1$. Then $p(x) \cup r(x,y) \vdash q(y)$ and

$$q(y) \cup r(x,y) \vdash \pi_M(x) := \bigcup_{f \in \mathcal{F}^{|x|,1}_{T(M)}} \operatorname{tp}_{w_f}(f(c)/\mathfrak{U}) \cup \left\{ w_f = f(x) \;\middle|\; f \in \mathcal{F}^{|x|,1}_{T(M)} \right\}$$

Proof.

① The element c_0 satisfies a product of length 1, hence a tuple $b \in \operatorname{dcl}(\mathfrak{U}c)$ satisfying a product of nonrealised invariant 1-types exists. Since $|b|$ is bounded above by $\dim(c/\mathfrak{U})$, there is a maximal such b.

② If, say, $q_0 \neq q_i$ but $q_0 \not\perp^{\mathrm{w}} q_i$, then by Lemma 4.1.12 there is a definable bijection f_i such that $(f_i)_* q_i = q_0$. By Lemma 2.1.19 and Corollary 2.2.8, we may replace every such q_i with $\tilde{q}_i := (f_i)_* q_i$ inside $q = q_0 \otimes \ldots \otimes q_n$ and obtain an equidominant product of 1-types. Now repeat this process on each $\equiv_{\mathrm{D}}$-class in $\{[\![q_0]\!], \ldots, [\![q_n]\!]\}$.

③ By Proposition 2.1.6, q is M-invariant. Apply Lemma 4.1.6 to obtain $\tilde{b} \in \operatorname{dcl}(Nc)$ realising q.

④ That $p \cup r \vdash q$ is trivial, so let $f \in \mathcal{F}^{|x|,1}_{T(M)}$ and consider $f(c)$. Note that $p \geq_{\mathrm{D}} \operatorname{tp}(f(c)/\mathfrak{U})$ and, since f is M-definable, $\operatorname{tp}(f(c)/\mathfrak{U})$ is M-invariant by Lemma 2.1.12. Let $p_0 := \operatorname{tp}(f(c)/\mathfrak{U}) \in S_1(\mathfrak{U}, M)$. If $p_0 \perp^{\mathrm{w}} q_i$ for every $i \leq n$, then by Lemma 4.1.15 $p_0 \perp^{\mathrm{w}} q$, hence $bf(c)$ is a tuple in $\operatorname{dcl}(\mathfrak{U}c)$ longer than b and satisfying a product of 1-types, against maximality of $|b|$. Therefore there is $i \leq n$ such that $p_0 \not\perp^{\mathrm{w}} q_i$. Since p_0 and all the q_i are M-invariant, by Lemma 4.1.12 there is an N-definable bijection g such that $g_* p_0 = q_i$. Let $b' \subseteq b$ be the subtuple of b consisting of points satisfying q_i.

Claim. There are $a_0, a_1 \in q_i(\operatorname{dcl}(N_1 b'))$ such that $a_0 < g(f(c)) < a_1$.

Proof of Claim. Otherwise, by Corollary 4.1.30 applied to N_1 instead of $\mathfrak{U}$, one between $g(f(c))b'$ and $b'g(f(c))$ satisfies $q_i^{(|b'|+1)} \restriction N_1$. Call $\hat{b}$ the one that does. Since $g \circ f$ is N-definable, and $b' \in \operatorname{dcl}(Nc)$ by point 3, the tuple $\hat{b}$ is the image under an N-definable function of c, hence it has N-invariant type; by uniqueness of invariant extensions, $\hat{b} \vDash q_i^{(|b'|+1)}$. By point 2, Corollary 4.1.16, and Fact 4.1.18, $g(f(c))b$ or $bg(f(c))$ satisfies a product of nonrealised invariant 1-types, against maximality of $|b|$. □ CLAIM

Write $a_j = h_j(b)$, where each $h_j(y)$ is N_1-definable. Then the formula $g^{-1}(h_0(y)) < f(x) < g^{-1}(h_1(y))$ is in r, and $q(y)$ shows that both $g^{-1}(h_j(y))$ realise p_0, so $q(y) \cup r(x,y) \vdash \operatorname{tp}(f(x)/\mathfrak{U})$, and we are done. □

Corollary 4.1.32. Assumption 4.1.28 implies Assumption 4.1.19.

Proof. Let $p(x) = \operatorname{tp}(c/\mathfrak{U})$ be M_0-invariant. By working on a basis c' of c and then adding to r the formulas isolating $\operatorname{tp}(c/\mathfrak{U}c')$ (see Lemma 4.1.7), we may assume that c is $\mathfrak{U}$-independent. Apply Proposition 4.1.31, and obtain a product $q(y)$ of invariant 1-types and a small $r \in S_{pq}(N_1)$ such that $p(x) \cup r(x,y) \vdash q(y)$ and $q(y) \cup r(x,y) \vdash \pi_M(x)$, and trivially $\pi_M(x) \vdash \pi(x)$. By Assumption 4.1.28, $\pi(x) \vdash p(x) = \operatorname{tp}(c/\mathfrak{U})$. □

We leave the following two related questions for future study.

Question 4.1.33. Let T be o-minimal, and suppose that p is M-invariant. Is p domination-equivalent to a product of M-invariant 1-types?

The next question asks whether the Idempotency Lemma can be improved.

Question 4.1.34. In an o-minimal T, let $p(x) \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$ have small cofinality on the right. If $b^0 \vDash p$, is $p(\operatorname{dcl}(Mb^0))$ cofinal in $p(\operatorname{dcl}(\mathfrak{U}b^0))$?

4.2 Examples

In this section we conclude the study of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ in some o-minimal theories. This is done in two steps: proving that Assumption 4.1.19 holds, and identifying a nice/informative set of representatives for domination-equivalence classes of 1-types. While it is not unreasonable to conjecture that Assumption 4.1.19 holds in every o-minimal theory, we leave the study of this problem to future work. At any rate, the direct proofs below shed some light on the meaning of domination in the theories under examination. Notably, we will see that in real closed fields the study of domination is "valuation theory in disguise".

4.2.1 Theories with no nonsimple types

In this subsection we deal with o-minimal theories with "few" definable functions, such as DLO. The main definition comes from [May88].

Definition 4.2.1. A 1-type $p \in S_1(A)$ is *simple* iff whenever there are an A-definable function $f(x_0, \ldots, x_n)$ and realisations $a_0, \ldots, a_n$ of p such that $f(a_0, \ldots, a_n) \vDash p$, then there is $j \leq n$ with $\bigcup_{i \leq n} p(x_i) \vdash f(x_0, \ldots, x_n) = x_j$.

Remark 4.2.2. A 1-type $p(x)$ is simple if and only if for all $k \in \omega$ the type $\{x_0 < \ldots < x_k\} \cup \bigcup_{i \leq k} p(x_i)$ is complete.

Proof. Left to right is immediate. Right to left, suppose that $a_0, \ldots, a_n$, and $f(a_0, \ldots, a_n)$ all model p, but for all $i \leq n$ we have $f(a_0, \ldots, a_n) \neq a_i$. Suppose for example that $f(a_0, \ldots, a_n) > a_n$, the other cases being analogous. Then $\{x_0 < \ldots < x_{n+1}\} \cup \bigcup_{i \leq n+1} p(x_i)$ is consistent with both $f(x_0, \ldots, x_n) = x_{n+1}$ and $f(x_0, \ldots, x_n) \neq x_{n+1}$. □

Note that, if p is invariant, modulo reversing the order of the variables the type above must be $p^{(k+1)}$.

Fact 4.2.3 ([RS17a, Corollary 2.6])**.** Let T be o-minimal. There is a nonsimple 1-type over $\emptyset$ if and only if there is one over some A, if and only if there is one over every A.

Proposition 4.2.4. Suppose that every $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ is simple. Then every invariant type is equidominant to a product of 1-types.

Proof. Let $\mathrm{tp}(a/\mathfrak{U})$ be invariant. By Lemma 4.1.12 we may assume that for all $i, j < |a|$ either $\mathrm{tp}(a_i/\mathfrak{U}) \perp^{\mathrm{w}} \mathrm{tp}(a_j/\mathfrak{U})$ or $\mathrm{tp}(a_i/\mathfrak{U}) = \mathrm{tp}(a_j/\mathfrak{U})$. Furthermore, by Corollary 4.1.16 and Fact 4.1.18, it is enough to show that if the a_i all have the same type p, then $\mathrm{tp}(a/\mathfrak{U}) \equiv_{\mathrm{D}} p^{(|a|)}$; equivalently, by the Idempotency Lemma, that $\mathrm{tp}_x(a/\mathfrak{U}) \equiv_{\mathrm{D}} p(y)$. This is immediate from the definition of simplicity, by taking as $r(x, y)$ a small type containing $y = x_i$, where $x_i = \min\{x_0, \ldots, x_{|x|-1}\}$ if p has small cofinality on the right, and $x_i = \max\{x_0, \ldots, x_{|x|-1}\}$ otherwise. □

Example 4.2.5. By quantifier elimination, every 1-type in DLO is simple and all pairs of distinct nonrealised 1-types are weakly orthogonal. Therefore Theorem 4.1.27 applies, and we may take as X the set $\mathrm{IC}(\mathfrak{U})$ defined in Example 2.2.22. Equivalently, in this case, X may be taken to be the set of all nonrealised 1-types.

We take the opportunity to record that dp-rank and regularity are not preserved by $\equiv_{\mathrm{D}}$, and a fortiori by $\sim_{\mathrm{D}}$: in DLO, let p be any nonrealised invariant 1-type. Then $p \equiv_{\mathrm{D}} p \otimes p$ even if the former has dp-rank 1 and is regular, and the latter has dp-rank 2 and is not regular.

Remark 4.2.6. Again in DLO, if $p(x) \in S_1^{\mathrm{inv}}(\mathfrak{U})$ has small cofinality on the right, say, and R_p has no minimum and is nonempty, then $p(x) \not\geq_{\mathrm{RK}} p(y) \otimes p(z)$: a formula $\varphi(x, y, z)$ consistent with $p(x)$ which implies $x \leq z$ does not prove $y < R_p$, and one which implies $x > z$ does not prove $z > L_p$.

Remark 4.2.7. While most o-minimal theories used in applications eliminate imaginaries, this is, by [Joh18], not always true. Therefore, it is in principle possible that there is an o-minimal theory where $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \neq \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$.

4.2.2 Divisible ordered abelian groups

Let $L = \{+, 0, -, <\}$, and define the L-theory DOAG of divisible ordered abelian groups by declaring that it does exactly what it says on the tin. It is well-known (see [vdD98]) that this theory is complete, eliminates quantifiers, and is o-minimal. Moreover, DOAG and all of its o-minimal expansions have definable choice. This in turns implies the existence of definable Skolem functions and elimination of imaginaries.

As I was saying at the beginning of this chapter, $\overline{\mathrm{Inv}}(\mathfrak{U})$ was computed for DOAG in [HHM08], and in fact the general strategy of proof in the previous section is inspired by this result. Unfortunately, the proof in [HHM08] has a gap, explained at the end of this subsection. In what follows, we still use ideas and results from [HHM08], but we avoid altogether the part of the [HHM08] proof containing the gap. There are other minor differences between the present approach and that of [HHM08]. For example, Proposition 4.2.18 is a consequence of [HHM08, Corollary 13.11], but the proof given here is easier to generalise, as we do in Proposition 4.2.31.

Remark 4.2.8. The theory DOAG is not weakly binary. Let $b > a > \mathfrak{U}$ be such that for every positive $d \in \mathfrak{U}$ the inequality $b - a < d$ holds, let $A \subset^+ \mathfrak{U}$, and take $d \in \mathfrak{U}$ such that, for every $c \in \mathrm{dcl}(A)$ with $c > 0$, we have $0 < d < c$. Then $\mathrm{tp}(a/\mathfrak{U}) \cup \mathrm{tp}(b/\mathfrak{U}) \cup \mathrm{tp}(ab/A) \nvdash b - a < d$.

Proposition 4.2.9. The theory DOAG satisfies Assumption 4.1.28.

Proof. By quantifier elimination and o-minimality, a type $p(x) \in S(\mathfrak{U})$ is determined once all formulas of the form $f(x, d) \geq 0$ are decided, where $f(x, d)$

is a $\mathbb{Q}$-linear combination $f(x,d) = \sum_{i<k} \lambda_i \cdot x_i + \sum_{j<\ell} \mu_j \cdot d_j$. Rearrange $f(x,d) \geq 0$ as

$$\sum_{i<k} \lambda_i \cdot x_i \geq -\sum_{j<\ell} \mu_j \cdot d_j \tag{4.2}$$

Since $\sum_{i<k} \lambda_i \cdot x_i$ is an $\emptyset$-definable function, whether (4.2) or its negation holds is decided by the partial type $\pi(x)$ in Assumption 4.1.28, thereby proving that the latter holds. □

By Corollary 4.1.32 we can therefore apply Theorem 4.1.27 to DOAG. We are now left with the task of identifying a nice maximal set of pairwise weakly orthogonal invariant 1-types.

Notation 4.2.10. If A is ordered and $a \in A$, denote $A_{>a} \coloneqq \{b \in A \mid b > a\}$.

Definition 4.2.11. Let H be a convex subgroup of $M \vDash \mathsf{DOAG}$. Let $q_H(x)$ denote the element of $S_1(M)$ defined by $\{x > d \mid d \in H\} \cup \{x < d \mid d > H\}$.

Definition 4.2.12. If H is a convex subgroup of $\mathfrak{U}$, we say that H is *invariant* iff there is a small A such that H is fixed setwise by $\operatorname{Aut}(\mathfrak{U}/A)$.

If $H \leq \mathfrak{U}$ is convex, it is easy to show, using convexity and Remark 2.1.2, that $q_H \in S_1(\mathfrak{U})$ is invariant if and only if H is.

Lemma 4.2.13. For all $b \vDash q_H$ and $\gamma \in \mathbb{Q}_{>0}$ we have $\gamma b \vDash q_H$.

Proof. Let $n \in \omega \setminus \{0\}$. Suppose that there is $c \in H$ such that $b/n < c$. Then $b < nc$, and since H is a convex subgroup this contradicts the definition of q_H. Similarly, if $c > H$ is such that $nb > c$, then $b > c/n$. By definition of q_H and convexity of H, we have $c/n \in H$. Therefore $c \in H$, a contradiction. □

Proposition 4.2.14. Whenever $H_0 \subsetneq H_1$ are distinct convex subgroups of $M \vDash \mathsf{DOAG}$, we have $q_{H_0} \perp^{\mathrm{w}} q_{H_1}$.

Proof. By quantifier elimination, we need to show that knowing $a \vDash q_{H_0}$ and $b \vDash q_{H_1}$ is enough to decide, for $d_i \in \operatorname{dcl}(Ma)$ and $e_i \in \operatorname{dcl}(Mb)$, all the inequalities of the form $d_0 + e_0 \leq d_1 + e_1$. Since the d_i and e_i are $\mathbb{Q}$-linear combinations of elements in Ma and Mb respectively, after some algebraic manipulation, we find $c \in M$ and $\gamma \in \mathbb{Q}$ such that $d_0 + e_0 \leq d_1 + e_1 \iff \gamma b - a \geq c$ or such that $d_0 + e_0 \leq d_1 + e_1 \iff \gamma b + a \geq c$. If $\gamma = 0$ then this information is in $\operatorname{tp}(a/M) = q_{H_0}$. If $\gamma > 0$ then, since $H_0 \subsetneq H_1$, we have

$$2\gamma b = \gamma b + \gamma b \geq \gamma b + a \geq \gamma b \geq \gamma b - a \geq \gamma b - \gamma b/2 = \gamma b/2$$

Since $2\gamma b$, γb, and $\gamma b/2$ have the same cut by Lemma 4.2.13, knowing $a \vDash q_{H_0}$ and $b \vDash q_{H_1}$ is enough to deduce $\gamma b \pm a \equiv_M b$, hence to decide whether $\gamma b \pm a \geq c$ holds or not. If $\gamma < 0$, argue similarly by showing that $\gamma b \pm a \equiv_M -b$. □

Definition 4.2.15. Let $A \leq B$ be ordered abelian groups.

1. If $b \in B$, let $\mathrm{ct}_A(b) := \{a \in A \mid 0 < a < b\}$.
2. We call B an *i-extension* of A iff there is no $b \in B$ such that $b > 0$ and $\mathrm{ct}_A(b)$ is closed under addition.
3. An ordered abelian group is *i-complete* iff it has no proper i-extensions.

Lemma 4.2.16. If $A \subseteq B$ and $B \subseteq C$ are i-extensions, then so is $A \subseteq C$. Consequently, i-extensions of A are closed under unions of chains.

Proof. Suppose that $c \in C$ is such that $c > 0$ and $\mathrm{ct}_A(c)$ is closed under sums. If there is $b \in B$ with $\mathrm{ct}_A(b) = \mathrm{ct}_A(c)$, then B is not an i-extension of A. Otherwise, by definition, $\mathrm{ct}_A(c)$ is cofinal in $\mathrm{ct}_B(c)$. Since addition is increasing in each coordinate, $\mathrm{ct}_B(c)$ is closed under sums and C is not an i-extension of B. The last part is standard. □

Proposition 4.2.17 ([HHM08, Lemma 13.9])**.** Every ordered abelian group A has an i-complete i-extension, of size at most $\beth_2(|A|)$.

Proof. We show that if B is an i-extension of A then $|B| \leq \beth_2(|A|)$. Since the union of a chain of i-extensions of A is an i-extension of A, it is then enough to keep extending A until the chain stabilises, as it must by cardinality reasons.

Colour the set of subsets of $B_{>0}$ of size 2 with $\mathrm{colour}(\{b_0, b_1\}) := \mathrm{ct}_A(|b_0 - b_1|)$. If $|B| > \beth_2(|A|)$, then by the Erdős–Rado Theorem there are $y < y' < y''$ in $B_{>0}$ such that $\mathrm{ct}_A(y' - y) = \mathrm{ct}_A(y'' - y) = \mathrm{ct}_A(y'' - y')$. Call the latter C. If we show that C is closed under addition, then B is not an i-extension of A. But if $c_0, c_1 \in C$ then $c_0 + c_1 \in C$, because

$$c_0 + c_1 < (y'' - y') + (y' - y) = y'' - y$$ □

Proposition 4.2.18. Let $p \in S_1(M)$, with $M \vDash \mathsf{DOAG}$ i-complete. There are an M-definable function f and a convex subgroup H of M such that $f_* p = q_H$.

Proof. Let p be a counterexample and let $a \vDash p$. By assumption, there is no $b > 0$ in $\mathrm{dcl}(Ma)$ such that $\mathrm{ct}_M(b)$ is closed under addition. But then $\mathrm{dcl}(Ma)$ is an i-extension of M. □

Corollary 4.2.19. In DOAG, for every invariant 1-type p there are a definable bijection f and an invariant convex subgroup H of $\mathfrak{U}$ such that $f_*p = q_H$.

Proof. Suppose that $p \in S^{\mathrm{inv}}(\mathfrak{U}, M)$. Up to enlarging M not beyond size $\beth_2(|M|)$, for some M-definable bijection f we have $f_*(p \upharpoonright M) = q_{H_0}$, where H_0 is some convex subgroup of M. Since by Lemma 2.1.12 f_*p is M-invariant, and extends q_{H_0}, it can only be one of two types; depending on whether it has small cofinality on the left or on the right, it will be the unique M-invariant type q_0 with H_0 cofinal in L_{q_0}, or the unique M-invariant q_1 with $(M \setminus H_0)_{>0}$ coinitial in R_{q_1}. Both of these are clearly of the form q_H, where H is an invariant convex subgroup of $\mathfrak{U}$: the convex hull of H_0 in the first case, and $\{d \in \mathfrak{U} \mid |d| < (M \setminus H_0)_{>0}\}$ in the second. □

We sum everything up as follows.

Theorem 4.2.20. In DOAG, the domination monoid $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined, coincides with $\overline{\mathrm{Inv}}(\mathfrak{U})$, and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}, \perp^{\mathrm{w}}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup, \supseteq, D)$, where X is the set of invariant convex subgroups of $\mathfrak{U}$.

Remark 4.2.21. By Remark 4.1.3, in DOAG there are only two $\equiv_{\mathrm{D}}$-classes of definable types, those of $q_{\mathfrak{U}}$ and $q_{(0)}$. Note how every definable 1-type different from $q_{\mathfrak{U}}$ and $-q_{\mathfrak{U}}$ is in definable bijection with $q_{(0)}$ via translations and reflections. In this case, the bulk of (the generators of) $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ really consists of classes of finitely satisfiable types.

It is natural to ask what happens if we consider the domination $\triangleright^{\mathrm{dcl}}$ induced on tuples by dcl-independence in a fashion analogous to Definition 3.1.5, and extend it to types as in Definition 3.1.7. After all, why do we care about $\geq_{\mathrm{D}}$ in the first place, when in o-minimal theories we have a well-behaved notion of independence? It is easy to see that $p \triangleright^{\mathrm{dcl}} q$ if and only if there are $a \models p$ and $b \models q$ such that $b \in \mathrm{dcl}(\mathfrak{U}a)$, if and only if (because of the existence of definable Skolem functions) $p \geq_{\mathrm{RK}} q$. Unfortunately, it is not true that every invariant type is Rudin–Keisler equivalent to a product of invariant 1-types.

Counterexample 4.2.22. In DOAG, let $a \models p(x) \vdash x > \mathfrak{U}$, let $\gamma \in \mathbb{R} \setminus \mathbb{Q}$, and let b be such that $b \models r(a, y) \coloneqq \{y > \beta \cdot a \mid \beta \in \mathbb{Q}, \beta \leq \gamma\} \cup \{y < \beta \cdot a \mid \beta \in \mathbb{Q}, \beta \geq \gamma\}$. Note that, by this very description, $q \coloneqq \mathrm{tp}(ab/\mathfrak{U}) \equiv_{\mathrm{D}} p$ in our usual sense. Now, $\bowtie^{\mathrm{dcl}}$ preserves dimension, so if q is equivalent to a product, we may assume it is a product of the form $q_0 \otimes q_1$, where the q_i are nonrealised invariant 1-types. By orthogonality considerations, both q_i must actually be interdefinable with p. But $p^{(2)}$ is not realised in $\operatorname{dcl}(\mathfrak{U}ab)$.

I have already said that the characterisation of $\overline{\operatorname{Inv}}(\mathfrak{U})$ in DOAG is not new. While we will not see all the details of how it is done in [HHM08], nor define all the notions involved, I would like to point out what is the gap that has been addressed above. The problem in the original proof resides, I believe, in an implicit use of symmetry of i-freeness in an unproven statement, or at least I have not been able to prove the latter without using symmetry.

In detail, the proof of [HHM08, Lemma 13.16] uses [HHM08, Lemma 13.14], in the proof of which it is assumed that A' is i-free from B over C. The only way I can see to show that a' exists is to use that B is i-free from A' over C. Unfortunately, i-freeness is not symmetric.

4.2.3 Real closed fields

Real closed fields are the ordered fields where every polynomial of odd degree has a zero and every positive element has a square root. Their theory, in the language $L = \{+, 0, -, \cdot, 1, <\}$, is called RCF, is complete, eliminates quantifiers, and is the theory of $\mathbb{R}$. It also has a famous, particularly monstrous monster model (it is a proper class), the field of surreal numbers.

To complete the study of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ in real closed fields, we need to show that classes of 1-types generate it, and to identify a nice representative for each such class. We do this by using a dash of valuation theory. The appearance of valuations is no coincidence: it will follow from the results in this subsection that, if Γ is the value group of $\mathfrak{U} \models \mathsf{RCF}$ with respect to the Archimedean valuation, then $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(\Gamma)$.

We point out once and for all that, in this subsection, valuations are only used "externally", i.e. are *not* part of the language of any structure under consideration. Besides the basics of valuation theory, we only use some properties of *Hahn fields* and of *maximally complete* valued fields. The reader is referred to [vdD14] but, if happy to take these as black boxes, might ignore the definitions of "Hahn field" and of "maximally complete".

Before we start, by identifying a maximal set of pairwise weakly orthogonal invariant 1-types, I would like to express my gratitude to Francesco Gallinaro for patiently listening to several versions of the proofs to come, most of which were irreparably wrong.

Lemma 4.2.23. Let H be a subring of the ordered field M. Then the convex hull $\hat{H}$ of H is a subring of M. In particular, $\hat{H}$ is a valuation ring.

Proof. Immediate from monotonicity of addition and multiplication by a fixed element. The fact that $\hat{H}$ is a valuation ring is also immediate: if $x \notin \hat{H}$, by convexity $|x| > 1$, so $|1/x| < 1$ and, by using convexity again, $1/x \in \hat{H}$. □

Definition 4.2.24. Let H be a convex subring of $M \vDash \mathsf{RCF}$. Let $q_H(x)$ denote the element of $S_1(M)$ defined by $\{x > d \mid d \in H\} \cup \{x < d \mid d > H\}$.

Definition 4.2.25. If H is a convex subring of $\mathfrak{U}$, we say that H is *invariant* iff there is a small A such that H is fixed setwise by $\operatorname{Aut}(\mathfrak{U}/A)$.

If H is a convex subring of $\mathfrak{U}$, it is easy to show, using convexity and Remark 2.1.2, that $q_H \in S_1(\mathfrak{U})$ is invariant if and only if H is.

Lemma 4.2.26. Let H be a convex subring of $M \vDash \mathsf{RCF}$, and let $c \vDash q_H$. Let v be the valuation on $\operatorname{dcl}(Mc)$ with valuation ring the convex hull of H. Then $v(c) \notin v(M)$.

Proof. Suppose that $d \in M$ is such that $v(c) = v(d)$ and, without loss of generality, that c and d are positive. By convexity of valuation balls, the fact that $v(c) < 0$ by assumption, and the fact that $v(c^k) = k \cdot v(c)$, we have that if $c < d$ then $c < d < c^2$, and if $d < c$ then $c^{\frac{1}{2}} < d < c$. The set $q_H(\operatorname{dcl}(Mc))$ is convex in $\operatorname{dcl}(Mc)$ and, since H is a subring, by arguing as in Lemma 4.2.13 we can show that it is closed under positive powers. This implies $d \vDash q_H$, which is absurd because $d \in M$. □

Proposition 4.2.27. Whenever $H_0 \subsetneq H_1$ are distinct convex subrings of $M \vDash \mathsf{RCF}$, we have $q_{H_0} \perp^{\mathrm{w}} q_{H_1}$.

Proof. Otherwise, by Lemma 4.1.11, there is a definable h such that $h_* q_{H_0} = q_{H_1}$. Let $a \vDash q_{H_0}$ and $b = h(a)$. On $\operatorname{dcl}(Ma)$, consider the valuation v_1 with valuation ring the convex hull of H_1. By the previous lemma $v_1(b) \notin v_1(M)$ so, in order to reach a contradiction, it is enough to show $v_1(\operatorname{dcl}(Ma)) = v_1(M)$.

Suppose we show that, when $f(x) \in M[x]$ is a polynomial, $v_1(f(a)) \in v_1(M)$. Then this holds too for f a rational function hence, if $M(a)$ is the field generated by Ma, we have $v_1(M(a)) = v_1(M)$. It is well-known (see [vdD14, Corollary 3.20]) that valuations extend uniquely to the algebraic closure, and in particular to the real closure, and that each embedding in such a closure induces an embedding of the value group in a divisible hull. Since $v_1(M(a)) = v_1(M)$ is already divisible, $v_1(\mathrm{dcl}(Ma)) = v_1(M)$.

So, suppose that $f(x) = \sum_{i \le n} d_i x^i$, with $n = \deg f$ and $d_i \in M$. Consider the valuation v_0 on $\mathrm{dcl}(Ma)$ with valuation ring the convex hull of H_0. We show by induction on n that there is $i \le n$ such that $v_0(f(a)) = v_0(d_i a^i)$. For $n = 0$ there is nothing to prove. Write $f(a) = d_{n+1}a^{n+1} + g(a)$, with $\deg g(a) \le n$. If $v_0(d_{n+1}a^{n+1})$ is different from $v_0(g(a))$, then $v_0(f(a))$ is the minimum of the two and we are done by inductive hypothesis. Otherwise, again by inductive hypothesis, there is $j \le n$ such that $v_0(d_{n+1}a^{n+1}) = v_0(g(a)) = v_0(d_j a^j)$. This implies $(n + 1 - j) \cdot v_0(a) = v_0(d_j) - v_0(d_{n+1}) \in v_0(M)$. Since $v_0(a) \notin v_0(M)$ by the previous lemma, we have $j = n + 1$, a contradiction.

Therefore, $v_0(f(a)) = v_0(d_i a^i)$ for some $i \le n$. Since v_0 is finer than v_1, we have that $v_0(z) = v_0(w)$ implies $v_1(z) = v_1(w)$, and since $a > 1$ is in the convex hull of H_1 we have $v_1(a) = 0$, hence $v_1(f(a)) = v_1(d_i a^i) = v_1(d_i) + i \cdot v_1(a) = v_1(d_i) \in v_1(M)$. □

Lemma 4.2.28. If $b > 2$ and $\mathrm{ct}_M(b)$ is closed under product, then it is also closed under sum.

Proof. Let $c, d \in \mathrm{ct}_M(b)$. If both $c, d \le 2$ then $c + d \le 4 = 2 \cdot 2 \in \mathrm{ct}_M(b)$ by hypothesis. Otherwise, say $d > 2$ and $d > c$, we have $\mathrm{ct}_M(b) \ni d \cdot d > 2 \cdot d = d + d > c + d$. □

Definition 4.2.29. If M is an ordered field, the *Archimedean valuation* v on M has valuation ring the convex hull of $\mathbb{Z}$.

Note that this is the finest convex valuation on M.

Fact 4.2.30. Every real closed field M embeds elementarily in a Hahn field $\mathbb{R}((t^\Gamma)) \models \mathsf{RCF}$, with Γ the value group of (M, v), where v is the Archimedean valuation. Moreover, Hahn fields $\mathbb{R}((t^\Gamma))$ are always maximally complete and have size at most $2^{|\Gamma|}$.

Proof sketch. An embedding exists by the field version of Hahn's Embedding Theorem[2]. It is easy to show that if $M \models \mathsf{RCF}$ then Γ is divisible, and it is well-known that if $\Gamma \models \mathsf{DOAG}$ then $\mathbb{R}((t^\Gamma)) \models \mathsf{RCF}$. Elementarity of the embedding follows from quantifier elimination. See [vdD14, Corollary 4.13] for maximal completeness, and for the size bound note that $|\mathbb{R}((t^\Gamma))| \le (2^{\aleph_0})^{|\Gamma|} = 2^{|\Gamma|}$. □

[2]This result has a somehow folkloric status. See [Ehr95, p.187] for an explanation of why it is difficult to attribute it.

Proposition 4.2.31. Let $\Gamma \vDash \mathsf{DOAG}$ be i-complete, and let $M := \mathbb{R}((t^\Gamma))$. For every $p \in S_1(M)$ there are an M-definable function f and a convex subring H of M such that $f_*p = q_H$.

Proof. Let p be a counterexample, let $a \vDash p$, and let $N := \operatorname{dcl}(Ma)$. Since every point of N is of the form $f(a)$, for f some M-definable function, it is enough to find $b \in N$ such that $\operatorname{tp}(b/M)$ is of the form q_H, and take f to be an M-definable function such that $b = f(a)$.

When we look at both M and N equipped with the Archimedean valuation, which we call v in both cases, they both have residue field $\mathbb{R}$. Since M is maximally complete by Fact 4.2.30, the value group $\Gamma(N)$ of (N, v) must be larger than Γ. Since Γ is i-complete, by Proposition 4.2.18 there must be $\gamma \in \Gamma(N)$ such that $\operatorname{tp}(\gamma/\Gamma) = q_{\tilde{H}}$, where $\tilde{H}$ is a convex subgroup of Γ. Let $b \in N$ be such that $v(b) = -\gamma$. Since $\gamma > 0$, we have $|b| > 1$, and in fact $|b| > \mathbb{R}$, hence by possibly replacing b with $-b$ we may assume that $b > 2$. Let $H := \{m \in M \mid |m| < b\}$. Since $\tilde{H}$ is a convex subgroup of Γ, and since $v(b) = -\gamma \notin v(M)$, we have that $\operatorname{ct}_M(b)$ is closed under products. By Lemma 4.2.28, H is a convex subring of M, and clearly $\operatorname{tp}(b/M) = q_H$. □

Corollary 4.2.32. In RCF, for every invariant 1-type p there are a definable bijection f and an invariant convex subring H of $\mathfrak{U}$ such that $f_*p = q_H$.

Proof. Suppose that $p \in S^{\mathrm{inv}}(\mathfrak{U}, M)$, and equip M with the Archimedean valuation. Note that an embedding of ordered groups $\Gamma(M) \hookrightarrow \Gamma$ induces an embedding of ordered fields $\mathbb{R}((t^{\Gamma(M)})) \hookrightarrow \mathbb{R}((t^\Gamma))$. Using this, Fact 4.2.30, Proposition 4.2.17 and the fact that $|\Gamma(M)| \le |M|$, up to enlarging M not beyond size $\beth_3(|M|)$ we may assume that it is of the form $\mathbb{R}((t^\Gamma))$, with $\Gamma \vDash \mathsf{DOAG}$ i-complete.

By Proposition 4.2.31 there are a convex subring H_0 of M and an M-definable bijection f such that $f_*(p \upharpoonright M) = q_{H_0}$. Since by Lemma 2.1.12 f_*p is M-invariant, and extends q_{H_0}, it can only be one of two types; depending on whether it has small cofinality on the left or on the right, it will be the unique M-invariant type q_0 with H_0 cofinal in L_{q_0}, or the unique M-invariant q_1 with $(M \setminus H_0)_{>0}$ coinitial in R_{q_1}. Both of these are clearly cuts of a type of the form q_H, where H is an invariant convex subring of $\mathfrak{U}$: the convex hull of H_0 in the first case, and $\{d \in \mathfrak{U} \mid |d| < (M \setminus H_0)_{>0}\}$ in the second. □

To conclude our study of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ in RCF, we need to show that every invariant type is domination-equivalent to a product of 1-types. The strategy of proof is to show that Assumption 4.1.28 holds at least in all cases of interest. In order to do this, we need some further help from valuation theory.

Definition 4.2.33. Let $M < N$ be an extension of nontrivially valued fields. A basis $e_0, \dots, e_n$ of a finite-dimensional M-vector subspace of N is *separated* iff for every $d_0, \dots, d_n \in M$ we have

$$v\Big(\sum_{i\le n} d_i e_i\Big) = \min_{i\le n} v(d_i e_i)$$

Fact 4.2.34. Let $M < N$ be an extension of nontrivially valued fields. If M is maximally complete, then every finite-dimensional M-vector subspace of N has a separated basis.

Proof. See [Bau82, Lemma 3] or [HHM08, Proposition 12.1(i)]. □

The following statement is well-known, but I could not find a reference.

Fact 4.2.35 (Folklore?). For every $M_0 \vDash \mathsf{RCF}$ there is an $|M_0|^+$-saturated $M \succ M_0$ (in the language of ordered rings) which is maximally complete with respect to the Archimedean valuation and of size at most $\beth_2(|M_0| + 2^{\aleph_0})$.

Proof sketch. By Fact 4.2.30, it is enough to show that if Γ is κ-saturated, then so is $\mathbb{R}((t^\Gamma))$. The cardinality bound then follows by first enlarging M_0 so that it contains $\mathbb{R}$, then using that $|\mathbb{R}((t^\Gamma))| \le (2^{\aleph_0})^{|\Gamma|} = \beth_1(|\Gamma|)$, and that there is an $|M_0|^+$-saturated $\Gamma \succ \Gamma(M_0)$ with $|\Gamma| \le \beth_1(|M_0|)$ by [CK90, Lemma 5.1.4].

Since $\mathbb{R}((t^\Gamma)) \vDash \mathsf{RCF}$, we must have $\Gamma \vDash \mathsf{DOAG}$. Take a partial type $\Phi(x) = \{x > a \mid a \in A\} \cup \{x < b \mid b \in B\}$, with $|AB| < \kappa$. We may assume that A, B only consist of positive elements. Moreover, since Γ is κ-saturated, no set of size less than κ is coinitial in $\mathbb{R}((t^\Gamma))_{>0}$ hence, up to adding a point to A, we may assume that $A \neq \emptyset$. Let v be the Archimedean valuation. If $v(A) > v(B)$, by divisibility and saturation of Γ there is γ_0 with $v(A) > \gamma_0 > v(B)$, and then $t^{\gamma_0} \vDash \Phi(x)$. Otherwise, large enough points of A are all of the form $r_a t^{\gamma_0} + \varepsilon_a$, and small enough points of B are all of the form $r_b t^{\gamma_0} + \varepsilon_b$, where $r_- \in \mathbb{R} \setminus \{0\}$, $v(\varepsilon_-) > \gamma_0$, and γ_0 is fixed. If there is r_0 such that $\sup_{a\in A} r_a < r_0 < \inf_{b\in B} r_b$, then $r_0 t^{\gamma_0} \vDash \Phi(x)$ and we are done. Otherwise, if $\sup_{a\in A} r_a = r_0 = \inf_{b\in B} r_b$, take $r_0 t^{\gamma_0}$ as our first approximant for a realisation of Φ. Replace A and B

with $A_1 := A - r_0 t^{\gamma_0}$ and $B_1 = B - r_0 t^{\gamma_0}$ and repeat the argument getting a second approximant $r_0 t^{\gamma_0} + r_1 t^{\gamma_1}$, with $v(\gamma_1) > v(\gamma_0)$. Iterate in the transfinite, where at limit stages we take infinite sums, which is possible since we are in a Hahn field and we are summing over a set with well-ordered support. After fewer than κ many steps, we have to realise $\Phi(x)$, because $|AB| < \kappa$. $\square$

Proposition 4.2.36. In the theory RCF, every invariant type is equidominant to a product of invariant 1-types.

Proof. Let $p(x) = \mathrm{tp}(c/\mathfrak{U})$ be M_0-invariant, and by enlarging M_0 assume that $M_0 \succ \mathbb{R}$. As usual we may assume that c is $\mathfrak{U}$-independent, by working with a basis of c and recovering the rest with a single formula. Let b be given by point 1 of Proposition 4.1.31, satisfying its point 2. Enlarge M_0 further so as to ensure that $\mathrm{tp}(b/\mathfrak{U})$ is M_0-invariant, then use Fact 4.2.35 to obtain a small $|M_0|^+$-saturated $M \succ M_0$ which is maximally complete with respect to the Archimedean valuation v.

Claim. Inside the ordered field $M(c)$ generated by Mc, let V be a finite-dimensional M-vector subspace generated by a finite set of monomials c^ℓ, for $\ell \in \omega^{|c|}$ a multi-index.[3] If e is a separated basis of the M-vector space V, then it is also a separated basis of the $\mathfrak{U}$-vector space generated by e inside $\mathfrak{U}(c)$, where $M, M(c), \mathfrak{U}, \mathfrak{U}(c)$ are equipped with the Archimedean valuation.

Proof of Claim. Take a linear combination $\sum_{i \le n} d_i e_i$, with $d_i \in \mathfrak{U}$. Since $e_i \in \mathrm{dcl}(Mc)$, we may write $e_i = h_i(c)$, for a suitable M-definable function $h_i(x)$. Let H be the (finite) set of parameters outside M_0 appearing in the functions $h_i(x)$. Since M is $|M_0|^+$-saturated, there is $\tilde{d} \in M$ with $\tilde{d} \equiv_{M_0 H} d$. Since e is a separated basis, up to reindexing we have $v(\sum_{i \le n} \tilde{d}_i h_i(c)) = v(\tilde{d}_n h_n(c))$. Therefore there is a real number $s \in \mathbb{R} \setminus \{0\}$ such that

$$\forall m \in \omega \setminus \{0\}\ p(x) \vdash \left| s - \frac{\sum_{i \le n} \tilde{d}_i h_i(x)}{\tilde{d}_n h_n(x)} \right| < \frac{1}{m}$$

By M_0-invariance of $p(x) = \mathrm{tp}(c/\mathfrak{U})$ we have

$$\forall m \in \omega \setminus \{0\}\ p(x) \vdash \left| s - \frac{\sum_{i \le n} d_i h_i(x)}{d_n h_n(x)} \right| < \frac{1}{m}$$

$\square$ CLAIM

[3] E.g. if $|c| = 2$ we could have $\ell = (2, 7)$ and $c^\ell = c_0^2 c_1^7$.

Apply the rest of Proposition 4.1.31, and work in its notation. So $p(x) = \operatorname{tp}(c/\mathfrak{U})$, $q(y) = \operatorname{tp}(b/\mathfrak{U})$, $r(x, y) = \operatorname{tp}(cb/N_1)$, $p(x) \cup r(x, y) \vdash q(y)$, and

$$q(y) \cup r(x,y) \vdash \pi_M(x) = \bigcup_{f \in \mathcal{F}^{|x|,1}_{T(M)}} \operatorname{tp}_{w_f}(f(c)/\mathfrak{U}) \cup \left\{ w_f = f(x) \;\middle|\; f \in \mathcal{F}^{|x|,1}_{T(M)} \right\}$$

We want to show that $q(y) \cup r(x, y) \vdash p(x)$.

By quantifier elimination it is enough to show that $q \cup r$ decides the sign of all polynomials $f(x, d') \in \mathfrak{U}[x]$, where d' is the tuple of coefficients. Note that, since c is $\mathfrak{U}$-independent, it is $\{d'\}$-independent, hence $p(x) \vdash f(x, d') \neq 0$, unless $f(x, d')$ is identically null (in which case there is nothing to do). By Fact 4.2.34, there is a separated basis $e_0, \ldots, e_n$ of the M-vector space generated by all the c^ℓ appearing in $f(c, d')$. We can write $e_i = h_i(c)$, where $h_i(x)$ is an M-definable function, and $c^\ell = \sum_{j \le n} \beta_{j,\ell} e_j$, for suitable $\beta_{j,\ell} \in M$. Note that $r(x, y) \vdash x^\ell = \sum_{j \le n} \beta_{j,\ell} h_j(x)$. After replacing, in $f(x, d')$, each x^ℓ with $\sum_{j \le n} \beta_{j,\ell} h_j(x)$, and collecting the monomials in each $h_j(x)$, we have

$$\vDash \forall x \left(\Big(\bigwedge_\ell x^\ell = \sum_{j \le n} \beta_{j,\ell} h_j(x) \Big) \to \big(f(x, d') = g(d, h(x)) \big) \right)$$

where $h(x) = (h_0(x), \ldots, h_n(x))$ and $g(d, z) = \sum_{j \le n} d_j z_j$, with $d_j = \sum_\ell d'_\ell \beta_{j,\ell}$. It follows that

$$r(x, y) \vdash f(x, d') = g(d, h(x)) \tag{4.3}$$

Now, since by the Claim e is also a separated basis of the $\mathfrak{U}$-vector space it generates, $v(f(c, d')) = v(g(d, e)) = \min_j v(d_j e_j)$. Suppose, by rearranging e, that this equals $v(d_n e_n)$. Define

$$a \coloneqq 1 + \sum_{j<n} \frac{d_j e_j}{d_n e_n} = \frac{f(c, d')}{d_n e_n}$$

Since $v(f(c, d')) = v(e_n d_n)$, we can write $a = s_a + \varepsilon_a$, where $s_a \in \mathbb{R} \setminus \{0\}$ and ε_a is $\mathbb{R}$-*infinitesimal*, i.e. $\forall m \in \omega \setminus \{0\}\ |\varepsilon_a| < 1/m$. Similarly, for all $j < n$, since $v(d_j e_j / d_n e_n) \ge 0$ there are $s_j \in \mathbb{R}$ (now possibly null) and ε_j such that for all $m \in \omega \setminus \{0\}$ we have $|\varepsilon_j| < 1/m$ and $(d_j e_j)/(d_n e_n) = s_j + \varepsilon_j$. Therefore

$$\forall m \in \omega \setminus \{0\}\ c \vDash \left| \frac{d_j}{d_n} \frac{h_j(x)}{h_n(x)} - s_j \right| < \frac{1}{m}$$

This information is, by assumption, in $\pi_M(x) \vdash \operatorname{tp}_w((h_j(c)/h_n(c))/\mathfrak{U}) \cup \{w = h_j(x)/h_n(x)\}$, the $h_i(x)$ being M-definable. It follows that

$$\forall m \in \omega \setminus \{0\}\ q(y) \cup r(x,y) \vdash \left|\frac{d_j}{d_n}\frac{h_j(x)}{h_n(x)} - s_j\right| < \frac{1}{m} \tag{4.4}$$

We can write

$$s_a + \varepsilon_a = \frac{f(c,d')}{d_n e_n} = 1 + \sum_{j<n} \frac{d_j e_j}{d_n e_n} = 1 + \sum_{j<n}(s_j + \varepsilon_j) = \varepsilon' + 1 + \sum_{j<n} s_j$$

where ε' is $\mathbb{R}$-infinitesimal. Hence $1 + \left(\sum_{j<n} s_j\right) - s_a$ is $\mathbb{R}$-infinitesimal and belongs to $\mathbb{R}$, so it is 0, yielding $s_a = 1 + \sum_{j<n} s_j$ and $\varepsilon' = \varepsilon_a$. Since ε_a is $\mathbb{R}$-infinitesimal, and $s_a \neq 0$, in particular $\varphi(\varepsilon_0, \dots, \varepsilon_{n-1})$ holds, where

$$\varphi(u_0, \dots, u_{n-1}) \coloneqq \left|\left(1 + \sum_{j<n}(s_j + u_j)\right) - s_a\right| < \frac{|s_a|}{2}$$

Note that $\varphi(\varepsilon_0, \dots, \varepsilon_{n-1})$ holds for *all* $\mathbb{R}$-infinitesimals $\varepsilon_0, \dots, \varepsilon_{n-1}$. Hence, if $\Phi(t) \coloneqq \{|t| < 1/m \mid m \in \omega \setminus \{0\}\}$, we have $\bigcup_{j<n} \Phi(u_j) \vdash \varphi(u_0, \dots, u_{n-1})$. Therefore, by compactness, for all sufficiently large m we have

$$\vDash \forall w \left(\bigwedge_{j<n} |w_j - s_j| < \frac{1}{m}\right) \to \left(\left|\left(1 + \sum_{j<n} w_j\right) - s_a\right| < \frac{|s_a|}{2}\right) \tag{4.5}$$

By (4.3), $r \vdash f(x,d')/(d_n h_n(x)) = 1 + \sum_{j<n} (d_j h_j(x))/(d_n h_n(x))$. This, together with (4.4) and (4.5), yields

$$q \cup r \vdash \left|\frac{f(x,d')}{d_n h_n(x)} - s_a\right| < \frac{|s_a|}{2}$$

which in turn implies $q \cup r \vdash |f(x,d') - s_a d_n h_n(x)| < |s_a d_n h_n(x)|/2$, and in particular $q \cup r$ proves that $f(x,d')$ and $s_a d_n h_n(x)$ have same sign. But $\mathrm{ED}(\mathfrak{U})$ decides the sign of both $s_a, d_n \in \mathfrak{U}$ and $\pi_M(x)$ decides the cut, hence the sign, of $h_n(x)$. Therefore, $q \cup r$ decides the sign of $f(x,d')$. □

We may therefore apply Theorem 4.1.27, and obtain the following result by Proposition 4.2.27 and Corollary 4.2.32.

Theorem 4.2.37. In RCF, the domination monoid $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is well-defined, coincides with $\overline{\operatorname{Inv}}(\mathfrak{U})$, and $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}, \perp^{\mathrm{w}}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup, \supseteq, D)$, where X is the set of invariant convex subrings of $\mathfrak{U}$.

Remark 4.2.38. By Theorem 4.2.37 and the fact that infinite and infinitesimal elements are in definable bijection, in RCF there is a unique nonrealised definable type up to equidominance.

Note how points of X, i.e. invariant convex subrings of $\mathfrak{U}$, correspond bijectively to invariant convex subgroups of its value group $\Gamma(\mathfrak{U})$ with respect to the Archimedean valuation. In turn, these correspond to invariant cuts in a linear order, itself associated with a valuation, as explained below.

A valuation on an abelian group is defined similarly to a valuation on a field, by dropping the requirements on the interaction with product; it takes values in a linear order, called its *rank*. The *Archimedean valuation* v' on an ordered abelian group Γ is defined by saying that $v'(x) > v'(y)$ iff for all $n \in \omega$ we have $n \cdot |x| < |y|$. The rank of Γ with respect to this valuation is exactly the set of its convex subgroups, ordered by reverse inclusion.

In conclusion, Theorem 4.2.37 tells us that the domination monoid of a monster model $\mathfrak{U}$ of RCF is the upper semilattice of finite subsets of the set of invariant cuts of the Archimedean rank of the Archimedean value group of $\mathfrak{U}$.

Finally, let me point out that in RCF and its o-minimal expansions invariant types have a particularly nice characterisation: a global n-type is A-invariant if and only if, whenever one of its realisations lies in a $\mathfrak{U}$-definable set, this set can be shrunk to have "one side" either in $\operatorname{dcl}(A)$, or at $\pm\infty$, in the sense specified below. By passing from A to $\operatorname{dcl}(A)$, because of the existence of definable Skolem functions, it is enough work over models. By [HP11, Proposition 2.1], a global type is M-invariant if and only if it only implies formulas that do not fork over M, and these have been characterised in [Dol04] as follows. A definable set does not fork over M if and only if it contains a set which is *halfway-definable* over M. In brief, a halfway-definable interval is exactly what the reader expects: an interval with at least one extreme in $M \cup \{\pm\infty\}$. In higher dimension, the definition is inductive and uses the Cell Decomposition Theorem: a set halfway-definable over M is either the graph of an M-definable function on a halfway-definable set, or an open cell based on a halfway-definable set where at least one of the two "boundary" functions is M-definable or $\pm\infty$.

Chapter 5

More examples and counterexamples

In this chapter we look at the domination monoid, or the lack thereof, in some unstable theories which are not o-minimal. Most of the theories under examination in the first two sections are NIP, and some of them already display undesirable (or interesting, depending on the reader's taste) properties. Nevertheless, the truly pathological behaviour of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in Section 5.3 (noncommutativity, non-well-definedness) will occur in the presence of the Independence Property. Whether this is a mere coincidence, a manifestation of some deeper phenomena, or follows from a triviality that I failed to notice, is at present unknown.

In the proofs to come, r will denote a (candidate) witness to domination, as usual. Recall that, if r is trying to witness $p \geq_{\mathrm{D}} q$, we implicitly assume $r \in S_{pq}(A)$, for A a basis of invariance for both p and q. We will not repeat this every time, and simply say e.g. "put in r the formula $\varphi(x, y)$".

5.1 Generic expansions of linear orders

The examples in our first batch are not too far from the realms of o-minimality. This section studies some theories obtained from dense linear orders by adding a single relation or, in the final subsection, "a single point". This is already enough to generate some interesting behaviour.

We will use standard facts about Fraïssé classes multiple times. See for instance [Hod93, Chapter 7] for the basics and terminology of Fraïssé theory.

Recall that, in the context of Fraïssé classes, the *Strong Amalgamation Property* (or *Disjoint Amalgamation Property*) is defined to be the strengthening of the Amalgamation Property obtained by requiring the amalgam to only identify points in the amalgamation base. The following is well-known.

Remark 5.1.1 (Folklore). Suppose that K_0, K_1 are two Fraïssé classes of finite structures with strong amalgamation, in relational languages L_0, L_1 respectively, such that $L_0 \cap L_1 = \emptyset$. If K is the class of all finite $L_0 \cup L_1$-structures such that, for each $A \in K$ and $i < 2$, we have $A \upharpoonright L_i \in K_i$, then K is a Fraïssé class with strong amalgamation as well.

Proof. Recall that Fraïssé classes are closed under taking substructures and isomorphisms, hence whenever $i < 2$, $B_0, B_1 \in K_i$, and $B_0 \cap B_1 = A$, the domain of a strong amalgam of B_0 and B_1 over A (with maps the inclusions) can be taken to be $B_0 \cup B_1$ (with maps the inclusions). Since $L_0 \cap L_1 = \emptyset$, it follows that we may amalgamate L_0-structures and L_1-structures on the same sets independently. □

5.1.1 A single predicate

Definition 5.1.2. Let $L := \{<, P\}$, and let DLOP be the L-theory asserting that $<$ is a DLO and P is a dense and codense unary predicate.

There are two reasons why this theory is interesting. It provides an example where $\sim_{\mathrm{D}}$ differs from $\equiv_{\mathrm{D}}$, and it is a NIP example where $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is commutative, but $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not. I am grateful to Ehud Hrushovski for pointing out this fact, and allowing me to include it here.

It is easy to see that DLOP is the theory of the Fraïssé limit of the class of finite linear orders with a predicate, to which Remark 5.1.1 applies, hence it is complete and eliminates quantifiers in L. From this we see immediately that it is binary, hence $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ and $\overline{\mathrm{Inv}}(\mathfrak{U})$ are well-defined by Corollary 2.2.17. Using quantifier elimination again, together with the fact that it is enough to test NIP up to Boolean combinations and in dimension 1 (see [Sim15, Lemma 2.9, Proposition 2.11]), DLOP can be shown to be NIP.

A 1-type is determined by a cut together with a "colour", where we say that two points *have the same colour* iff they are both in P or both in $\neg P$. The invariant 1-types will be those with small cofinality on exactly one side, in the terminology of Definition 4.1.1.

Proposition 5.1.3. Let $p(x)$ be the type at $+\infty$ in P, and $q(y)$ the type at $+\infty$ in $\neg P$. Then $p \sim_{\mathrm{D}} q$ but $p \not\equiv_{\mathrm{D}} q$.

Proof. Note that both types are $\emptyset$-invariant, and in fact $\emptyset$-definable. To show domination-equivalence, observe that $p(x) \geq_{\mathrm{D}} q(y)$ [resp. $p(x) \leq_{\mathrm{D}} q(y)$] can be witnessed by any r containing $y > x$ [resp. $y < x$]. As for $p \not\equiv_{\mathrm{D}} q$, take any small A and any $r \in S_{pq}(A)$. Since $(p(x) \restriction \emptyset) \cup (q(y) \restriction \emptyset) \vdash P(x) \land \neg P(y)$ we have $r \vdash x \neq y$, and since A is small there is $b \in \mathfrak{U}$ such that $b > A$. It follows from quantifier elimination that, if $r \vdash x > y$, then $p \cup r \nvdash y > b$, and a fortiori $p \cup r \nvdash q$. Similarly, if $r \vdash y > x$, then $q \cup r \nvdash p$. □

Proposition 5.1.4. Suppose $p_0(x) \in S^{\mathrm{inv}}(\mathfrak{U})$ asserts that $x_0 < \ldots < x_{|x|-1}$ all lie in the cut C. Then $p_0 \equiv_{\mathrm{D}} p_1$ where, if C has small cofinality on the right [resp. left], $p_1(y)$ is the type of an element in C of the same colour as x_0 [resp. $x_{|x|-1}$].

Proof. Use $y = x_0$ in the first case, and $y = x_{|x|-1}$ in the second. □

Proposition 5.1.5 (Hrushovski)**.** In DLOP, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not commutative.

Proof. Let p and q be the non-equidominant types from Proposition 5.1.3. By Proposition 5.1.4 we have $(p \otimes q) \equiv_{\mathrm{D}} q$ and $(q \otimes p) \equiv_{\mathrm{D}} p$. Therefore $(p \otimes q) \equiv_{\mathrm{D}} q \not\equiv_{\mathrm{D}} p \equiv_{\mathrm{D}} (q \otimes p)$. □

This counterexample exploits crucially $\equiv_{\mathrm{D}}$, as opposed to $\sim_{\mathrm{D}}$. In fact, in DLOP $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is the same as in the restriction of $\mathfrak{U}$ to $\{<\}$, and in DLO the monoid $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes) \cong (\mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U})), \cup)$ is commutative. The following proposition is another reason why $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is better behaved than $\overline{\mathrm{Inv}}(\mathfrak{U})$.

Proposition 5.1.6. In DLOP, the monoid $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ cannot be endowed with any order $\leq$ compatible with $\otimes$ in which $[\![0]\!]$ is the minimum.

Proof. Let p and q be as in Proposition 5.1.3. If we had an order $\leq$ as above then we would get

$$[\![p]\!] = [\![p]\!] \otimes [\![0]\!] \leq [\![p]\!] \otimes [\![q]\!] = [\![q]\!] = [\![q]\!] \otimes [\![0]\!] \leq [\![q]\!] \otimes [\![p]\!] = [\![p]\!]$$

contradicting $[\![p]\!] \neq [\![q]\!]$. □

Lemma 5.1.7. Let $p_x, q_y \in S(\mathfrak{U})$, and suppose that there are cuts $C_p \neq C_q$ such that for all $i < |x|$ and $j < |y|$ the type p proves that the cut of x_i is C_p and the type q proves that the cut of y_j is C_q. Then $p \perp^{\mathrm{w}} q$.

Proof. Immediate from quantifier elimination. □

Proposition 5.1.3 also holds, with essentially the same proof, for every cut C. Putting everything together, we obtain a description of $\overline{\mathrm{Inv}}(\mathfrak{U})$. Let $\mathrm{IC}(\mathfrak{U})$ be as in Example 2.2.22. Let $p_{C,0}(x)$ [resp. $p_{C,1}(x)$] denote the 1-type saying that x is in the cut C and $P(x)$ holds [resp. $\neg P(x)$ holds].

Theorem 5.1.8. In DLOP, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is generated by $\{[\![p_{C,i}]\!] \mid C \in \mathrm{IC}(\mathfrak{U}), i < 2\}$ with relations

- $[\![p_{C,i}]\!] \otimes [\![p_{C,j}]\!] = [\![p_{C,j}]\!]$
- $[\![p_{C_0,i}]\!] \otimes [\![p_{C_1,j}]\!] = [\![p_{C_1,i}]\!] \otimes [\![p_{C_0,j}]\!]$ for $C_0 \neq C_1$.

Proof. By the previous lemma and Fact 4.1.18, up to discarding duplicate and realised coordinates, every type can be written as a product $p_0(x^0) \otimes \ldots \otimes p_n(x^n)$, where the cut of x^i_j only depends on i and if $i_0 \neq i_1$ then $x^{i_0}_0$ has a different cut than $x^{i_1}_0$. Types in different cuts are weakly orthogonal, hence commute, and we conclude by Proposition 5.1.4. □

5.1.2 An equivalence relation

Let $L = \{<, E\}$. The classes of finite linear orders and of finite equivalence relations fall in the setting of Remark 5.1.1. Therefore, the finite L-structures where $<$ is a linear order and E is an equivalence relation form a Fraïssé class with strong amalgamation. In this subsection, we study the theory of their Fraïssé limit.

Definition 5.1.9. Let DLODE be the L-theory where $<$ is a dense linear order without endpoints and E is an equivalence relation with infinitely many classes, all of which are dense, in the sense that they meet every open interval.

For reasons completely analogous to those explained at the beginning of the previous subsection, DLODE is complete, eliminates quantifiers, is NIP, and $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined. A 1-type $p(x)$ is then determined by the cut of x together with its equivalence class. Given a cut C and a point $a \in \mathfrak{U}$, we denote with $p_{C,a}$ the type concentrating in the cut C and in the equivalence class of a, and with $p_{C,\mathrm{g}}$ the type concentrating in the cut C and in a new equivalence class.

Remark 5.1.10. By quantifier elimination, two nonrealised 1-types are non-weakly orthogonal if and only if they are of the forms $p_{C,a}$ and $p_{C,a'}$, of the forms $p_{C,a}$ and $p_{C,\mathrm{g}}$, or the forms $p_{C,\mathrm{g}}$ and $p_{C',\mathrm{g}}$.

Counterexample 5.1.11. Let C be the cut at $+\infty$. For any $a \in \mathfrak{U}$, the type $p_{C,a}$ is dominated by an $\emptyset$-invariant type, but is not domination-equivalent to any $\emptyset$-invariant type, as we now show. It is easy to see that $p_{C,\mathrm{g}} \geq_{\mathrm{D}} p_{C,a}$. On the other hand, $p_{C,a}$ cannot be domination-equivalent to any $\emptyset$-invariant type: such a type cannot place any of its variables in any realised equivalence class, hence dominates some type of the form $p_{C',\mathrm{g}}$, but by quantifier elimination $p_{C,a}$ does not dominate any of these types.

The reader might object that we may have fabricated such a counterexample in the theory of a pure set, by taking the unique nonrealised 1-type, and observing it dominates any realised type, none of which can be $\emptyset$-invariant. Anyway, this is not a counterexample, because all realised types are equidominant to the unique 0-type.

Besides this pedantry — the reader may object again — we could have gone to the theory T_2 of two cross-cutting generic equivalence relations E_0, E_1 and produced another instance of this behaviour by observing that if p is the 1-type in a new class for both E_i, and q is the 1-type in a new E_0-class, but in the E_1-class of a point a, then $p \geq_{\mathrm{D}} q$. This is true, but then the problem disappears as soon as we pass to T_2^{eq}, as then q is domination-equivalent to the unique nonrealised type in the sort $\mathfrak{U}/E_0$. By contrast, in $\mathsf{DLODE}^{\mathrm{eq}}$, the $\sim_{\mathrm{D}}$-class of $p_{C,a}$ still fails to contain any $\emptyset$-invariant types.

Since we are already working in DLODE, we take the time to observe a phenomenon which cannot arise in stable contexts. This requires an easy preparatory lemma.

Lemma 5.1.12. In DLODE, all the generically stable types are realised.

Proof. Let $p(x)$ be nonrealised and pick any nonrealised coordinate, say x_0. To contradict generic stability over A, it is enough to build a Morley sequence $(a^i)_{i<\omega+\omega}$ of p over A, and then consider the formula $x_0 < b$, where b is any point between a_0^ω and $a_0^{\omega+1}$. □

Corollary 5.1.13. In DLODE, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ changes when passing to T^{eq}.

Proof. Let p be the unique nonrealised type in the sort $\mathfrak{U}/E$. The structure on this sort is that of a pure set, hence p is generically stable. By Theorem 2.3.7 and Lemma 5.1.12, p cannot be domination-equivalent to any type with all variables in the home sort. □

So, what is $\widetilde{\mathrm{Inv}}(\mathfrak{U})$? If we add a sort for $\mathfrak{U}/E$, it can be described as $\mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U})) \oplus \mathbb{N}$, where $\mathrm{IC}(\mathfrak{U})$ is as in Example 2.2.22, and the generic type of $\mathfrak{U}/E$ corresponds to $(\emptyset, 1)$. This can be shown using quantifier elimination, Remark 5.1.10, and the fact that the sort $\mathfrak{U}/E$ is stably embedded and the structure induced on it is that of a pure set.

Even without adding the sort $\mathfrak{U}/E$, the domination-equivalence class of a type $p(x)$ is determined by the cuts of the x_i, together with the number of equivalence classes represented in x but not in $\mathfrak{U}$. As we essentially already observed above, if $n > 0$, no type in any cartesian power of the home sort corresponds to $(\emptyset, n) \in \mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U})) \oplus \mathbb{N}$, as such a type would need to be generically stable and this is forbidden by Lemma 5.1.12. In other words, if x_i is in a new equivalence class, then it must fill a nonrealised cut. It is then easy to see that, if we just work in the sort $\mathfrak{U}$, we have

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \{(\emptyset, 0)\} \cup \{(A, n) \in \mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U})) \oplus \mathbb{N} \mid A \neq \emptyset\}$$

5.1.3 Another order

Consider the language $L := \{<_0, <_1\}$. Finite L-structures M in which both $<_i$ are interpreted as total orders are sometimes called *finite permutation structures* since, if $|M| = n$, then the unique pair of isomorphisms from $(M, <_i)$ to $n = \{0, \ldots, n-1\}$ with the usual ordering codes an element of the symmetric group S_n on n. By Remark 5.1.1, finite permutation structures form a Fraïssé class with strong amalgamation, whose Fraïssé limit is called the *generic permutation structure*. Its theory is the model completion of the theory of two linear orders, eliminates quantifiers, and is NIP. Essentially, we are dealing with two "independent" DLO, in the sense that each $<_i$-open interval is dense in each $<_{1-i}$-open interval. Nonrealised 1-types are given by specifying cuts for the two orders (see. [Poi00, p. 268]), and they are invariant if and only if both cuts have small cofinality on exactly one side (which, of course, could be different for the two orders, e.g. the left side for $<_0$ and the right side for $<_1$). It is (see [Sim15, after Example 4.16]) a theory where no type has dp-rank exactly 1. By quantifier elimination in a binary language, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined.

Naively, we could expect something like $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathfrak{U}\restriction<_0) \times \widetilde{\mathrm{Inv}}(\mathfrak{U}\restriction<_1)$ to hold, but this not the case. In fact, our interest in this theory arises from its display of a rather pathological behaviour.

Counterexample 5.1.14. In the theory of the generic permutation structure, there are two invariant types p_0, p_1 such that if $p_0 \geq_{\mathrm{D}} q$ and $p_1 \geq_{\mathrm{D}} q$, then q is realised, yet $p_0 \not\perp^{\mathrm{w}} p_1$.

Proof. The crucial observation is that it is not possible for the $<_i$-cut of a type to be realised, unless its $<_{1-i}$-cut is realised as well.

Let $p_0(x)$ and $p_1(y)$ be two 1-types that concentrate in the same $<_0$-cut but in different $<_1$-cuts. Since $p_0(x) \cup p_1(y)$ has a completion with $x <_0 y$ and one with $y <_0 x$, we have $p_0 \not\perp^{\mathrm{w}} p_1$. Yet, the only types which are dominated simultaneously by p_0 and p_1 are realised, as we now show. Assume that $p_0(x) \geq_{\mathrm{D}} q(z)$, and fix a coordinate z_i. As can be seen directly using quantifier elimination, the $<_1$-cut of z_i is either realised, or it must be the same as that of p_0. If even a single z_i is not realised, then $p_1 \not\geq_{\mathrm{D}} q$. □

The same reader who was raising objections after Counterexample 5.1.11 may want to point out that we could have done something similar in DLODE by considering $p_{C_0,\mathrm{g}}$ and $p_{C_1,\mathrm{g}}$ for distinct cuts C_0 and C_1. As the reader will probably expect, in that case the problem disappears after passing to T^{eq}, since both types dominate the unique nonrealised type in the sort $\mathfrak{U}/E$.

This phenomenon is also observable in the theory of the Random Graph (see Remark 5.3.7), but we have just shown that it is not excluded by NIP. This is an obstruction that needs to be dealt with if one wants to try and generalise certain results from stable to NIP theories. See Subsection 7.2.4.

As we observed above, it is not possible to have a nonrealised cut for exactly one of the orders, and it is easy to show that there is an isomorphism

$$f\colon \widetilde{\mathrm{Inv}}(\mathfrak{U}) \to \{(\emptyset,\emptyset)\} \cup \big((\mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U},<_0)) \setminus \{\emptyset\}) \oplus (\mathscr{P}_{\mathrm{fin}}(\mathrm{IC}(\mathfrak{U},<_1)) \setminus \{\emptyset\})\big)$$

such that, whenever $f(\llbracket p \rrbracket) = (a,b)$ and $f(\llbracket q \rrbracket) = (c,d)$, we have

$$\llbracket p \rrbracket \perp^{\mathrm{w}} \llbracket q \rrbracket \iff a \cap c = b \cap d = \emptyset$$

5.1.4 The dense circular order

The purpose of this very short subsection is to present a theory which is weakly binary, but not binary.[1] Although it becomes binary after naming a point, it can be used as a basis to build a weakly binary theory that does not become binary after naming constants. See Example 2.2.13.

Given the very little space that we will devote to this theory, instead of giving axioms, we will give a "definition by example", and invite the reader to think of the circle $S^1(\mathbb{R})$ with the relation $C(x, y, z)$ holding if, starting from x and moving clockwise, y is encountered not after z. The theory of *dense circular orders* is then $\mathrm{Th}(S^1(\mathbb{R}), C)$. This theory eliminates quantifiers in $\{C\}$. See [BMMN98] for more details.

We can think about a circular order as a reduct of a linear order, or as a sort of "one-point compactification" of one.

Fact 5.1.15 ([BMMN98, Theorem 11.9])**.** If $(M, <)$ is a linear order, then

$$C(x, y, z) \coloneqq (x \le y \le z) \lor (z \le x \le y) \lor (y \le z \le x)$$

defines a circular order on M. If (N, C) is a circular order and $a \in N$, then $y \le z \coloneqq C(a, y, z)$ defines a linear order on $N \setminus \{a\}$. If we extend this order to N by declaring that $\forall y\; y \le a$, then the circular order induced by $\le$ is C.

It follows immediately from this fact that, after naming a point, the dense circular order becomes binary. On the other hand, by quantifier elimination, if we just work in the language $\{C\}$, this theory is not binary. For example, take any nonrealised $p(x) \in S_1(\mathfrak{U})$, take two distinct realisations b_0, b_1 of p, and let d be any point of $\mathfrak{U}$. Then, $\pi(x) \coloneqq p(x_0) \cup p(x_1) \cup \mathrm{tp}_{x_0x_1}(b_0b_1/\emptyset)$ does not decide which one between $C(b_0, b_1, d)$ and $C(b_0, d, b_1)$ holds.

5.2 Trees and related structures

From Chapter 4 until now we have been dealing with structures based on linear orders. In this section, we focus on structures built around trees. These are not the same trees we encountered in Subsection 3.2.4, but rather dense lower semilinear orders with finite meets and everywhere infinite ramification.

[1] Nor, strictly speaking, a generic expansion of a linear order, despite the title of this section. Nevertheless, see Fact 5.1.15.

Proofs regarding trees have a tendency to split in cases and subcases. As they become incredibly easier to follow if the objects in them are drawn as soon as they appear in the proof, the reader is encouraged to reach for writing devices, preferably capable of producing different colours.

5.2.1 Dense meet-trees

Definition 5.2.1. Let $L = \{<, \sqcap\}$, where $<$ is a binary relation symbol and $\sqcap$ is a binary function symbol. A *meet-tree* is an L-structure M such that $(M, <)$ is a lower semilinear order, i.e. a poset in which the order induced on all sets of the form $\{x \in M \mid x \leq a\}$ is linear, and where every pair of elements a, b has a greatest common lower bound, the *meet* $a \sqcap b$. If M is a meet-tree and $g \in M$, classes of the equivalence relation defined on $\{x \in M \mid x > g\}$ by $E(x, y) \coloneqq x \sqcap y > g$ are called *open cones above* g.

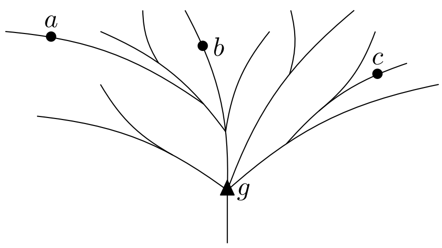


Figure 5.1: the point a is in the same open cone above g as the point b, while c is in a different open cone above g.

It can be shown that finite meet-trees form a Fraïssé class.

Definition 5.2.2. A *dense meet-tree* is a model of the theory of the Fraïssé limit of finite meet-trees.

Fact 5.2.3. The theory of dense meet-trees is axiomatised by saying that

1. $(M, <, \sqcap)$ is a meet-tree;
2. for every $a \in M$, the structure $(\{x \in M \mid x < a\}, <)$ is a DLO; and
3. for every $a \in M$, there are infinitely many open cones above a.

This theory is not binary, due to the presence of the binary function symbol $\sqcap$. In fact, it is easy to see that it stays non-binary even after adding constants. We will show in Theorem 5.2.15 that dense meet-trees are weakly binary.

Remark 5.2.4. The operation $\sqcap$ is associative, idempotent, and commutative. Using this and quantifier elimination, and observing for example that for every a, b the set defined by $x \sqcap a = b$ is either empty or infinite, it is easy to see that the definable closure of every set A coincides with its closure under meets. In particular, if A is finite, then so is $\operatorname{dcl}(A)$: by the properties of $\sqcap$ we just pointed out, its size cannot exceed that of the powerset of A.

Definition 5.2.5. Define the *cut* C_p of a type $p(x) \in S_1(M)$ to be $\{c \in M \mid p \vdash x \geq c\}$ and the cut in M of an element b to be $C_b^M := C_{\operatorname{tp}(b/M)}$. We say that C_p is *bounded* iff it is bounded from above in M.

This meaning of the word "cut" is a bit more general than the one we used for linear orders: now cuts have no right/upper part, only a left/lower one.

It can be shown using standard techniques that dense meet-trees eliminate quantifiers in L and are NIP, in fact dp-minimal. This makes them amenable to an analysis of invariant types using indiscernible sequences, and it turns out that invariant 1-types are necessarily of one of the six kinds below, as shown by using *eventual types* (see [Sim15, Subsection 2.2.3]). We refer the reader to [Sim11] and [Sim15, Subsection 2.3.1]. Alternatively, it is possible to prove this directly via quantifier elimination by considering, for a fixed $p(x) \in S_1^{\mathrm{inv}}(\mathfrak{U})$, what are the possible values of each $d_p\varphi$, as $\varphi(x; y)$ ranges among L-formulas.

Definition 5.2.6. Let $p(x) \in S_1(\mathfrak{U})$. We say that p is of kind

(0) iff p is realised;

(Ia) iff there is a small linearly ordered set A such that $p(x) \vdash \{x < a \mid a \in A\} \cup \{x > b \mid b < A\}$;

(Ib) iff there is a small linearly ordered set A with no maximum such that $p(x) \vdash \{x > a \mid a \in A\} \cup \{x < b \mid b > A\}$, or if there are a and c such that $p(x) \vdash \{a < x < c\} \cup \{x < b \mid a < b < c\}$;

(II) iff there is g such that $p(x) \vdash \{x > g\} \cup \{x \sqcap b = g \mid b > g\}$;

(IIIa) iff $p(x) \vdash \{x \not\leq b \mid b \in \mathfrak{U}\}$ and there is $c \in \mathfrak{U}$ such that $\operatorname{tp}(x \sqcap c/\mathfrak{U})$ is of kind (Ia);

(IIIb) iff $p(x) \vdash \{x \not\leq b \mid b \in \mathfrak{U}\}$ and there is $c \in \mathfrak{U}$ such that $\operatorname{tp}(x \sqcap c/\mathfrak{U})$ is of kind (Ib).

So types of kind (0), (Ia), or (Ib) correspond to cuts in a linearly ordered subset of the tree, where in kind (Ib), if the cut of p has a maximum a, we are specifying an existing open cone above a. Kinds (II), (IIIa), and (IIIb) are the corresponding "branching" versions. Kind (II) is the type of an element in a new open cone above an existing point. See Figure 5.2.

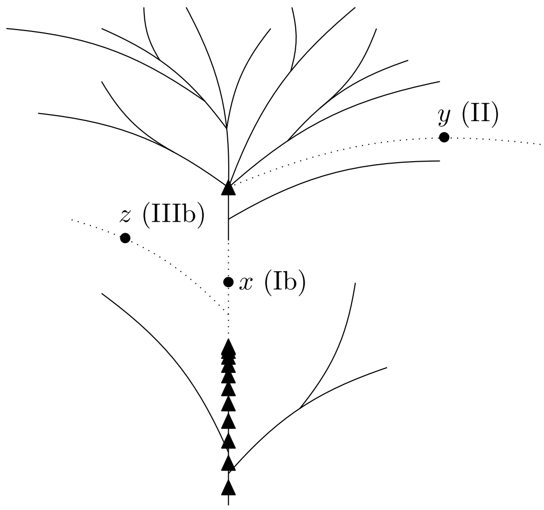


Figure 5.2: some nonrealised B-invariant types, where points of B are denoted by triangles. In this picture, the set of triangles below x has no maximum, solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 \mathrel{^+{\succ}} \mathfrak{U}$. The type of x is of kind (Ib), that of y of kind (II), and that of z of kind (IIIb).

It can be shown that generically stable 1-types are of kind (0) or of kind (II), while other kinds are definable or finitely satisfiable, but not both.

Lemma 5.2.7.

1. Let $b_0, b_1 \in N \succ M$. If $C_{b_0}^M \subseteq C_{b_1}^M$ then $C_{b_0 \sqcap b_1}^M = C_{b_0}^M$. If none of $C_{b_0}^M$ and $C_{b_1}^M$ is included in the other, then $b_0 \sqcap b_1 \in M$.

2. For all $b_0, \dots, b_n \in N \succ M$, points of $\operatorname{dcl}(Mb_0, \dots, b_n)$ are either in M or have the same cut as one of the b_i.

3. If $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ then C_p is bounded.

Proof. The first part is clear from the definitions of cut and meet, and the second one follows by induction. The last part follows from the characterisation of invariant 1-types. □

Proposition 5.2.8. In dense meet-trees the following statements hold.

1. Suppose all coordinates of $p \in S(\mathfrak{U})$ have the same cut C_0, all coordinates of $q \in S(\mathfrak{U})$ have the same cut C_1, and $C_0 \neq C_1$. Then $p \perp^{\mathrm{w}} q$.

2. Let C be a cut with a maximum g. Suppose that all 1-subtypes of p are of kind (Ib) with cut C and all 1-subtypes of q are of kind (II) with cut C, or that all 1-subtypes of p, q are of kind (Ib) with cut C, but no open cone above g contains both a coordinate of p and a coordinate of q. Then $p \perp^{\mathrm{w}} q$.

3. Every type of kind (IIIa) is domination-equivalent to a type of kind (Ia). Similarly for kind (IIIb).

In particular, if $p, q \in S_1^{\mathrm{inv}}(\mathfrak{U})$, then either $p \perp^{\mathrm{w}} q$ or $p \sim_{\mathrm{D}} q$.

Proof. ① Immediate from quantifier elimination and Lemma 5.2.7.

② This does not follow from the previous point because such types have the same cut, but it is still easy from quantifier elimination and the fact that the open cones in which types of kind (II) concentrate are new, while those of types of kind (Ib) are realised.

③ We only prove the statement for kind (IIIa); the statement for kind (IIIb) has an analogous proof. Suppose that $c \in \mathfrak{U}$ and $A \subset^+ \mathfrak{U}$ are such that

$$p(x) \vdash \{x \not\leq b \mid b \in \mathfrak{U}\} \cup \{x \sqcap c < a \mid a \in A\} \cup \{x \sqcap c > b \mid b < A\}$$

Let $q(y)$ be the pushforward of $p(x)$ under the definable function $y = x \sqcap c$. By this very description, $p(x) \geq_{\mathrm{D}} q(y)$ and, by definition of type of kind (IIIa), the type q is of kind (Ia). To prove $q(y) \geq_{\mathrm{D}} p(x)$, use $\{a > (x \sqcap c) > y \mid a \in A\}$, and recall that r contains $p(x) \restriction A$, which proves $x \not\leq a$ for all $a \in A$. See Figure 5.3. □

In the previous proposition, it is important that we work with $\sim_{\mathrm{D}}$, as opposed to $\equiv_{\mathrm{D}}$. While using some r containing[2] $x \sqcap c = y$ would still work to show that every type of kind (IIIa) is equidominant to one of type (Ia), trees are not symmetric, and this would not work for kind (IIIb), as shown below. This is another reason why $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is better behaved than $\overline{\mathrm{Inv}}(\mathfrak{U})$.

[2]As usual, remember that the domain A of r has to be large enough for p, q to be A-invariant. Using $q(y) \cup \{x \sqcap c = y\}$ alone is not enough to show $x \neq c$, and if $\{a \in \mathfrak{U} \mid p \vdash x \sqcap c < a\}$ does not have a minimum then no single formula is enough to show $q \geq_{\mathrm{D}} p$.

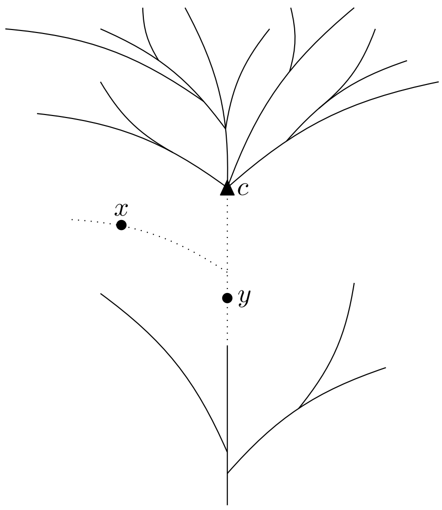


Figure 5.3: proof of Proposition 5.2.8, how to show that $q(y) \geq_{\mathrm{D}} p(x)$. In this picture A only contains the point c, denoted with a triangle. Solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 \mathrel{^{+}\!\succ} \mathfrak{U}$.

Remark 5.2.9. Let $p(x)$ and $q(y)$ be the types respectively of kind (IIIb) and (Ib) with cut $\emptyset$. Then $p \not\equiv_{\mathrm{D}} q$.

Proof. Suppose that $r(x, y)$ witnesses equidominance. If $r(x, y) \vdash x \sqcap y < y$, then $p(x) \cup r(x, y) \nvdash q(y)$, since by quantifier elimination and compactness it cannot prove all formulas $y < d$, for $d \in \mathfrak{U}$. If $r(x, y) \vdash x \sqcap y = y$, then $q(y) \cup r(x, y) \nvdash p(x)$, since it cannot prove all formulas $x \not\leq d$. □

Proposition 5.2.10. In the theory of dense meet-trees the following hold.

1. Types of kind (Ia) and (Ib) are idempotent modulo equidominance.
2. If p is of kind (II) and $m < n \in \omega$ then $p^{(m)} \not\geq_{\mathrm{D}} p^{(n)}$.

Proof. ① This is similar to the o-minimal case, but the structure here is simpler and we do not need the full Idempotency Lemma. It follows easily from quantifier elimination that to show $p(x_1) \otimes p(x_0) \equiv_{\mathrm{D}} p(y)$ it is enough to use a suitable r containing the formula $x_0 = y$.

② For notational simplicity we show the case $m = 1$, $n = 2$, the general case being analogous. Suppose that p is the type of a new open cone above g, i.e. $p(y) \vdash \{y > g\} \cup \{y \sqcap b = g \mid b > g\}$. We want to show that there is no small $r(y, x_0, x_1)$ such that $p(y) \cup r \vdash p(x_1) \otimes p_0(x_0)$. Since $p^{(2)} \upharpoonright \{g\}$

proves $x_0 \sqcap x_1 = g$, i.e. that the cones of x_0 and x_1 are distinct, there is $i < 2$ such that $r \vdash y \sqcap x_i = g$. Since r is small there is $d > g$ in $\mathfrak{U}$ such that $p(y) \cup r \nvdash x_i \sqcap d = g$; in other words it is not possible, with a small type, to say that x_i is in a new open cone, unless it is the same cone as y, but y cannot be in the open cones of x_0 and x_1 simultaneously. □

It will follow from Theorem 5.2.15 that dense meet-trees are weakly binary, hence $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is well-defined. We assume this for now, and characterise the domination monoid. By the results above, nonrealised generically stable 1-types, i.e. those of kind (II), generate a copy of $\mathbb{N}$, while all other 1-types are idempotent. We have also seen that if p, q are nonrealised 1-types, then either $p \perp^{\mathrm{w}} q$ or $p \sim_{\mathrm{D}} q$. In particular, all pairs of 1-types commute modulo domination-equivalence. To complete our study we need one last ingredient.

Proposition 5.2.11. In the theory of dense meet-trees, every invariant type is domination-equivalent to a product of invariant 1-types.

Proof. By Proposition 5.2.8 and Fact 4.1.18 we reduce to showing the conclusion for types $p(x)$ consisting of elements all with the same cut C_p.

Assume first that C_p does not have a maximum, let $c \vDash p(x)$ and let $d \in \mathfrak{U}$ be such that $d > C_p$. Let $H = \{h_0(c), \dots, h_n(c)\}$ be the (finite, by Remark 5.2.4) set of points in $\operatorname{dcl}(cd)$ such that $d > h_i(c)$, where each $h_i(x)$ is an $\emptyset$-definable function. Suppose that $h_0(c) = \min H$ and $h_n(c) = \max H$. We have two subcases. If C_p has small cofinality, let $q(y)$ be of kind (Ib) with $C_q = C_p$. Let A be such that $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$, let $r(x, y) \in S_{pq}(Ad)$ contain the formula $h_n(x) < y$, and note that $q(y) \cup r(x, y)$ implies the type over $\mathfrak{U}$ of each point of $\operatorname{dcl}(xd)$, i.e. of the closure of xd under meets. It follows from quantifier elimination that $q \cup r \vdash p$. To prove $p \cup r \vdash q$, use instead some r containing the formula $y < h_0(x)$. In the other subcase, $\{e \in \mathfrak{U} \mid C_p < e < d\}$ has small coinitiality. The argument is analogous, except we need to use an r containing $h_0(x) > y$ to show $q \cup r \vdash p$ and one containing $h_n(x) < y$ to show $p \cup r \vdash q$.

Suppose now that C_p has maximum g. Assume without loss of generality that $c_0, \dots, c_{k-1}$ are the points of c such that there is $d_i \in \mathfrak{U}$ such that $d_i \sqcap c_i > g$. In other words, these are the points in an existing open cone above g, and $c_k, \dots, c_{|c|-1}$ are in new open cones. Again by quantifier elimination, we have $\operatorname{tp}(c_0, \dots, c_{k-1}/\mathfrak{U}) \perp^{\mathrm{w}} \operatorname{tp}(c_k, \dots, c_{|c|-1}/\mathfrak{U})$, so we can deal with the two subtypes separately. Similarly, by using weak orthogonality we may split $c_{<k}$

further, and we may assume that for $i < \ell$, say, all c_i are in the same open cone, say that of the point $d \in \mathfrak{U}$. It is now enough to proceed as in the previous case, by taking $q(y)$ to be the type of kind (Ib) with the same cut and open cone above g. As for $c_k, \ldots, c_{|c|-1}$, let H be the set of minimal elements of $\operatorname{dcl}(c_k, \ldots, c_{|c|-1}) \setminus \mathfrak{U}$. Let $q(y)$ be the type of kind (II) above g. To conclude, let r identify elements of H with coordinates of a realisation of $q^{(|H|)}$. $\square$

The previous results (together with Theorem 5.2.15, which we still have to prove) yield the following characterisation of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ in dense meet-trees.

Theorem 5.2.12. In dense meet-trees, $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{\mathrm{fin}}(X) \oplus \bigoplus_{g\in\mathfrak{U}} \mathbb{N}$. Generators of copies of $\mathbb{N}$ correspond to types of new open cones above a point $g \in \mathfrak{U}$, i.e. to types of kind (II), while each point of X corresponds to, either:

1. a linearly ordered subset of $\mathfrak{U}$ with small coinitiality, modulo mutual coinitiality; this corresponds to types of kind (Ia)/(IIIa);
2. a cut with no maximum, but with small cofinality; this corresponds to some types of kind (Ib)/(IIIb);
3. an existing open cone above an existing point; this corresponds to the rest of the types of kind (Ib)/(IIIb).

5.2.2 Binary cone-expansions

Here we show that certain expansions of dense meet-trees are weakly binary. I would like to thank Itay Kaplan for bringing them to my attention.

The main result of this subsection, Theorem 5.2.15, applies for instance to the theory obtained by equipping every set of open cones above a point with a structure elementarily equivalent to the Random Graph. This theory was studied in [EK19], which also contains a more general study of theories of trees with relations on sets of open cones.

Notation 5.2.13. We write $x \parallel y$ to mean that $x \not\leq y$ and $y \not\leq x$.

Lemma 5.2.14. In the theory of dense meet-trees, let b be a finite tuple such that each $C_{b_i}^{\mathfrak{U}}$ is bounded. Then there is a finite tuple d such that $\mathfrak{U}bd$ is closed under meets. Moreover, d can be chosen such that additionally, if we let $c := \mathfrak{U} \cap d$, then $d \in \operatorname{dcl}(bc)$, and for every $e \in bd \setminus \mathfrak{U}$ the following happens.

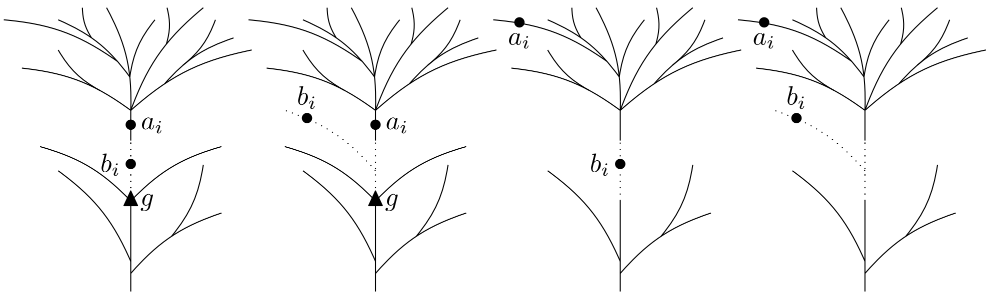


Figure 5.4: how to choose a_i in the proof of Lemma 5.2.14. In the first two pictures, $C^{\mathfrak{U}}_{b_i}$ has a maximum, g, denoted by a triangle. In the last two it does not have one. Solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 \ {}^{+}\!\succ \mathfrak{U}$.

1. There is $a_e \in c$ such that $a_e > C^{\mathfrak{U}}_e$.

2. If $C^{\mathfrak{U}}_e$ has a maximum g and e is in an existing open cone above g, then this is the open cone of a_e.

Proof. If $C^{\mathfrak{U}}_{b_i}$ has a maximum g and b_i is in an open cone above g which intersects $\mathfrak{U}$, let $a_i \in \mathfrak{U}$ be such that $a_i \sqcap b_i > g$ (see first half of Figure 5.4); otherwise (second half of the same figure), choose any $a_i > C^{\mathfrak{U}}_{b_i}$. The closure of ba under meets is finite by Remark 5.2.4; enumerate the points of this closure which are not in b in a tuple d. Recall that we defined $c \coloneqq \mathfrak{U} \cap d$, and note that, by construction, $d \in \operatorname{dcl}(bc)$.

We now prove the "moreover" part, and then show how closure under meets of $\mathfrak{U}bd$ follows. Let $e \in bd \setminus \mathfrak{U}$. By Lemma 5.2.7, construction, and the fact that $e \notin \mathfrak{U}$, there is $i < |b|$ such that e is obtained as the meet of b_i with other points with the same cut (possibly none). In particular $e \le b_i$ and $C^{\mathfrak{U}}_e = C^{\mathfrak{U}}_{b_i}$.

① Let i be as above. Since $C^{\mathfrak{U}}_e = C^{\mathfrak{U}}_{b_i}$, we have $a_e \coloneqq a_i > C^{\mathfrak{U}}_e$.

② Let i and a_e be as above. By choice of $a_i = a_e$, we have $a_i \sqcap b_i > g$. By construction and the fact that $e \notin \mathfrak{U}$, we have $g < e \le b_i$, so $e \sqcap b_i = e > g$ and e and b_i are in the same open cone above g, which is that of a_i. This completes the proof of 2, hence of the "moreover" part.

We are left to prove that $\mathfrak{U}bd$ is closed under meets. As both $\mathfrak{U}$ and bd are, and $\sqcap$ is commutative, all we need to show is that if $e \in bd \setminus \mathfrak{U}$ and $f \in \mathfrak{U}$ then $f \sqcap e \in \mathfrak{U}bd$. If e and f and comparable there is nothing to prove, so assume they are not, i.e. that $e \parallel f$.

Claim. To conclude, it is enough to show that $f \sqcap e \le f \sqcap a_e$.

Proof of Claim. By assumption, commutativity, and idempotency of $\sqcap$ we have $f \sqcap e = (f \sqcap e) \sqcap (f \sqcap a_e) = (f \sqcap a_e) \sqcap (a_e \sqcap e)$. Since $f \sqcap a_e$ and $a_e \sqcap e$ are both predecessors of a_e they are comparable, so their meet is one of them. But $a_e \sqcap e \in bd$ and $f \sqcap a_e \in \mathfrak{U}$, so $f \sqcap e \in \mathfrak{U}bd$. $\square_{\text{CLAIM}}$

We prove that $f \sqcap e \leq f \sqcap a_e$ by cases. Note that, since $f \sqcap a_e$ and $f \sqcap e$ are both predecessors of f, they are comparable.

1. If $f > C_e^{\mathfrak{U}}$ then $C_{f\sqcap e}^{\mathfrak{U}} = C_e^{\mathfrak{U}}$. Suppose additionally that $f \sqcap a_e > C_e^{\mathfrak{U}} = C_{f\sqcap e}^{\mathfrak{U}}$. Since $f \sqcap a_e \in \mathfrak{U}$, having $f \sqcap a_e \leq f \sqcap e$ would contradict $f \sqcap a_e > C_{f\sqcap e}^{\mathfrak{U}}$, and therefore $f \sqcap e < f \sqcap a_e$.

2. If $f > C_e^{\mathfrak{U}}$ and we are not in the previous case, then C_e has a maximum g and $f \sqcap a_e = g$, i.e. f and a_e are in different open cones above g. Now, e can be either in the same open cone as a_e, or in a new one, but in both cases $f \sqcap e = g = f \sqcap a_e$.

3. If $f \not> C_e^{\mathfrak{U}}$ then there is $h \in C_e^{\mathfrak{U}}$ such that $f \not> h$, and then $f \sqcap h = f \sqcap (h \sqcap e) = f \sqcap e$. As $a_e > C_e^{\mathfrak{U}}$ in particular $a_e > h$, hence by definition of meet we must have $f \sqcap a_e = f \sqcap h = f \sqcap e$. $\square$

Theorem 5.2.15. Let $L_0 = \{<, \sqcap\}$ and $L = L_0 \cup \{R_j, P_{j'} \mid j \in J, j' \in J'\}$, where every $P_{j'}$ is a unary relation symbol, and every R_j is a binary relation symbol. Suppose that T is a completion in L of the theory of dense meet-trees with the following properties.

1. T eliminates quantifiers in L.

2. Every R_j is on open cones, in the following sense: $R_j(x, y) \to x \parallel y$ and if x', y' are such that $x \sqcap x' > x \sqcap y$ and $y \sqcap y' > x \sqcap y$ then $R_j(x, y) \leftrightarrow R_j(x', y')$.

3. If $p \in S_1^{\text{inv}}(\mathfrak{U})$, then C_p is bounded.

Then T is weakly binary.

Proof. Let b^0, b^1 be tuples each having invariant global type. By quantifier elimination it is enough to find a finite tuple $c \in \mathfrak{U}$ such that $\operatorname{tp}_x(b^0/\mathfrak{U}) \cup \operatorname{tp}_y(b^1/\mathfrak{U}) \cup \operatorname{tp}(b^0 b^1/c)$ decides all the atomic relations in L between points of b^0, b^1, $\mathfrak{U}$, and their meets.

Let $b := b^0b^1$ and note that by assumption each $C^{\mathfrak{U}}_{b_i}$ is bounded above, so we may apply Lemma 5.2.14 to b. Let d be the resulting tuple and set $c := d \cap \mathfrak{U}$. We want to show that

$$\pi := \operatorname{tp}(b^0/\mathfrak{U}) \cup \operatorname{tp}(b^1/\mathfrak{U}) \cup \operatorname{tp}(b/c) \vdash \operatorname{tp}(b/\mathfrak{U})$$

If $e \in d$ and $f \in b$, since $e \in \operatorname{dcl}^{L_0}(bc)$ then $\operatorname{tp}(ef/\emptyset)$ is decided by $\operatorname{tp}(b/c)$. This in particular settles everything about the unary predicates $P_{j'}$.

Claim. We have $\pi \vdash \operatorname{tp}^{L_0}(b/\mathfrak{U})$.

Proof of Claim. Since $\mathfrak{U}bd$ is closed under meets we only need to show that the position of all the $e \in d \setminus \mathfrak{U}b$ with respect to $\mathfrak{U}$ is determined. By Lemma 5.2.7 and the fact that $e \in \operatorname{dcl}^{L_0}(bc) \setminus \mathfrak{U}$ there is $i < |b|$ such that $e < b_i$ and $C^{\mathfrak{U}}_e = C^{\mathfrak{U}}_{b_i}$; note that this information is deduced by π, because e is a meet of points in bc. It follows that, if $a_e \in c$ is as in Lemma 5.2.14, all we need to decide is whether e is below or incomparable with $\{h \in \mathfrak{U} \mid h > a_e \sqcap e\}$. This is decided by whether $a_e > e$ or not, and this information is in $\operatorname{tp}(b/c)$. □ CLAIM

We then need to take care of formulas of the form $R_j(e, f)$ for $e \in d \setminus \mathfrak{U}b$ and $f \in \mathfrak{U}$; the argument for formulas of the form $R_j(f, e)$ is identical *mutatis mutandis*. If $e \le f$ or $f \le e$, by hypothesis we must have $\neg R_j(e, f)$, so we may assume that $e \parallel f$. We distinguish three cases; the fact that, by the Claim, π implies the position of e with respect to $\mathfrak{U}$ will be used tacitly.

1. Assume first $e \sqcap f > C^{\mathfrak{U}}_e$. Some subcases of this case are depicted in Figure 5.5. If $a_e \in c$ is as in Lemma 5.2.14, we have $a_e \sqcap f > C^{\mathfrak{U}}_e = C^{\mathfrak{U}}_{e \sqcap f}$. Since $a_e \sqcap f$ and $e \sqcap f$ must be comparable, this implies $a_e \sqcap f > e \sqcap f$, so a_e and f are in the same open cone above $e \sqcap f$. By our hypotheses on T then $R_j(e, f) \leftrightarrow R_j(e, a_e)$, but $a_e \in c$ and $e \in \operatorname{dcl}^{L_0}(bc)$, so since $\pi \vdash \operatorname{tp}(b/c)$ we are done.

2. Assume now that $e \sqcap f \not> C^{\mathfrak{U}}_e$ and there is $h \in \mathfrak{U}$ such that $e \sqcap h > e \sqcap f$. Then e is in the same open cone above $e \sqcap f$ as h, hence $R_j(e, f) \leftrightarrow R_j(h, f)$. Since $f, h \in \mathfrak{U}$ we are done.

3. If $e \sqcap f \not> C^{\mathfrak{U}}_e$ but there is no h as in the previous point, then $C^{\mathfrak{U}}_e$ must have a maximum g, which needs to equal $e \sqcap f$, and since $e \parallel f$ we need to have $f > g$. If e is in an existing open cone above g, since the R_j

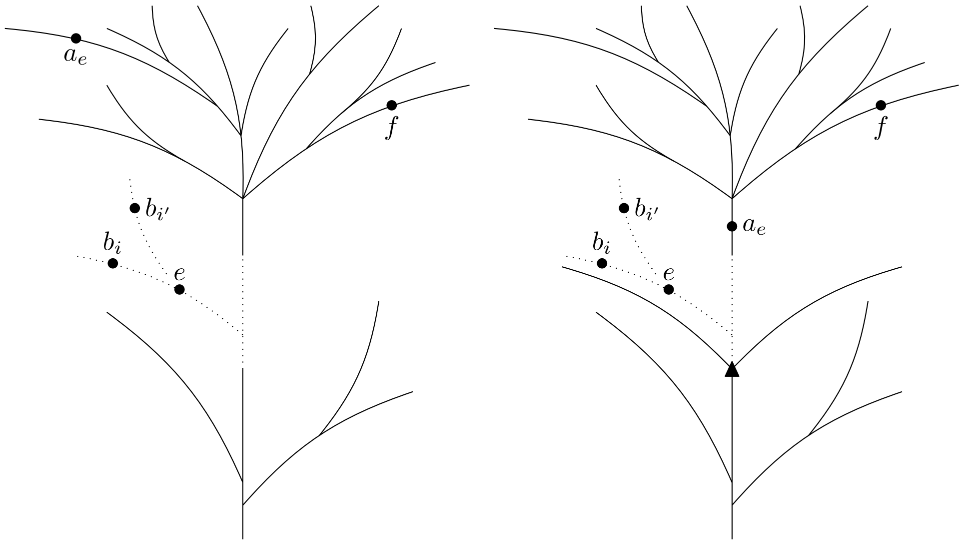


Figure 5.5: two subcases of case 1 in the proof of Theorem 5.2.15, where $e \sqcap f > C_e^{\mathfrak{U}}$. In the first picture, $C_e^{\mathfrak{U}}$ does not have a maximum. In the second picture it has one, denoted by a triangle. Solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 \mathrel{^+\!\succ} \mathfrak{U}$. Other subcases are similar, and correspond to different arrangements of a_e and f, e.g. $a_e > f$.

are on open cones, we are done, so assume it is in a new one. Since $e \in \operatorname{dcl}^{L_0}(bc)$, by Lemma 5.2.7 this can only happen if there is $i < |b|$ such that $e \le b_i$, hence e shares the same open cone above g as b_i. Again, since the R_j are on open cones, we are done. □

If for instance L contains a unary predicate symbol P, and $P(\mathfrak{U})$ is a branch, i.e. a maximal linearly ordered subset, then hypothesis 3 in Theorem 5.2.15 is clearly not satisfied, since there is an $\emptyset$-invariant type with cut $P(\mathfrak{U})$. However, unary predicates are the only possible obstruction, as we now show.

Remark 5.2.16. Hypothesis 3 follows from the other assumptions whenever $J' = \emptyset$, i.e. whenever L does not contain any unary predicate symbol.

Proof. Let $G_g \coloneqq \{b \in \mathfrak{U} \mid b \ge g\}$ be the *closed cone* above g. For the purposes of this proof, call a formula $\varphi(x)$ with $|x| = 1$ *tame* iff it has the following property: there is a finite set $D \subseteq \mathfrak{U}$ such that, for every $a \in \varphi(\mathfrak{U})$, either there is $d \in D$ such that $a \le d$, or $G_a \subseteq \varphi(\mathfrak{U})$.

It is clear that every atomic and negated atomic $\varphi(x) \in L_0(\mathfrak{U})$ is tame. Fix a point c, and consider $\varphi(x) \coloneqq R_j(x, c)$; if $a \in \varphi(\mathfrak{U})$, by assumption we also have $\varphi(b)$ for every $b > a$, hence $G_a \subseteq \varphi(\mathfrak{U})$. Consider now $\varphi(x) \coloneqq \neg R_j(x, c)$, and let $D = \{c\}$. Suppose that $a \not\leq c$. If $a \parallel c$ and $\varphi(a)$ holds, we can argue as above, so assume that $a > c$. For any $b \geq a$ we have in particular $b > c$, hence $\varphi(b)$ holds by assumption and $G_a \subseteq \varphi(\mathfrak{U})$; therefore $\neg R_j(x, c)$ is tame. The formula $R_j(x, x \sqcap c)$ and its negation are tame, because $R_j(x, x \sqcap c)$ is always false. As for the formula $\varphi(x) \coloneqq R_j(x \sqcap c_0, x \sqcap c_1)$, take $D = \{c_0, c_1\}$. If $a \not\leq c_0 \wedge a \not\leq c_1$, then for every $b > a$ and $i < 2$ we have $a \sqcap c_i = b \sqcap c_i$, hence $b > a \rightarrow (\varphi(a) \leftrightarrow \varphi(b))$, proving tameness of both $\varphi(x)$ and $\neg\varphi(x)$. Since the same arguments apply to $R_j(c, x)$, $R_j(x \sqcap c, x)$, and their negations, we conclude that all atomic and negated atomic formulas are tame.

Tame formulas in the variable x are easily seen to be closed under conjunctions and disjunctions: if D_φ and D_ψ witness tameness of $\varphi(x)$ and $\psi(x)$ respectively, then $D_\varphi \cup D_\psi$ witnesses tameness of both $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \vee \psi(x)$. By quantifier elimination, every formula in one free variable is tame.

Similarly, by taking unions of witnesses, if $\Phi(x)$ is a small disjunction of small types, then it satisfies the analogue of tameness where we allow D to have size $|\Phi|$. By saturation, if $\Phi(\mathfrak{U})$ is linearly ordered, it must be bounded.

To conclude, let $p \in S_1^{\mathrm{inv}}(\mathfrak{U}, A)$. By invariance, C_p is the set of realisations of a disjunction of 1-types over A. By what we just proved, C_p is bounded. $\square$

The previous remark applies for example to the case where $L = L_0 \cup \{R\}$, and R induces on every set of open cones above a point a structure elementarily equivalent to the Random Graph; see [EK19] for a study of this theory.

Whether theories constructed this way are NIP or not only depends on the R_j, by [EK19, Corollary 4.14] and the fact that unary predicates, in a language with quantifier elimination, cannot introduce the Independence Property by [Sim15, Lemma 2.9].

5.2.3 Algebraically or real closed valued fields

We now look at the domination monoid in algebraically closed valued fields, as well as in their real closed counterpart. The reader probably knows, or will know after reading [Hol01], what valued fields are and why we talk about them in the same section where we talk about trees. At any rate, we are not going to recall any of this here. For the basic model theory of valued fields, see [vdD14].

There is a variety of languages in which the theory of algebraically closed valued fields can be formulated. One common choice is to work with three sorts K, k, Γ, where K is the actual valued field, k is the residue field, and Γ the value group. The first two sorts are equipped with the field structure, while Γ comes with the structure of an ordered abelian group, together with a constant for the valuation of p in the case where K has characteristic 0 and k has characteristic $p > 0$. Strictly speaking, Γ also includes a constant symbol for the valuation of 0, which is not part of the ordered group structure. It is customary to abuse the notation, talking of Γ as if it were an ordered group. The sorts are connected by the valuation $v\colon K \to \Gamma$ and the modified residue map $\operatorname{Res}\colon K^2 \to k$ sending (x, y) to the residue class of x/y if the latter is in the valuation ring, and to 0 otherwise.

It is well-known that such a language is not enough to eliminate imaginaries, and extra sorts are needed for this purpose. We refer the reader to [HHM06] or [HHM08] for a description of an appropriate language where elimination of imaginaries holds for algebraically closed valued fields.

The situation in real closed valued fields is similar, except K and k are *ordered* fields, and the valuation ring is required to be convex. See [Mel06] for elimination of imaginaries.

Notation 5.2.17. We denote by ACVF a *complete*[3] theory of algebraically closed valued fields, *in a language with elimination of quantifiers and imaginaries* including the sorts K, k, and Γ. Similarly, RCVF denotes the theory of real closed valued fields whose valuation ring is convex, *in a language with elimination of quantifiers and imaginaries* including the sorts K, k, and Γ.

Note that, by Remark 2.3.33, we may as well work in T^{eq}. For the following fact see [vdD14], [HHM06, Proposition 2.1.3], and [Mel06, Lemma 3.13].

Fact 5.2.18. In ACVF [resp. RCVF] the following hold.

1. We have $\Gamma \vDash \mathsf{DOAG}$ and $k \vDash \mathsf{ACF}$ [resp. $k \vDash \mathsf{RCF}$].
2. If $p \in S_{k^n}(\mathfrak{U})$ and $q \in S_{\Gamma^m}(\mathfrak{U})$, then $p \perp^{\mathrm{w}} q$.
3. The structures $k(\mathfrak{U})$ and $\Gamma(\mathfrak{U})$ are fully embedded, as in Definition 2.3.24.

In these theories we have a decomposition of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ in terms of $\widetilde{\operatorname{Inv}}(k(\mathfrak{U}))$ and $\widetilde{\operatorname{Inv}}(\Gamma(\mathfrak{U}))$ of Ax–Kochen–Eršov flavour. The reduction to k and Γ was

[3]So we should really write $\mathsf{ACVF}_{0,0}, \mathsf{ACVF}_{0,p}$ or $\mathsf{ACVF}_{p,p}$, but we will avoid doing this in order not to burden the notation too much.

done in [HHM08] for ACVF, and in [EHM19] for RCVF. We will not give full details here, and only explain how to proceed modulo the following statements.

Notation 5.2.19. If H is a sort, let $H(B)$ denote $\operatorname{dcl}(B) \cap H$.

Fact 5.2.20. Suppose that all coordinates of $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ are in the valued field sort K and M is maximally complete. If $(a, b) \vDash p \otimes q$, then

$$k(Mab) = k\big(k(Ma), k(Mb)\big) \qquad \Gamma(Mab) = \Gamma\big(\Gamma(Ma), \Gamma(Mb)\big)$$

Proof sketch. For $X \subseteq k$ and $Y \subseteq \Gamma$, denote by $F(X)$ the field generated by X, and by $G(Y)$ the group generated by Y. By invariance, $k(Ma)$ and $k(Mb)$ are linearly disjoint over $k(M)$, and $\Gamma(Ma) \cap \Gamma(Mb) = \Gamma(M)$. Therefore, we may apply [HHM08, Proposition 12.11(ii)] to $\operatorname{dcl}(Ma)$ and $\operatorname{dcl}(Mb)$, and obtain

$$k(Mab) = F\big(k(Ma), k(Mb)\big) \qquad \Gamma(Mab) = G\big(\Gamma(Ma), \Gamma(Mb)\big)$$

To conclude, observe that

$$k\big(k(Ma), k(Mb)\big) \subseteq k(Mab) = F\big(k(Ma), k(Mb)\big) \subseteq k\big(k(Ma), k(Mb)\big)$$
$$\Gamma\big(\Gamma(Ma), \Gamma(Mb)\big) \subseteq \Gamma(Mab) = G\big(\Gamma(Ma), \Gamma(Mb)\big) \subseteq \Gamma\big(\Gamma(Ma), \Gamma(Mb)\big) \quad \square$$

Fact 5.2.21 ([HHM08, Corollary 12.12] and [EHM19, Corollary 2.8]). Let $M, B, \mathfrak{U}$ be contained in a monster model of ACVF or RCVF. Let M be maximally complete, $M \subseteq B = \operatorname{dcl}(B)$, and $M \subseteq \mathfrak{U}$, with $k(B), k(\mathfrak{U})$ linearly disjoint over $k(M)$, and $\Gamma(B) \cap \Gamma(\mathfrak{U}) = \Gamma(M)$. Then $\operatorname{tp}(\mathfrak{U}/M, k(B), \Gamma(B)) \vdash \operatorname{tp}(\mathfrak{U}/B)$.

The statement and proof below are essentially [HHM08, Corollary 12.14], the only differences being that the latter only worked in ACVF, considered $\overline{\operatorname{Inv}}(\mathfrak{U})$ only, and took its well-definedness and commutativity for granted. The part about RCVF follows from results in [EHM19]. Recall that we know how to characterise $\widetilde{\operatorname{Inv}}(k(\mathfrak{U}))$ and $\widetilde{\operatorname{Inv}}(\Gamma(\mathfrak{U}))$: see Proposition 3.2.1 or Theorem 4.2.37 for k, and Theorem 4.2.20 for Γ.

Theorem 5.2.22. In ACVF and RCVF we have

$$\overline{\operatorname{Inv}}(\mathfrak{U}) = \widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(k(\mathfrak{U})) \oplus \widetilde{\operatorname{Inv}}(\Gamma(\mathfrak{U}))$$

Proof. Let $p(x) \in S^{\mathrm{inv}}(\mathfrak{U}, M)$; by [vdD14, Proposition 3.6 and Corollary 4.14], up to enlarging M not beyond size $\beth_1(|M|)$, we may assume it is maximally complete. In some $\mathfrak{U}_1 \mathrel{^+\!\!\succ} \mathfrak{U}$, let $b \vDash p$ and $B = \operatorname{dcl}(Mb)$. By M-invariance of p, the fields $k(B), k(\mathfrak{U})$ are linearly disjoint over $k(M)$, and $\Gamma(B) \cap \Gamma(\mathfrak{U}) = \Gamma(M)$. Apply Fact 5.2.21, recalling that its conclusion is to be understood modulo the elementary diagram $\mathrm{ED}(\mathfrak{U}_1)$. By making explicit which parts of it we are using, we find that, working just modulo T,

$$\operatorname{tp}(\mathfrak{U}, M, k(B), \Gamma(B)/\emptyset) \cup \operatorname{tp}(k(B), \Gamma(B), B, M/\emptyset) \vdash \operatorname{tp}(\mathfrak{U}, B/\emptyset)$$

When we work modulo the elementary diagram $\mathrm{ED}(\mathfrak{U})$, this becomes

$$\operatorname{tp}(k(B), \Gamma(B)/\mathfrak{U}) \cup \operatorname{tp}(k(B), \Gamma(B), B/M) \vdash \operatorname{tp}(B/\mathfrak{U}) \tag{5.1}$$

Recall that $B = \operatorname{dcl}(Mb)$; since b is finite, by (the proofs of) [HHM08, Corollary 11.9 and Corollary 11.16] there is a finite tuple $\tilde{b}$ from K such that $K(Mb) = K(M\tilde{b})$. By [HHM06, Proposition 2.1.3(iv)] [resp. [Mel06, Proposition 8.1(i)]], inside K, acl coincides with acl in the sense of the restriction to the [ordered] field language. Recall that if $X \subseteq k \vDash \mathsf{ACF}$ then $\operatorname{dcl}(X)$ is the perfect field generated by X, and if $X \subseteq K \vDash \mathsf{ACVF}$ then $K(X)$ is the perfect Henselian field generated by X. It follows that, in both ACVF and RCVF, there are finite tuples b_k and b_Γ, in a cartesian power of $k(B)$ and $\Gamma(B)$ respectively, such that $k(B) = k(k(M)b_k)$ and $\Gamma(B) = \Gamma(\Gamma(M)b_\Gamma)$. Note that there are M-definable functions sending b to b_k and b_Γ. Let $p_k \coloneqq \operatorname{tp}(b_k/\mathfrak{U})$ and $p_\Gamma \coloneqq \operatorname{tp}(b_\Gamma/\mathfrak{U})$; since $p_k \perp^{\mathrm{w}} p_\Gamma$, we have $p_k \otimes p_\Gamma = p_k \cup p_\Gamma = p_\Gamma \otimes p_k$. Define $r \coloneqq \operatorname{tp}(b_k, b_\Gamma, b/M)$. By (5.1) we have $p_k \cup p_\Gamma \cup r \vdash p$. Since $p \cup r \vdash p_k \cup p_\Gamma$, as r says the latter is a pushforward of p, we obtain $p \equiv_{\mathrm{D}} p_k \otimes p_\Gamma$.

Claim. For all $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$, we have $p \otimes q \equiv_{\mathrm{D}} q \otimes p$.

Proof of Claim. Let $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$. Again by [vdD14, Corollary 4.14], we may assume that M is maximally complete. Assume first that all coordinates of both p and q are in K, and choose suitable $p_k, q_k, p_\Gamma, q_\Gamma$ as above; these are not unique, but since we need to use them in Fact 5.2.21, we only care about their realisations up to definable closure. The same applies to e.g. $(p\otimes q)_k$, and Fact 5.2.20, ensures that we may take $(p\otimes q)_k \coloneqq p_k \otimes q_k$ and $(p\otimes q)_\Gamma \coloneqq p_\Gamma \otimes q_\Gamma$, and have $p \otimes q \equiv_{\mathrm{D}} p_k \otimes q_k \otimes p_\Gamma \otimes q_\Gamma$. By applying the same argument to $q \otimes p$, *and using the same* $p_k, q_k, p_\Gamma, q_\Gamma$, we obtain

$$p \otimes q \equiv_{\mathrm{D}} p_k \otimes q_k \otimes p_\Gamma \otimes q_\Gamma \qquad q \otimes p \equiv_{\mathrm{D}} q_k \otimes p_k \otimes q_\Gamma \otimes p_\Gamma \tag{5.2}$$

Before concluding that these two types are equidominant, we show that a similar situation may be arranged in the case where some coordinates of p, q are in an imaginary sort. By [HHM08, Corollary 11.16] and [EHM19, Theorem 4.5], for ACVF and RCVF respectively, if a is an imaginary tuple then there is a tuple $\tilde{a}$ from the sort K such that $a \in \operatorname{dcl}(M\tilde{a})$ and

$$k(Ma) = k(M\tilde{a}) \qquad \Gamma(Ma) = \Gamma(M\tilde{a}) \tag{5.3}$$

Let $a \vDash p$, let f be an M-definable function such that $f(\tilde{a}) = a$, denote $\tilde{p} := \operatorname{tp}(\tilde{a}/\mathfrak{U})$, and observe that $f_*\tilde{p} = p$. Given $b \vDash q$, define $\tilde{q}$ and g analogously.

Let $(a', b') \vDash \tilde{p} \otimes \tilde{q}$, and let $a := f(a')$ and $b := g(b')$; by Lemma 2.1.17 $a \vDash p \mid \mathfrak{U}b'$, so in particular $a \vDash p \mid \mathfrak{U}b$ and $(a, b) \vDash p \otimes q$. Since the fact that (5.3) holds is a property of $\tilde{p}$, and similarly for $\tilde{q}$, we may take $\tilde{a} := a'$ and $\tilde{b} := b'$. Then, by Fact 5.2.20 and (5.3),

$$k(k(Ma)k(Mb)) \subseteq k(Mab) \subseteq k(M\tilde{a}\tilde{b}) = k(k(M\tilde{a})k(M\tilde{b})) = k(k(Ma)k(Mb))$$

and similarly for Γ. Therefore

$$k(Mab) = k(M\tilde{a}\tilde{b}) \qquad \Gamma(Mab) = \Gamma(M\tilde{a}\tilde{b}) \tag{5.4}$$

or, in other words, we may take $\tilde{a}\tilde{b}$ as $\widetilde{ab}$.

Let $B := \operatorname{dcl}(M\tilde{a}\tilde{b})$, and let $\tilde{p}_k, \tilde{q}_k, \tilde{p}_\Gamma, \tilde{q}_\Gamma$ be defined as above. By (5.2), $\tilde{p} \otimes \tilde{q}$ is equidominant to $\tilde{p}_k \otimes \tilde{q}_k \otimes \tilde{p}_\Gamma \otimes \tilde{q}_\Gamma$ and, using again (5.1), so is $p \otimes q$, because by (5.4) we may take $\tilde{p}_k \otimes \tilde{q}_k$ as $(p \otimes q)_k$, and similarly for Γ. Use *the same* four types, and obtain similarly that $\tilde{q} \otimes \tilde{p}$ and $q \otimes p$ are equidominant to $\tilde{q}_k \otimes \tilde{p}_k \otimes \tilde{q}_\Gamma \otimes \tilde{p}_\Gamma$. Therefore, if we set $p_k := \tilde{p}_k$, and similarly for the other three types, then (5.2) holds for imaginary tuples as well.

To conclude, recall that k and Γ are both fully embedded, so we may use the results in Subsection 2.3.4. In all completions of ACF the product $\otimes$ is commutative by stability, and in RCF and DOAG it is commutative modulo $\equiv_D$ by Theorem 4.2.37 and Theorem 4.2.20 respectively. Using this, we obtain

$$p \otimes q \equiv_D p_k \otimes q_k \otimes p_\Gamma \otimes q_\Gamma \equiv_D q_k \otimes p_k \otimes q_\Gamma \otimes p_\Gamma \equiv_D q \otimes p$$

□ CLAIM

By the Claim, every pair of invariant types commutes modulo equidominance, hence $\overline{\operatorname{Inv}}(\mathfrak{U})$ is well-defined by Corollary 2.1.22.

By weak orthogonality and stable embeddedness the only types concentrating in a cartesian power of k which are dominated by p are those that are already dominated by p_k, and similarly for Γ. Moreover, recall that by Proposition 3.2.1, Theorem 4.2.37, and Theorem 4.2.20, equidominance and domination-equivalence coincide in all of completions of ACF, in RCF, and in DOAG. It follows that, if $p \sim_{\mathrm{D}} q$, then

$$p \equiv_{\mathrm{D}} p_k \otimes p_\Gamma \equiv_{\mathrm{D}} q_k \otimes q_\Gamma \equiv_{\mathrm{D}} q$$

Therefore $\sim_{\mathrm{D}}$ equals $\equiv_{\mathrm{D}}$, hence $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined and equals $\overline{\mathrm{Inv}}(\mathfrak{U})$.

Because $k(\mathfrak{U})$ and $\Gamma(\mathfrak{U})$ are fully embedded in $\mathfrak{U}$, by Proposition 2.3.31 we have embeddings $\widetilde{\mathrm{Inv}}(k(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$ and $\widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$. By point 2 of Fact 5.2.18 and Corollary 2.3.17 we actually have an embedding $\widetilde{\mathrm{Inv}}(k(\mathfrak{U})) \oplus \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$. By the decomposition $p \equiv_{\mathrm{D}} p_k \otimes p_\Gamma$, this embedding is surjective, and we are done. □

By the previous theorem and Theorem 2.3.7, in ACVF, the generically stable types are precisely those domination-equivalent to a type concentrating in a cartesian power k^n of the residue field. Types concentrating in k^n are *stable* types. This is related to the concepts of *stable domination* and *metastability*, for which we refer the reader to [HHM08]. It is *not* true in general that generically stable types are those domination-equivalent to a stable type; see [Usv09, Example 6.13].

5.3 Pathologies

This section is completely devoted to pathological behaviour. We begin with a mild misdemeanour, noncommutativity, and we then encounter a theory where the domination monoid is not well-defined. This is the main counterexample in this thesis, and the same theory also shows that our notion of domination does not coincide with domination in the sense of forking in simple theories. A variation of it shows that the property of being *generically NIP*, contrary to generic stability, is not preserved by domination. We conclude by recalling a standard example from [ACP12], and seeing how it relates to Example 2.1.30.

5.3.1 The Random Graph

In this subsection we prove that, in the theory of the Random Graph, the Fraïssé limit of finite graphs, $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ coincides with $\overline{\operatorname{Inv}}(\mathfrak{U})$ and is not commutative. To begin, note that this theory is binary, hence $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is well-defined by Corollary 2.2.17. This also follows from the next proposition.

Proposition 5.3.1. The Random Graph has degenerate domination.

Proof. Suppose that $r \in S_{pq}(A)$ witnesses $p(x) \geq_{\mathrm{D}} q(y)$ and assume that q has no realised or duplicate coordinates. Up to a permutation of the y_j, assume that r identifies $y_0, \ldots, y_{n-1}$ with some variables in x and for all j such that $n \leq j < |y|$ and all $i < |x|$ we have $r \vdash x_i \neq y_j$. If $n = |y|$ then, in the notation of Definition 2.2.9, we can let r_0 be a suitable restriction of r and we are done. Assume that $n < |y|$, hence for every $i < |x|$ we have $r \vdash y_n \neq x_i$. Pick any $b \in \mathfrak{U} \setminus A$; by the Random Graph axioms $p \cup r$ is consistent with both $E(y_n, b)$ and $\neg E(y_n, b)$, contradicting $p \cup r \vdash q$. □

Corollary 5.3.2 ([Men20, Corollary 2.12]). In the theory of the Random Graph, $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ equals $\overline{\operatorname{Inv}}(\mathfrak{U})$ and is not commutative.

Proof. The first part follows from degenerate domination by Proposition 2.2.11. As for noncommutativity, consider the global types $p(x) := \{\neg E(x, a) \mid a \in \mathfrak{U}\}$ and $q(y) := \{E(y, a) \mid a \in \mathfrak{U}\}$. Both are clearly $\emptyset$-invariant, in fact $\emptyset$-definable, and it follows straight from the definitions that $p(x) \otimes q(y) \vdash \neg E(x, y)$ and $q(y) \otimes p(x) \vdash E(x, y)$. The conclusion now follows from degenerate domination and Remark 2.2.10. □

Corollary 5.3.3. The following hold in the theory of the Random Graph.

1. There is no $n < \omega$ such that $S_n^{\mathrm{inv}}(\mathfrak{U})/\sim_{\mathrm{D}}$ generates $\widetilde{\operatorname{Inv}}(\mathfrak{U})$.
2. $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ cannot be generated by pairwise weakly orthogonal elements.
3. For every nonrealised p the submonoid generated by $[\![p]\!]$ is infinite.

Proof. These are all easy consequences of degenerate domination combined with the Random Graph axioms. □

In the guise of Section 5.1, we may look at the domination monoid in the theory of the Random Ordered Graph, the Fraïssé limit of ordered graphs. The proof of degenerate domination in Proposition 5.3.1 can be adapted to work in this case too.

Question 5.3.4. Let T be NIP and $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$. Is $p \otimes q \sim_{\mathrm{D}} q \otimes p$?

Note that, by Corollary 2.1.22, a positive answer would imply that $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined in every NIP theory. There is also a weaker reformulation.

Question 5.3.5. If T is NIP and $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined, is it commutative?

For $\overline{\mathrm{Inv}}(\mathfrak{U})$ the answer is negative by Proposition 5.1.5.

Question 5.3.6. If $p \otimes q \sim_{\mathrm{D}} q \otimes p$ always holds, is T necessarily NIP?

Remark 5.3.7. In some cases, e.g. by Theorem 3.1.24 if T is thin, if $p \not\perp^{\mathrm{w}} q$ there is a nonrealised s such that $p \geq_{\mathrm{D}} s$ and $q \geq_{\mathrm{D}} s$. This is false in general: in the theory of the Random Graph two nonrealised types p, q are never weakly orthogonal. Anyway, as soon as they do not have any common 1-subtype, if r is dominated by both p and q then it must be realised. This is easily seen to be the case for instance with $p(x) \coloneqq \{E(x,a) \mid a \in \mathfrak{U}\}$ and $q(y) \coloneqq \{\neg E(y,a) \mid a \in \mathfrak{U}\}$.

This may also fail to happen under NIP: see Counterexample 5.1.14.

5.3.2 A theory with no domination monoid

In [HHM08, p. 18] it was claimed without proof that $\overline{\mathrm{Inv}}(\mathfrak{U})$ is well-defined and commutative in every first-order theory. As we saw in the previous subsection, the second statement is unfortunately not true. The main result of this subsection is a counterexample to the first one.

Theorem 5.3.8 ([Men20, Theorem 2.1])**.** There is a ternary, ω-categorical, supersimple theory of SU-rank 2 with degenerate algebraic closure in which none of $\sim_{\mathrm{D}}$,$\equiv_{\mathrm{D}}$, $\sim_{\mathrm{RK}}$, or $\equiv_{\mathrm{RK}}$ is a congruence with respect to $\otimes$.

We present the theory promised above as a Fraïssé limit, and provide an explicit axiomatisation in Proposition 5.3.12. We then show in Proposition 5.3.15 that in this theory $\otimes$ does not respect $\geq_{\mathrm{D}}$, nor any of the four equivalence relations above.

Recall that S_3 denotes the group of permutations of $\{0, 1, 2\}$.

Definition 5.3.9. Let L be the relational language $L := \{E^{(2)}, R_2^{(2)}, R_3^{(3)}\}$, with arities of symbols indicated as superscripts, and let $\Lambda := \Lambda_0 \wedge \Lambda_1$, where

$$\Lambda_0(x_0, x_1, x_2) := \bigvee_{\sigma \in S_3} \big(R_2(x_{\sigma 0}, x_{\sigma 1}) \wedge R_2(x_{\sigma 0}, x_{\sigma 2}) \wedge \neg R_2(x_{\sigma 1}, x_{\sigma 2})\big)$$

$$\Lambda_1(x_0, x_1, x_2) := \bigwedge_{0 \le i < j < 3} \neg E(x_i, x_j)$$

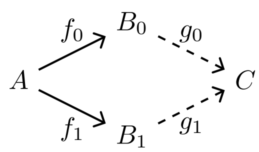
Let K be the class of finite L-structures where

1. E is an equivalence relation,
2. R_2 is symmetric, irreflexive, and E-equivariant, in the sense that the formula $(E(x_0, x_1) \wedge E(y_0, y_1)) \to (R_2(x_0, y_0) \leftrightarrow R_2(x_1, y_1))$ holds,
3. R_3 is symmetric, i.e. $R_3(x_0, x_1, x_2) \to \bigwedge_{\sigma \in S_3} R_3(x_{\sigma 0}, x_{\sigma 1}, x_{\sigma 2})$, and
4. $R_3(x_0, x_1, x_2) \to \Lambda(x_0, x_1, x_2)$ is satisfied, i.e. if $R_3(x_0, x_1, x_2)$ holds then between the x_i there are precisely two R_2-edges and their E-classes are pairwise distinct.

Note that in particular R_2 is still symmetric irreflexive on the quotient by E. We do not add an imaginary sort for this quotient; it will be notationally convenient to mention it anyway but, formally, every reference to the quotient by E, the relative projection π, etc, is to be understood as a mere shorthand.

Proposition 5.3.10. K is a Fraïssé class with strong amalgamation.

Proof. It is clear that K is closed under isomorphism and taking substructures, and it only remains to show that K has the Strong Amalgamation Property (since the empty structure is in K, the Joint Embedding Property follows from amalgamation). Given $A, B_0, B_1 \in K$ and embeddings $f_i : A \to B_i$ we want to find $C \in K$ and embeddings $g_i : B_i \to C$ making the diagram below commute.



Assume without loss of generality that $A \subseteq B_i$, that the maps f_i are inclusions, and that $B_0 \cap B_1 = A$. We define C in the obvious way, by taking $\operatorname{dom} C = \operatorname{dom} B_0 \cup \operatorname{dom} B_1$ and adding only the E-edges and R_2-edges necessary to stay in K. Formally, we let E^C be the transitive closure of $E^{B_0} \cup E^{B_1}$, declare $R_3^C := R_3^{B_0} \cup R_3^{B_1}$, and stipulate that R_2^C is the symmetrisation of the union of $R_2^{B_0} \cup R_2^{B_1}$ with the set of all pairs (c, d) such that, for some $i < 2$, we have $c \in B_i \setminus A$, $d \in B_{1-i} \setminus A$, and

$$\exists a \in A\ E^{B_i}(c, a) \wedge R_2^{B_{1-i}}(a, d)$$

Note that E^C is an equivalence relation and R_2^C is symmetric irreflexive. Furthermore, it is easy to see that, if for $i < 2$ we have $b_i \in B_i$, then $E^C(b_0, b_1)$ holds if and only if there is $e \in A$ such that $E^C(b_i, e)$ holds for both $i < 2$.

We now check that R_2^C is E^C-equivariant. The proof below is by cases, and it is easier to follow if the reader draws the usual V-shaped Venn diagram and adds points and edges to the drawing as they are mentioned in the proof (see e.g. Figure 5.6). Using different colours for R_2 and E also helps.

Assume $E^C(c_0, c_1)$.

1. Suppose first that for some i we have $c_0, c_1 \in B_i$. If $d \in B_i$ as well then we have $R_2^C(c_0, d) \Leftrightarrow R_2^{B_i}(c_0, d) \Leftrightarrow R_2^{B_i}(c_1, d) \Leftrightarrow R_2^C(c_1, d)$. If $d \in B_{1-i} \setminus A$ and $R_2^C(c_0, d)$ holds, by construction we must be in one of two subcases.

   (a) This subcase is depicted in Figure 5.6. There is $a \in A$ such that $E^{B_i}(c_0, a)$ and $R_2^{B_{1-i}}(a, d)$ hold. But then $E^{B_i}(c_1, a)$ holds as well, hence $R_2^C(c_1, d)$ holds by construction.

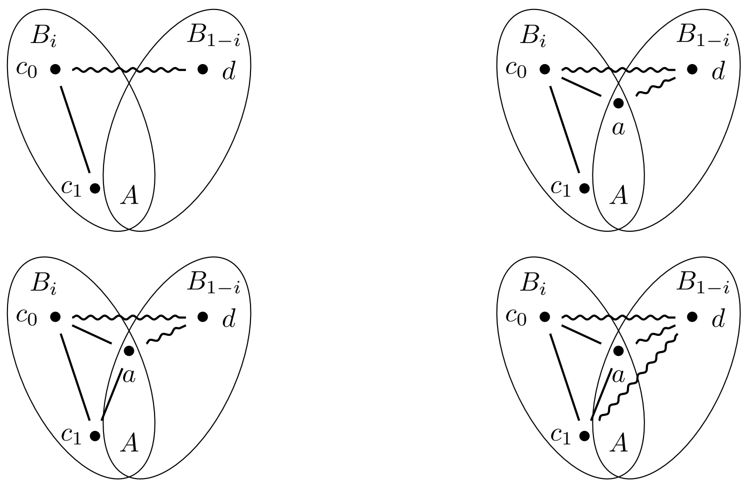


Figure 5.6: subcase (1a) of Proposition 5.3.10. Straight lines denote E-edges, oscillating lines denote R_2-edges.

(b) There is $a \in A$ such that $E^{B_{1-i}}(a,d)$ and $R_2^{B_i}(c_0,a)$ hold. But then $R_2^{B_i}(c_1,a)$ holds, and again $R_2^C(c_1,d)$ holds by construction.

Therefore $R_2^C(c_0,d) \to R_2^C(c_1,d)$ holds. By repeating the argument after swapping the roles of c_1 and c_0 we obtain $R_2^C(c_0,d) \leftrightarrow R_2^C(c_1,d)$.

2. Suppose now that $c_0, d \in B_i$ and $c_1 \in B_{1-i} \setminus A$. Since we are assuming $E^C(c_0,c_1)$, this means that there is $e \in A$ such that $E^C(c_i,e)$ for $i < 2$. If $R_2^C(c_0,d)$ holds, then $R_2^{B_i}(c_0,d)$ also holds, hence we also have $R_2^{B_i}(e,d)$ and $R_2^C(c_1,d)$ holds by construction. This shows $R_2^C(c_0,d) \to R_2^C(c_1,d)$.

   If instead $R_2^C(c_1,d)$ holds, then we are again in one of two subcases.

   (a) There is $a \in A$ such that $E^{B_i}(a,d)$ and $R_2^{B_{1-i}}(a,c_1)$. Then we must have $R_2^A(e,a)$, hence $R_2^{B_i}(c_0,a)$ holds, therefore $R_2^{B_i}(c_0,d)$ holds.

   (b) There is $a \in A$ such that $E^{B_{1-i}}(a,c_1)$ and $R_2^{B_i}(a,d)$. Then $E^A(e,a)$ holds, hence we must have $R_2^{B_i}(e,d)$, and therefore $R_2^{B_i}(c_0,d)$.

Hence $R_2^C(c_1,d) \to R_2^C(c_0,d)$; the proof of E^C-equivariance of R_2^C is complete.

Note that we are adding no new R_3-hyperedges, so if $C \vDash R_3(c_0,c_1,c_2)$ then there is $i < 2$ such that for all $j < 3$ we have $c_j \in B_i$. As the only new E-edges added are between points of $B_0 \setminus A$ and of $B_1 \setminus A$, and the same holds for new R_2-edges, from the fact that $\Lambda(c_0,c_1,c_2)$ holds in B_i we conclude that it must also hold in C.

Take g_i to be the inclusion $B_i \hookrightarrow C$. As we have not identified any points of $B_0 \setminus A$ with anything in $B_1 \setminus A$, i.e. the image of g_0 meets that of g_1 precisely in the image of $g_0 \circ f_0$, the class K has the Strong Amalgamation Property. □

Let T be the theory of the Fraïssé limit of K.

Proposition 5.3.11. *T is ω-categorical, eliminates quantifiers in L and has degenerate algebraic closure, i.e. for all sets $A \subseteq M \vDash T$ we have $\operatorname{acl} A = A$. Moreover T is ternary: every formula is a Boolean combination of formulas with at most 3 free variables.*

Proof. The first part is immediate from the previous proposition and [Hod93, Theorem 7.1.8 and Corollary 7.3.4], and T is ternary because it eliminates quantifiers in a ternary relational language. □

Proposition 5.3.12. T can be axiomatised as follows:

(I) E is an equivalence relation with infinitely many classes, all of which are infinite.

(II) Whether $R_2(x_0, x_1)$ holds only depends on the E-classes of x_0, x_1. Furthermore, the structure induced by R_2 on the quotient by E is elementarily equivalent to the Random Graph.

(III) T satisfies $R_3(x_0, x_1, x_2) \to \Lambda(x_0, x_1, x_2)$, i.e. if $R_3(x_0, x_1, x_2)$ holds then between the x_i there are precisely two R_2-edges and their E-classes are pairwise distinct.

(IV) Denote by $[x_i]_E$ the E-class of x_i. If $\Lambda(x_0, x_1, x_2)$ holds, then $R_3 \restriction [x_0]_E \times [x_1]_E \times [x_2]_E$ is a symmetric generic tripartite 3-hypergraph, i.e. for every $i < j < 3$ and $k \in \{0, 1, 2\} \setminus \{i, j\}$, if $U, V \subseteq [x_i]_E \times [x_j]_E$ and $U \cap V = \emptyset$ then there is $z \in [x_k]_E$ such that for every $(x, y) \in U$ we have $R_3(x, y, z)$ and for every $(x, y) \in V$ we have $\neg R_3(x, y, z)$.

Proof. Easy back-and-forth between any $M \vDash T$ and the Fraïssé limit of K. □

It is possible to view T as a generalisation of the lexicographical product of theories studied in [Mei16]. I am grateful to Nadav Meir for the useful discussions on this topic.

Denote with π the projection to the quotient by E. Recall that this is only a notational shorthand, and there is no quotient sort in the language.

Proposition 5.3.13. The theory T is simple and $a \mathop{\smile\!\!\!\!\!\!|}_C b \iff (a \cap b \subseteq C) \wedge (\pi a \cap \pi b \subseteq \pi C)$. In particular, T has SU-rank 2.

Proof. We use the Kim–Pillay Theorem [KP97, Theorem 4.2]; see also [TZ12, Theorem 7.3.13]. The relation we promised to be forking independence in T, call it $\mathop{\smile\!\!\!\!\!\!|}^0$, trivially satisfies Invariance, Monotonicity, Transitivity, Symmetry, Finite Character, Local Character. To prove Existence, we have to show that for every a, B, C there is $a' \equiv_B a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^0_B C$ i.e., setting $p(x) \coloneqq \operatorname{tp}(a/B)$, we want to prove that $p(x) \cup$ "$x \cap C \subseteq B$" $\cup$ "$\pi x \cap \pi C \subseteq \pi B$" is consistent. Let $L_0 \coloneqq \{E, R_2\}$. By the Random Graph axioms on the relation induced by R_2 on the quotient by E, the partial type $(p(x) \restriction L_0) \cup$ "$\pi x \cap \pi C \subseteq \pi B$" has unboundedly many realisations, and by genericity of R_3 there is no problem in realising p while avoiding points of $C \setminus B$.

To prove Independence over Models, suppose $A, B \supseteq M$, $a \equiv_M b$, $A \mathop{\smile\hskip-0.9em\raise0.1ex\hbox{|}}^0_M B$, $a \mathop{\smile\hskip-0.9em\raise0.1ex\hbox{|}}^0_M A$, and $b \mathop{\smile\hskip-0.9em\raise0.1ex\hbox{|}}^0_M B$. We want to find c such that $c \equiv_A a$, $c \equiv_B b$ and $c \mathop{\smile\hskip-0.9em\raise0.1ex\hbox{|}}^0_M AB$. By hypothesis $(\pi A \setminus \pi M) \cap (\pi B \setminus \pi M) = \emptyset$, and as the Random Graph is simple, so satisfies the Independence Theorem, there is a tuple $\tilde{c}$ satisfying $\tilde{c} \equiv_A^{L_0} a$ and $\tilde{c} \equiv_B^{L_0} b$; indeed, there are unboundedly many (suitable $|\tilde{c}|$-tuples of) E-classes of them, so we can assume $(\pi\tilde{c} \cap \pi A \pi B) \subseteq \pi M$ with no loss of generality. Fix the E-classes of the elements of such a tuple $\tilde{c}$. As $(A \setminus M) \cap (B \setminus M) = (a \setminus M) \cap (A \setminus M) = (b \setminus M) \cap (B \setminus M) = \emptyset$ and $a \equiv_M b$ by hypothesis, genericity of R_3 ensures that we can find some c such that $c_i E \tilde{c}_i$ for all $i < |c|$ satisfying the requirements on R_3 imposed by $\operatorname{tp}(a/A) \cup \operatorname{tp}(b/B)$. Again, as there are unboundedly many such c, we may finally find one such that $c \cap AB \subseteq M$, and we are done.

From the characterisation of forking we immediately see that the SU-rank of every 1-type in T is at most 2, and that any 1-type in a new equivalence class has SU-rank 2. □

Note that in this theory the formula $R_3(x, y, z)$ clearly has IP, and even IP_2 (see [She14, Definition 5.63] or [CPT19, Definition 2.1] for the definition of IP_2). Since T is ternary, by [CPT19, Example 2.2.3] it is NIP_3.

Definition 5.3.14. In T, define the global types

$$\begin{aligned} p(x) &\coloneqq \{R_2(x, d) \mid d \in \mathfrak{U}\} \cup \{\neg R_3(x, d, e) \mid d, e \in \mathfrak{U}\} \\ q_0(y) &\coloneqq \{\neg R_2(y, d) \mid d \in \mathfrak{U}\} \\ q_1(z_0, z_1) &\coloneqq \{\neg R_2(z_0, d) \mid d \in \mathfrak{U}\} \cup \{E(z_0, z_1) \land z_0 \neq z_1\} \end{aligned}$$

These three types are complete by quantifier elimination and the axioms of T: for instance, in the case of q_1, the lack of R_2-edges forces the equivalence class of the z_i to be new, and for all $d \in \mathfrak{U}$ we have $\neg\Lambda_1(z_0, z_1, d)$, hence $\neg R_3(z_0, z_1, d)$. Moreover, the condition $E(z_0, z_1)$ together with the restriction of q_1 to z_0 decides all the R_2-edges of z_1, and for all $d, e \in \mathfrak{U}$ and $i < 2$ we have $\neg\Lambda_0(z_i, d, e)$, hence $\neg R_3(z_i, d, e)$.

It follows easily from their definition that p, q_0 and q_1 are all $\emptyset$-invariant. To conclude the proof of Theorem 5.3.8 we only need to take one last step: proving the proposition below.

Proposition 5.3.15. We have $q_0 \equiv_{\mathrm{RK}} q_1$, but $p(x) \otimes q_0(y) \not\geq_{\mathrm{D}} p(w) \otimes q_1(z)$.

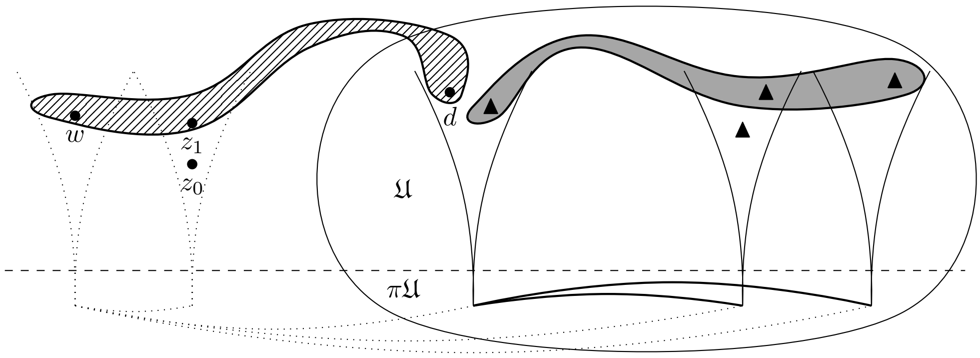


Figure 5.7: in Proposition 5.3.15, if for instance $r \in S_{(p\otimes q_0)(p\otimes q_1)}(A)$ proves $(x = w) \wedge (y = z_0)$, and $d \in \mathfrak{U} \setminus A$, then $(p(x) \otimes q_0(y)) \cup r$ is consistent with $R_3(w, z_1, d)$ (left shaded area). Vertical lines enclose E-classes; the quotient by E lies below the dashed line. Triangles denote points of A, horizontal curved lines R_2-edges, shaded areas R_3-edges. Dotted lines lie in a bigger $\mathfrak{U}_1 \ {}^{+}\!\succ \mathfrak{U}$.

Proof. Let $\varphi(y, z) \coloneqq (y = z_0) \wedge (z_0 \neq z_1) \wedge E(z_0, z_1)$. For $i < 2$ we have $q_i \cup \{\varphi\} \vdash q_{1-i}$, hence $q_0 \equiv_{\mathrm{RK}} q_1$. Note $p(x) \otimes q_0(y)$ is axiomatised by

$$p(x) \cup q_0(y) \cup \{R_2(x, y)\} \cup \{\neg R_3(x, y, d) \mid d \in \mathfrak{U}\}$$

and similarly $p(w) \otimes q_1(z)$ is axiomatised by

$$p(w) \cup q_1(z) \cup \{R_2(w, z_0) \wedge R_2(w, z_1)\} \cup \{\neg R_3(w, z_j, d) \mid j < 2, d \in \mathfrak{U}\}$$

Let $A \subset^{+} \mathfrak{U}$ and $r(x, y, w, z) \in S_{(p\otimes q_0)(p\otimes q_1)}(A)$. Pick any $d \in \mathfrak{U} \setminus A$ and any $i < 2$ such that $(p(x) \otimes q_0(y)) \cup r \vdash y \neq z_i$. By genericity of R_2, the set

$$\Phi \coloneqq (p(x) \otimes q_0(y)) \cup r \cup \{R_2(w, d) \wedge R_2(w, z_i) \wedge \neg R_2(z_i, d)\}$$

is consistent.[4] By genericity of R_3 so is $\Phi \cup \{R_3(w, z_i, d)\}$ (as well as $\Phi \cup \{\neg R_3(w, z_i, d)\}$). This shows that

$$p(x) \otimes q_0(y) \cup r \nvdash \{\neg R_3(w, z_j, d) \mid j < 2, d \in \mathfrak{U}\} \subseteq p(w) \otimes q_1(z) \qquad \square$$

As an aside, note that anyway $p(x) \otimes q_0(y) \leq_{\mathrm{RK}} p(w) \otimes q_1(z)$ by Corollary 2.2.8, taking as the map f the projection on the coordinate $y = z_0$.

[4]If $r \vdash x = w \wedge y = z_0$ we even have $p(x) \otimes q_0(y) \cup r \vdash R_2(w, d) \wedge R_2(w, z_1) \wedge \neg R_2(z_1, d)$. See Figure 5.7.

Remark 5.3.16. Definition 3.1.5 makes sense also in simple theories, and more generally in rosy theories if we replace forking by þ-forking (see [OU11a]). One can then define $\triangleright$ as in Definition 3.1.7 even for types that are not stationary but, in the unstable case, the relation $\triangleright$ on global types need not coincide with $\geq_{\mathrm{D}}$, and example being the theory T above.

Proof. In the notation of Definition 5.3.14, let $(b,c) \vDash q_1$ and $a \vDash p \mid \mathfrak{U}bc$, and recall that in T forking is characterised as

$$e \mathop{\downarrow}_C d \iff (e \cap d \subseteq C) \wedge (\pi e \cap \pi d \subseteq \pi C)$$

Let $B \supseteq \mathfrak{U}$ and d be such that $abc \mathop{\downarrow}_{\mathfrak{U}} B$ and that $ab \mathop{\downarrow}_B d$. We want to show that $abc \mathop{\downarrow}_B d$, and by the characterisation of forking it is enough to show that for all $i < |d|$ we have $c \mathop{\downarrow}_B d$. By definition of q_1, we have $c \notin \mathfrak{U}$, and from $abc \cap B \subseteq \mathfrak{U}$ we conclude $c \notin B$. If $\pi c = \pi d_i$, then since $b \mathop{\downarrow}_B d$ and $\pi b = \pi c$, we must have $\pi d_i \in \pi B$. But then $\pi d_i = \pi c \in \pi\mathfrak{U}$ because $abc \mathop{\downarrow}_{\mathfrak{U}} B$, contradicting $\pi c \notin \pi\mathfrak{U}$, which holds by definition of q_1. Moreover, $\pi c \neq \pi d_i$ clearly implies that we must also have $c \neq d_i$. Therefore, $ab \triangleright_{\mathfrak{U}} abc$.

Since $\operatorname{tp}(a, b/\mathfrak{U}) = p \otimes q_0$ and $\operatorname{tp}(a, bc/\mathfrak{U}) = p \otimes q_1$ this shows $p \otimes q_0 \triangleright p \otimes q_1$, but by Proposition 5.3.15 $p \otimes q_0 \not\geq_{\mathrm{D}} p \otimes q_1$. □

Question 5.3.17. The types in Definition 5.3.14 are definable. Are there finitely satisfiable types p, q_0, q_1 such that $q_0 \geq_{\mathrm{D}} q_1$ and $p \otimes q_0 \not\geq_{\mathrm{D}} p \otimes q_1$?

Question 5.3.18. Is $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ well-defined in every NIP theory?

Note that by Corollary 2.1.22, a positive answer to this question would follow from one to Question 5.3.4. As a first step in the direction of looking for a counterexample, we note that in T we may replace R_2 with a DLOP, and Λ with another suitable formula, and still obtain a theory T' where $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is not well-defined. We define this in a two-sorted language for later convenience. Similarly to T, it is an easy task (which we leave to the reader) to prove that T' is complete and eliminates quantifiers, by showing that it is the theory of a Fraïssé limit of finite structures.

Definition 5.3.19. Work in a language with two sorts H and D. Let $L' = \{<^{(D^2)}, P^{(D)}, \pi^{(H \to D)}, E^{(H^2)}, R_3^{(H^3)}\}$, where arities of symbols are indicated as superscripts. Let T' be the L'-theory below.

(I) E is an equivalence relation on H with infinitely many classes, all of which are infinite, and π is the projection to the quotient D.

(II) $(D, <, P) \vDash \mathsf{DLOP}$.

(III) T satisfies $R_3(x, y, z) \to \Gamma(x, y, z)$, where

$$\Gamma(x, y, z) \coloneqq P(\pi x) \wedge \neg P(\pi y) \wedge P(\pi z) \wedge \pi(x) < \pi(y) < \pi(z)$$

(IV) Denote by $[x_i]_E$ the E-class of x_i. If $\Gamma(x_0, x_1, x_2)$ holds, then $R_3 \upharpoonright [x_0]_E \times [x_1]_E \times [x_2]_E$ is a generic tripartite 3-hypergraph.

Definition 5.3.20. Define global $\emptyset$-definable types $p(x)$, $q_0(y)$ and $q_1(z)$, where x and z are of sort H and y is of sort D, as follows:

$$p(x) \coloneqq \{P(\pi x) \wedge (\pi x < d) \mid d \in D(\mathfrak{U})\} \cup \{\neg R_3(x, d, e) \mid d, e \in H(\mathfrak{U})\}$$
$$q_0(y) \coloneqq \{\neg P(y) \wedge (y < d) \mid d \in D(\mathfrak{U})\}$$
$$q_1(z) \coloneqq \{\neg P(\pi z) \wedge (\pi z < d) \mid d \in D(\mathfrak{U})\}$$

Similarly to what happened in T, we have $q_0 \equiv_{\mathrm{RK}} q_1$ but $p \otimes q_0 \not\geq_{\mathrm{D}} p \otimes q_1$. The formula $R_3(x, y, z)$ has IP_2 in T' as well. It would be interesting to know if a similar construction can be carried out under NIP, or even just NIP_2 .

This theory can be used to show that point 3 of Theorem 2.3.7 does not generalise to preservation of the following property.

Definition 5.3.21. Let $p(x) \in S^{\mathrm{inv}}(\mathfrak{U}, A)$. We say that p is *generically NIP over A* iff there are no $\varphi(x; w) \in L(A)$, $b \in \mathfrak{U}$, and Morley sequence $(a^i \mid i < \omega)$ of p over A such that $\vDash \varphi(a^i; b)$ if and only if i is even.

The difference with the definition of a NIP type is that we require a^i to be a Morley sequence, not just an indiscernible sequence in p. So if a type is NIP, it is generically NIP.

Counterexample 5.3.22. Let T' be as in Definition 5.3.19 and q_0, q_1 as in Definition 5.3.20. Fix any small set A and let $(a^i \mid i < \omega)$ be a Morley sequence of q_1 over A, i.e. a sequence such that $\neg P(\pi a^i)$ and $\pi a^j < \pi a^i < \pi A$ hold for all $i < j < \omega$. By genericity of R_3, compactness, and saturation, we may find b_0 and b_1 such that $\pi b_0 < \pi a^i$ for all i, that $\pi b_1 > \pi A$, that $P(\pi b_0) \wedge P(\pi b_1)$, and that $R_3(b_0, a^i, b_1)$ holds precisely when i is even. Therefore, q_1 is not generically NIP. On the other hand, q_0 is generically NIP over $\emptyset$, and in fact even NIP, because the structure induced on the sort D is that of a pure DLOP, which is NIP. Since $q_0 \equiv_{\mathrm{RK}} q_1$, and in particular $q_0 \geq_{\mathrm{D}} q_1$, the property of being generically NIP is not preserved by domination.

5.3.3 Parameterised finite equivalence relations

We briefly recall an example from [ACP12] for the reader's convenience.

Definition 5.3.23. Let L be the two-sorted language with sorts P, E, a relation symbol $R^{(P^2\times E)}$ and a function symbol $f^{(P\times E\to P)}$. Let K be the class of finite structures where, for every $e\in E$, $R(x,y,e)$ defines an equivalence relation on P with all equivalence classes of size 2, and $f(x,e)$ is the unique $y\neq x$ such that $R(x,y,e)$.

The class K is a Fraïssé class; the theory of its Fraïssé limit is called T^*_{feq2}.

Definition 5.3.24. Let $p_x \in S^{\mathrm{inv}}_P(\mathfrak{U},\emptyset)$ be the type $\{x\neq a \mid a\in P(\mathfrak{U})\}\cup \{t_0(x,e)\neq t_1(x,e) \mid e\in E(\mathfrak{U}),\ t_i(x,w)\ \{f\}\text{-terms}\}$.

For example, given $e_0,e_1,e_2\in E(\mathfrak{U})$, we have $p(x)\vdash f(f(x,e_0),e_1)\neq f(x,e_2)$. Completeness of $p(x)$ follows from quantifier elimination and the fact that for all $a\in P(\mathfrak{U})$ and $e\in E(\mathfrak{U})$ there are only two points satisfying $R(x,a,e)$, namely a and $f(a,e)$, and they are in $\mathfrak{U}$.

Proposition 5.3.25 ([ACP12, Example 1.7])**.** The type p is generically stable, but $p\otimes p$ is not.

Proof. Generic stability of p over $\emptyset$ follows easily from quantifier elimination. Let $(a^i_0a^i_1)_{i<\omega}$ be a Morley sequence of $p^{(2)}$ over any small set A. By compactness and saturation, there is $e\in E(\mathfrak{U})$ such that $R(a^i_0,a^i_1,e)$ holds if and only if i is even, so $p^{(2)}$ is not generically stable. □

Finally, we explain how this theory is related to Example 2.1.30. The theory T^*_{feq} is defined analogously to T^*_{feq2}, except the language does not contain f, and in K there is no bound on the size of $R(x,y,e)$-classes. Consequently, in the Fraïssé limit each equivalence class has infinite size. A conceptual way to make sense of Example 2.1.30, or rather of its variant where digraphs are replaced by equivalence relations, is to imagine unboundedly many models or T^*_{feq} which share the sort E, and where the other sort carries a generic predicate.

Chapter 6

Changing monster model

In strongly minimal theories, by Proposition 3.2.1, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N}$ regardless of $\mathfrak{U}$. On the other hand, in the Random Graph $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is very close to $S^{\mathrm{inv}}(\mathfrak{U})$ by Proposition 5.3.1 and Remark 2.2.10: the former is obtained from the latter by identifying types that only differ because of realised, duplicate, or permuted coordinates. It is natural to ask whether and how much $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ depends on the choice of $\mathfrak{U}$, and we now investigate the matter.

The problem at hand makes sense even when $\otimes$ does not respect $\geq_{\mathrm{D}}$, and some of the results below apply also to theories with no domination monoid. In that case, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ will denote the domination poset. Similarly for $\overline{\mathrm{Inv}}(\mathfrak{U})$.

6.1 The extension map

Let $\mathfrak{U}_1 \,{}^{+}\!\succ \mathfrak{U}_0$. The map $p \mapsto p \mid \mathfrak{U}_1$ shows that, for every tuple of variables x, a copy of $S^{\mathrm{inv}}_x(\mathfrak{U}_0)$ sits inside $S^{\mathrm{inv}}_x(\mathfrak{U}_1)$; for instance, if T is stable, this is nothing more than the classical identification of types over $\mathfrak{U}_0$ with types over $\mathfrak{U}_1$ that do not fork over $\mathfrak{U}_0$. This induces a map between the respective domination posets, which we now study. While a good portion of this section has already appeared in [Men20], the notion of *F-local domination* (Definition 6.1.7) is new, and so is most of Corollary 6.1.9.

Unless otherwise stated, in this section T is arbitrary.

6.1.1 Homomorphism properties

Definition 6.1.1. Let $\mathfrak{U}_0 \prec^{+} \mathfrak{U}_1$ be two monster models of an arbitrary T. We define the *extension map* $\mathfrak{e}\colon \widetilde{\mathrm{Inv}}(\mathfrak{U}_0) \to \widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$ by setting $\mathfrak{e}(\llbracket p \rrbracket) := \llbracket p \mid \mathfrak{U}_1 \rrbracket$.

We need to show that $\mathfrak{e}([\![p]\!])$ does not depend on p. This is indeed the case.

Proposition 6.1.2. The map $\mathfrak{e}$ is

1. well-defined and weakly increasing: $[\![p]\!] \geq_{\mathrm{D}} [\![q]\!] \implies \mathfrak{e}([\![p]\!]) \geq_{\mathrm{D}} \mathfrak{e}([\![q]\!])$;
2. a homomorphism for both $\perp^{\mathrm{w}}$ and $\not\perp^{\mathrm{w}}$; and
3. a homomorphism of monoids, provided $\otimes$ respects $\geq_{\mathrm{D}}$.

Proof. ① If $p \geq_{\mathrm{D}} q$, as witnessed by r, by Lemma 2.1.17 we have $(p \mid \mathfrak{U}_1) \cup r \vdash (q \mid \mathfrak{U}_1)$, and the first part follows.

② It is well-known that, if $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $M \prec^+ N \prec^+ \mathfrak{U}$, then $p \perp^{\mathrm{w}} q \iff (p \restriction N) \perp^{\mathrm{w}} (q \restriction N)$. We give a proof for the sake of completeness.

Suppose $p(x) \perp^{\mathrm{w}} q(y)$, and let $\varphi(x, y; w) \in L(M)$ and $d \in N$ be such that $p \cup q \vdash \varphi(x, y; d)$. By compactness there is $\theta(y; e) \in q$ with $\theta(y; t) \in L(M)$ such that $p \vdash \forall y\ (\theta(y; e) \to \varphi(x, y; d))$. Since $d \in N$ and $N \,{}^+\!\!\succ M$, there is $\tilde{e} \in N$ such that $\tilde{e} \equiv_{Md} e$. So by invariance $p \vdash \forall y\ (\theta(y; \tilde{e}) \to \varphi(x, y; d))$. As this is an $L(N)$ formula, it is in $p \restriction N$. Since $q \vdash \theta(y; e)$, by invariance $q \vdash \theta(y; \tilde{e})$; as the latter is an $L(N)$-formula it is in $q \restriction N$, and we have left to right.

Conversely, suppose $(p \restriction N) \perp^{\mathrm{w}} (q \restriction N)$. Let $\varphi(x, y; w) \in L(M)$ and $d \in \mathfrak{U}$ and choose some $\tilde{d} \equiv_M d$ in N, by saturation. Suppose without loss of generality that $(p \restriction N) \cup (q \restriction N) \vdash \varphi(x, y; \tilde{d})$. So for some $\theta(y; \tilde{e}) \in q \restriction N$ with $\theta(y; t) \in L(M)$ we have $p \restriction N \vdash \forall y\ \theta(y; \tilde{e}) \to \varphi(x, y; \tilde{d})$. By an M-automorphism argument, obtain $e \in \mathfrak{U}$ such that $ed \equiv_M \tilde{e}\tilde{d}$. Then by invariance $p \vdash \forall y\ \theta(y; e) \to \varphi(x, y; d)$. Moreover, still by invariance, $q \vdash \theta(y; e)$ and this completes the proof of right to left.

③ Suppose now that $\otimes$ respects $\geq_{\mathrm{D}}$ and denote for brevity $p \mid \mathfrak{U}_1$ with $\widetilde{p}$. Recall that, if $\varphi(x, y) \in L(\mathfrak{U}_0)$, then $\varphi(x, y) \in p(x) \otimes q(y)$ if and only if for *any* $b \vDash q$ we have $\varphi(x, b) \in p \mid \mathfrak{U}_0 b$. Since we may take b realising $\widetilde{q}$, this shows that $(\widetilde{p} \otimes \widetilde{q}) \restriction \mathfrak{U}_0 = p \otimes q$, or in other words $(p \otimes q) \mid \mathfrak{U}_1 = \tilde{p} \otimes \tilde{q}$. Therefore

$$\mathfrak{e}([\![p]\!]) \otimes \mathfrak{e}([\![q]\!]) = [\![\widetilde{p}]\!] \otimes [\![\widetilde{q}]\!] = [\![(p \otimes q) \mid \mathfrak{U}_1]\!] = \mathfrak{e}([\![p \otimes q]\!])$$

so $\mathfrak{e}$ is a homomorphism of semigroups. As $\mathfrak{e}$ clearly sends $[\![0]\!]$ to $[\![0]\!]$, because the unique extension of a realised type is realised, we have the conclusion. □

Remark 6.1.3. If Question 2.3.9 has a positive answer then the image of $\mathfrak{e}$ is downward closed.

With regard to invariant types, domination, and products, the restriction map $S(\mathfrak{U}_1) \to S(\mathfrak{U}_0)$ is not as well behaved as the map $p \mapsto p \mid \mathfrak{U}_1$. To begin with, the restriction of even a realised type to $\mathfrak{U}_0$ need not be invariant. Even if this is the case (e.g. if T is stable, because then every type is invariant by Fact 3.1.1), types realised in $\mathfrak{U}_1$ but not in $\mathfrak{U}_0$ are in $[\![0]\!] \in \widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$, but not in $[\![0]\!] \in \widetilde{\mathrm{Inv}}(\mathfrak{U}_0)$, so domination is not preserved: the point is that every r witnessing domination-equivalence between a type realised in $\mathfrak{U}_0$ and one realised in $\mathfrak{U}_1 \setminus \mathfrak{U}_0$ needs to mention points outside of $\mathfrak{U}_0$, so Lemma 2.1.16 does not apply. Moreover, say again in the stable case, taking restrictions is not even a homomorphism $(S^{\mathrm{inv}}(\mathfrak{U}_1), \otimes) \to (S^{\mathrm{inv}}(\mathfrak{U}_0), \otimes)$: if p is realised in $\mathfrak{U}_1$ but not in $\mathfrak{U}_0$, then $(p(x) \otimes p(y)) \upharpoonright \mathfrak{U}_0 \vdash x = y$, but $(p(x) \upharpoonright \mathfrak{U}_0) \otimes (p(y) \upharpoonright \mathfrak{U}_0) \vdash x \neq y$.

Question 6.1.4. Is $\mathfrak{e}$ always injective?

The question below can be asked in different flavours, depending on the structure we endow $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ with; this can be any combination of $\geq_{\mathrm{D}}$, $\perp^{\mathrm{w}}$, and $\otimes$ (when it preserves $\geq_{\mathrm{D}}$).

Question 6.1.5. Let $\mathfrak{U}_0 \prec^+ \mathfrak{U}_1$. Are $\widetilde{\mathrm{Inv}}(\mathfrak{U}_0)$ and $\widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$ elementarily equivalent? Assuming that $\mathfrak{e}$ is injective, is it an elementary embedding?

6.1.2 Injectivity

We now turn to conditions that ensure injectivity of $\mathfrak{e}$. Fix $\mathfrak{U}_0$ and $\mathfrak{U}_1$ and, for $p \in S^{\mathrm{inv}}(\mathfrak{U}_0)$, denote $p \mid \mathfrak{U}_1$ with $\tilde{p}$.

Lemma 6.1.6. Suppose that every time $\tilde{p}, \tilde{q} \in S(\mathfrak{U}_1)$ are A_0-invariant for some $A_0 \subset^+ \mathfrak{U}_0$ and $\tilde{p} \geq_{\mathrm{D}} \tilde{q}$ then this can be witnessed by some $r' \in S(A')$ such that $|A'|$ is small in the sense of $\mathfrak{U}_0$.[1] Then $\mathfrak{e}([\![p]\!]) \geq_{\mathrm{D}} \mathfrak{e}([\![q]\!])$ implies $[\![p]\!] \geq_{\mathrm{D}} [\![q]\!]$, and in particular $\mathfrak{e}$ is injective.

Proof. If $\widetilde{p} \geq_{\mathrm{D}} \widetilde{q}$ can be witnessed by some r with parameters in some $A \subset^+ \mathfrak{U}_0$, then we are done: by Lemma 2.1.16 $p \cup r \vdash q$.

In the general case, as $|A_0 \cup A'|$ is still small in the sense of $\mathfrak{U}_0$, by taking unions we may assume $A' \supseteq A_0$, and we can find an A_0-isomorphic copy A of A' inside $\mathfrak{U}_0$. Let $f \in \mathrm{Aut}(\mathfrak{U}_1/A_0)$ be such that $A = f(A')$ and define

$$\mathfrak{U}_0' := f^{-1}(\mathfrak{U}_0) \qquad p' := f^{-1}(p) \in S(\mathfrak{U}_0') \qquad q' := f^{-1}(q) \in S(\mathfrak{U}_0')$$

[1]Note that A' need not be a subset of $\mathfrak{U}_0$.

As $\widetilde{p}$ and $\widetilde{q}$ are A_0-invariant they are fixed by f, so $p' \subseteq \widetilde{p}$ and $q' \subseteq \widetilde{q}$; by Lemma 2.1.16 we have $p' \cup r' \vdash q'$, and so $r := f(r')$ witnesses $p \geq_{\mathrm{D}} q$. □

Definition 6.1.7. Let F be a function between infinite cardinals. Say that in $\mathfrak{U}$ domination is *F-local*, or that $\mathfrak{U}$ has *F-local domination*, iff $\kappa(\mathfrak{U})$ is closed under F, i.e. $\lambda < \kappa(\mathfrak{U})$ implies $F(\lambda) < \kappa(\mathfrak{U})$, and if $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $p \geq_{\mathrm{D}} q$ then domination can be witnessed by a small type over a set of size at most $F(|M| + |T|)$. Say that in T domination is *local*, or that T has *local domination* iff in every $\mathfrak{U} \models T$ domination is F-local for F the identity function.

Corollary 6.1.8. Suppose that $\kappa(\mathfrak{U}_0)$ is closed under F and that in $\mathfrak{U}_1$ domination is F-local. Then $\mathfrak{e}\colon \widetilde{\mathrm{Inv}}(\mathfrak{U}_0) \to \widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$ is injective and $\mathfrak{e}([\![p]\!]) \geq_{\mathrm{D}} \mathfrak{e}([\![q]\!])$ implies $[\![p]\!] \geq_{\mathrm{D}} [\![q]\!]$. In particular, this holds if T has local domination.

Proof. Immediate from Lemma 6.1.6. □

Corollary 6.1.9. If T is stable, has algebraic domination, or is binary, then $\mathfrak{e}$ is injective and $\mathfrak{e}([\![p]\!]) \geq_{\mathrm{D}} \mathfrak{e}([\![q]\!])$ implies $[\![p]\!] \geq_{\mathrm{D}} [\![q]\!]$.

Proof. In all three cases, we show that T has local domination. In the stable case, use point 3 of Proposition 3.1.8. The second case is immediate, so assume that T is binary. Let $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$, suppose that $r \in S(N)$ witnesses $p \geq_{\mathrm{D}} q$, and let $ab \in \mathfrak{U}$ realise r. If C is such that $M \subseteq C \subseteq \mathfrak{U}$ and $a \models p \upharpoonright C$ then, since T is binary, we automatically have $a \models p \upharpoonright NC$. By Lemma 2.1.16 we have $b \models q \upharpoonright NC$, and in particular $b \models q \upharpoonright C$. By Lemma 2.4.4, $p \cup \mathrm{tp}(ab/M) \vdash q$. □

Remark 6.1.10. The fact that, by Proposition 6.1.2, $\mathfrak{e}$ is a $\perp^{\mathrm{w}}$-homomorphism, can sometimes be used to show that $\mathfrak{e}$ is injective. This is for instance the case in those o-minimal theories where every invariant type is domination-equivalent to a product of 1-types, by Theorem 4.1.27.

6.2 Independence from the choice of monster model

6.2.1 Dimensionality

Let T be stable with elimination of imaginaries. At least in the thin case independence of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ from the choice of $\mathfrak{U}$ is equivalent to *dimensionality* of T. As in Chapter 3, the content of this subsection is not really new.

Definition 6.2.1. Let $T = T^{\mathrm{eq}}$ be stable. We say that T is *dimensional* iff for every nonrealised global type p there is a global type q that does not fork over $\emptyset$ and such that $p \not\perp q$. We say that T is *bounded* iff $|\widetilde{\mathrm{Inv}}(\mathfrak{U})| < |\mathfrak{U}|$. If T is not dimensional we say that it is *multidimensional*.

In the literature, the term *non-multidimensional* is also used, sometimes shortened to *nmd*. The meaning is the same as "dimensional". The idea behind the name is that, say in the thin case, generators of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ correspond to "dimensions" of T.

Conjecture 6.2.2. Let $T = T^{\mathrm{eq}}$ be stable. The following are equivalent.

1. T is bounded.
2. T is dimensional.
3. $\mathfrak{e}$ is surjective.

In Conjecture 6.2.2, 1 $\Rightarrow$ 2 follows from [Bue17, Proposition 5.6.2]. Since 3 $\Rightarrow$ 1 is trivial, it remains to prove 2 $\Rightarrow$ 3, namely that if there is a type over $\mathfrak{U}_1$ not domination-equivalent to any type that does not fork over $\mathfrak{U}_0$, then there is a type orthogonal to every type that does not fork over $\emptyset$.

Proposition 6.2.3. If T is thin then Conjecture 6.2.2 holds.

Proof. Suppose $\mathfrak{U}_0 \prec^+ \mathfrak{U}_1$ and, for $j < 2$, let $f_j \colon \widetilde{\mathrm{Inv}}(\mathfrak{U}_j) \to \bigoplus_{\kappa_j} \mathbb{N}$ be given by Theorem 3.1.24. Let

$$g \coloneqq f_1 \circ \mathfrak{e} \circ f_0^{-1} \colon \bigoplus_{\kappa_0} \mathbb{N} \to \bigoplus_{\kappa_1} \mathbb{N}$$

Since weight is, by definition, preserved by nonforking extensions, $\mathfrak{e}$ sends types of weight 1 to types of weight 1. Therefore by Remark 3.1.25 we may decompose the codomain of g as

$$\bigoplus_{\kappa_1} \mathbb{N} \cong \bigoplus_{i<\kappa_0} \mathbb{N} \oplus \bigoplus_{\kappa_0 \le i < \kappa_1} \mathbb{N}$$

where the direct summand $\bigoplus_{i<\kappa_0} \mathbb{N}$ may be assumed to coincide with $\operatorname{Im} g$. Hence, if $\mathfrak{e}$ is not surjective, we can find $[\![p]\!] \notin \operatorname{Im} \mathfrak{e}$ such that p has weight 1. Again by Theorem 3.1.24, such a p needs to be orthogonal to every type in the union of $\operatorname{Im} \mathfrak{e}$, which is the set of types that are domination-equivalent to some type that does not fork over $\mathfrak{U}_0$. In particular, p is orthogonal to every type that does not fork over $\emptyset$. □

If T is thin and dimensional, then the number of copies of $\mathbb{N}$ required in Theorem 3.1.24 is bounded by $2^{|T|}$, and by $|T|$ if T is totally transcendental, by standard bounds on the number of types over $\emptyset$ and on the number of nonforking extensions. See e.g. [Bue17, Corollary 7.1.1], although this source only states this fact for superstable theories. In fact, some sources define boundedness only in the superstable case, essentially as boundedness of the number of copies of $\mathbb{N}$ given by Theorem 3.1.24.

A possible strategy to prove Conjecture 6.2.2 in the general stable case could be, assuming $\mathfrak{e}\colon \widetilde{\operatorname{Inv}}(\mathfrak{U}_0) \to \widetilde{\operatorname{Inv}}(\mathfrak{U}_1)$ is not surjective, to try to find a type p of weight 1 whose class is not in the image of $\mathfrak{e}$. Then p will be either orthogonal to every type that does not fork over $\mathfrak{U}_0$, or dominated by one of them by Fact 3.1.21. If we knew a positive answer to Question 2.3.9 at least in the stable case, and if we managed to find a type as above, then we would be done.

A possibly related notion is the *strong compulsion property* (see [Hyt95, Definition 2]); it implies that every type over $\mathfrak{U}_1 \mathrel{^{+}\!\succ} \mathfrak{U}_0$ is either orthogonal to every type that does not fork over $\mathfrak{U}_0$ or dominates one of them. Whether all countable stable T^{eq} have a weakening of this property is [Hyt95, Conjecture 18]. If T has the strong compulsion property and $\mathfrak{e}$ is not surjective, but there are no types orthogonal to every $\mathfrak{U}_0$-invariant type, then there is a type p_1 over $\mathfrak{U}_1$ which dominates, but is not domination-equivalent to, a $\mathfrak{U}_0$-invariant type. If this can be iterated, i.e. we can find p_2 over $\mathfrak{U}_2 \mathrel{^{+}\!\succ} \mathfrak{U}_1$ which dominates $p_1 \mid \mathfrak{U}_2$ but is not domination-equivalent to any $\mathfrak{U}_1$-invariant type, etc., then we can produce an arbitrarily long domination chain, hence T would have *unboundedness of types*, a notion introduced in [Her91]. To the best of my knowledge, it is unknown whether such a theory exists.

6.2.2 Independence implies NIP

It might not be unreasonable to conjecture that, if $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ does not depend on the choice of $\mathfrak{U}$, then T is stable. We now take a first step towards this.

As we will see in Section 7.1, the preorder $\geq_{\mathrm{D}}$ is the result of a series of generalisations, ultimately going back to the Rudin–Keisler preorder on ultrafilters. It is not surprising, therefore, that some classical arguments involving the latter object generalise as well. We show in Theorem 6.2.9 that, in the case of theories with IP, one of them is the abundance of pairwise Rudin–Keisler inequivalent ultrafilters on ω; the classical proof goes through for $\sim_{\mathrm{D}}$ as well, and shows that even the cardinality of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ depends on the choice of $\mathfrak{U}$.

In this subsection $[\![p]\!]$ denotes the $\sim_{\mathrm{D}}$-class of p. Even if we state everything for $\sim_{\mathrm{D}}$ and its quotient $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, the same arguments work if we replace $\sim_{\mathrm{D}}$ by $\equiv_{\mathrm{D}}$, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ by $\overline{\mathrm{Inv}}(\mathfrak{U})$, and interpret $[\![p]\!]$ as the class of p modulo $\equiv_{\mathrm{D}}$.

The following result is classical; see for instance [Hod93, Exercise 4(a) of Section 10.1 and Theorem 10.2.1].

Fact 6.2.4. Let T be any theory and $\kappa \geq |T|$. Then T has a κ^+-saturated and κ^+-strongly homogeneous model of cardinality at most 2^κ.

We now drop our usual conventions on $\mathfrak{U}$. For the rest of this subsection, $\mathfrak{U}$ is a κ^+-saturated and κ^+-strongly homogeneous model of cardinality at most $\mu := |\mathfrak{U}| \leq 2^\kappa$, we denote by σ the least cardinal such that $\mathfrak{U}$ is not σ^+-saturated, and with *small* we mean "of size strictly less than σ". Thus $\kappa^+ \leq \sigma \leq \mu \leq 2^\kappa$.

Lemma 6.2.5. For every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ we have $|[\![p]\!]| \leq |\{q \mid q \leq_{\mathrm{D}} p\}| \leq \mu^{<\sigma}$.

Proof. Clearly $[\![p]\!] \subseteq \{q \mid q \leq_{\mathrm{D}} p\}$. For every $q \leq_{\mathrm{D}} p$, there is some small r_q such that $p \cup r_q \vdash q$. If $r_q = r_{q'}$ then $q = q'$, and therefore $|\{q \mid q \leq_{\mathrm{D}} p\}|$ is bounded by the number of small types. As "small" means "of cardinality strictly less than σ", the number of such types is at most the size of $\bigcup_{A \subset \mathfrak{U}, |A| < \sigma} S(A)$, which cannot exceed $\mu^{<\sigma} \cdot 2^{<\sigma} = \mu^{<\sigma}$. □

Corollary 6.2.6. For every p, we have $|\{[\![q]\!] \mid [\![q]\!] \leq_{\mathrm{D}} [\![p]\!]\}| \leq \mu^{<\sigma}$.

Lemma 6.2.7. If T has IP, then $2^\kappa = \mu = 2^{<\sigma} = \mu^{<\sigma}$.

Proof. If $\varphi(x; y)$ witnesses IP, then over a suitable model of cardinality κ, which we may assume to be embedded in $\mathfrak{U}$, there are 2^κ-many φ-types, and a fortiori types. This gives the first equality, and the same argument applied to those λ such that $\kappa \leq \lambda < \sigma$ gives the second one. The third one follows from $2^\kappa = \mu$ by cardinal arithmetic:

$$\mu^{<\sigma} = \sup_{\lambda<\sigma} \mu^\lambda = \sup_{\lambda<\sigma} 2^{\kappa \cdot \lambda} = \sup_{\lambda<\sigma} 2^\lambda = 2^{<\sigma} \qquad \square$$

Recall the following property of theories with IP.

Fact 6.2.8. If T has IP, then for every $\kappa \geq |T|$ there is a type p over some $M \vDash T$ such that $|M| = \kappa$ and p has 2^{2^κ}-many M-invariant extensions. Moreover, such extensions can be chosen to be over any κ^+-saturated model.

Proof sketch. The first statement is [Poi00, Theorem 12.28]. The "moreover" part follows from the proof in the referenced source: in its notation, it is enough to realise the f-types of the b_w over $\{a_\alpha \mid \alpha < \kappa\}$. □

Theorem 6.2.9 ([Men20, Proposition 4.6])**.** If T has IP and $\mathfrak{U}$ is κ^+-saturated and κ^+-strongly homogeneous of cardinality 2^κ, then $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ has size $2^{|\mathfrak{U}|}$. In particular, if $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ does not depend on the choice of $\mathfrak{U}$ then T is NIP.

Proof. Since κ^+-saturation implies κ^+-universality, we may assume that the M given by Fact 6.2.8 is an elementary submodel of $\mathfrak{U}$, and by the "moreover" part of Fact 6.2.8 we have $|S^{\mathrm{inv}}(\mathfrak{U})| \geq 2^{2^\kappa}$. But then by Lemma 6.2.5

$$2^{2^\kappa} \leq |S^{\mathrm{inv}}(\mathfrak{U})| = \sum_{[\![p]\!] \in \widetilde{\operatorname{Inv}}(\mathfrak{U})} |[\![p]\!]| \leq |\widetilde{\operatorname{Inv}}(\mathfrak{U})| \cdot \mu^{<\sigma}$$

Using Lemma 6.2.7 we obtain

$$2^\mu = 2^{2^\kappa} \leq |\widetilde{\operatorname{Inv}}(\mathfrak{U})| \cdot \mu^{<\sigma} = |\widetilde{\operatorname{Inv}}(\mathfrak{U})| \cdot \mu$$

and therefore $|\widetilde{\operatorname{Inv}}(\mathfrak{U})| = 2^\mu = 2^{|\mathfrak{U}|}$.

For the last part, note that if T has IP and $\mathfrak{U}_1$ is, say, $|\mathfrak{U}_0|^+$-saturated of cardinality $2^{|\mathfrak{U}_0|}$, then $|\widetilde{\operatorname{Inv}}(\mathfrak{U}_1)| = 2^{2^{|\mathfrak{U}_0|}}$. □

Hence, 1 and 3 from Conjecture 6.2.2 are equivalent in every IP theory as they are both false, yet the equivalence with 2 fails in the theory of the Random Graph, where no two nonrealised types are weakly orthogonal. This, along with other similar phenomena, may anyway just be evidence that weak orthogonality on invariant types is not a meaningful notion in the IP case.

Note that, should $\mathfrak{e}$ fail to be injective, we could still in principle have two monster models $\mathfrak{U}_0$ and $\mathfrak{U}_1$ of different cardinalities such that $|\widetilde{\operatorname{Inv}}(\mathfrak{U}_0)| = |\widetilde{\operatorname{Inv}}(\mathfrak{U}_1)|$. For instance, even in a theory with IP, the previous results do not prevent this from happening in the case where $|\mathfrak{U}_0|$ and $|\mathfrak{U}_1|$ are strong limits.

In the NIP unstable case, if we assume, say, the Generalised Continuum Hypothesis (in particular we may dispense with Lemma 6.2.7), similar arguments show that the quotient of the space of *all* types by domination-equivalence depends on the choice of $\mathfrak{U}$. The questions below, which can also be asked for $\overline{\operatorname{Inv}}(\mathfrak{U})$, remain nonetheless open. Recall that we only consider theories with infinite models and that, by Proposition 3.1.31, in every stable theory $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is infinite.

Question 6.2.10. Is there an unstable NIP theory where $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ does not depend on $\mathfrak{U}$? Is there one where it is finite?

Another classical set-theoretic result is the abundance of pairwise Rudin–Keisler incomparable ultrafilters. This is related to *independent families*, and it seems plausible for it to generalise to theories with IP.

Question 6.2.11. Suppose that T is a theory with IP. Are there $2^{|\mathfrak{U}|}$ pairwise $\geq_{\mathrm{D}}$-incomparable invariant types over $\mathfrak{U}$?

Finally, I would like to ask the reader to forgive me for the heavy overloading of the word *independent*, the most outrageous instance of which being, perhaps, the title of this subsection.

Chapter 7

Past and future

We conclude this thesis by trying to put it in context. In the first section we will see where does it come from, and in the last one where does it go.

7.1 Historical notes

Below is a brief, and by no means meant to be exhaustive, overview of the genesis of the notions of domination and product of types.

7.1.1 Ultrafilters

Types are ultrafilters on Boolean algebras of definable sets. On the other hand, ultrafilters are types: if X is any set, one can equip it with the "full" language, mentioning every subset of X^n for every n, and consider ultrafilters on X^n as n-types over X in this theory.[1] Of course, as soon as one has a bijection $X \to X \times X$ in the language, it is enough to just mention all subsets of X, and this is why set-theorists studying ultrafilters appear to be mostly considering 1-types in these theories. Note that, even if we are not working over a saturated model, this context is not too different from that of global invariant types: we are considering every possible subset of every X^n to be definable, hence every type over X is definable.

Two points of a topological space can be considered equivalent iff they are conjugate by a homeomorphism. In the case of the topological space βX of ultrafilters on X, i.e. $S_1(X)$ if X is equipped with the structure described

[1] Or as n-types over $\emptyset$. Since all singletons are named, the difference is subtle, but it becomes relevant when talking e.g. of definable types, as we are about to do.

above, a homeomorphism $\beta X \to \beta X$ is essentially the same as a bijection $X \to X$, since the principal ultrafilters coincide with the isolated points of βX, whose set must be preserved by every homeomorphism. So, in this order of ideas, two ultrafilters p, q are considered equivalent when there is a bijection $f\colon X \to X$ such that $f_*p = q$. It turns out (see [CN74, Corollary 9.3]) that in this very particular theory this is the same as requiring the existence of two functions $f, g\colon X \to X$, not necessarily bijective, such that $f_*p = q$ and $g_*q = p$. Hence the definition of the *Rudin–Keisler preorder* on ultrafilters: we say that $p \geq_{\mathrm{RK}} q$ iff there is f such that $f_*p = q$.

Let $p, q \in \beta X$. Some sources, e.g. [Boo70], define $p \otimes q \in \beta X^2$ as

$$\{A \subseteq X^2 \mid \{b \mid \{a \mid (a,b) \in A\} \in p\} \in q\} \tag{7.1}$$

On the other hand, most of the contemporary literature on ultrafilters writes the product in the opposite order, i.e. defines $p \otimes q$ as

$$\{A \subseteq X^2 \mid \{a \mid \{b \mid (a,b) \in A\} \in q\} \in p\} \tag{7.2}$$

We will see later how the model-theoretic notion of product we have studied in this thesis has followed a similar fate.

One of several ways ultrafilters can be thought about is as finitely additive measures which can only take value 0 or 1. In this sense, the definition above reads "for q-almost every point b, the "slice" A_b has p-measure 1". Note the analogy with the Fubini–Tonelli Theorem or, in a different (but related) context, with the Kuratowski–Ulam Theorem. From this viewpoint, $p \geq_{\mathrm{RK}} q$ reads "the measure q can be obtained as a pushforward of the measure p".

7.1.2 Domination

The Rudin–Keisler preorder was generalised to types by Lascar in [Las75, Las76]. He noticed that, if we interpret ultrafilters as types in the fashion above, then $p \geq_{\mathrm{RK}} q$ if and only if every model that realises p also realises q. In the presence of prime models, say for T countable and ω-stable, this is then the same as q being isolated over p, and the name "Rudin–Keisler preorder" often refers to this other notion. In ω-stable context, or in *small* theories for types over $\emptyset$, the distinction is blurred but, as pointed out in [Tan15], in general semi-isolation, i.e. Definition 2.1.28, tends to be more adequate.

At any rate, in several places across the model-theoretic literature the name "Rudin–Keisler preorder" refers to isolation, which is studied in the presence of prime models. Shelah defined several generalised notions of isolation, which the reader can find in [She90, Chapter 4]. In the stable case, he proved the existence of *a-prime* models, and defined domination, although not with this name, which to the best of my knowledge first appears in [Las82].

As for the notions we used here, namely those in Definition 2.1.9, equidominance appears in [HHM08], under the name "domination-equivalence" (see Remark 2.1.14). I am not aware of any use the word "domination" in the current sense anywhere prior to [Men20], but it is noted in [HHM08] that a notion of domination can be defined in full generality for invariant types, and my guess is that Definition 2.1.9 is what the authors had in mind. The closest analogue is Shelah's small-type isolation $\mathrm{F}^{\mathrm{s}}_{\kappa}$, in the notation of [She90] (or $\mathrm{F}^{\mathrm{t}}_{\kappa}$; the difference disappears for $\kappa \geq |T|$). Anyway, as we saw in Example 2.1.30, this is not the same as Definition 2.1.9.

Several other notions of domination have been considered in the past. It is beyond the scope of this thesis to consider them in detail, so I will be brief. Whenever one has a nice enough independence relation, Definition 3.1.5 can be generalised. This has of course been done in simple theories, and more generally in rosy theories in [OU11a]. A general definition in terms of functions and ideals can be found in [HHM08, p. 6]. *Stable domination*, introduced in [HHM08], was studied also in [OU11b] and corresponds to the nonforking ideal. When used with the ideal of measure-zero sets on a compact group equipped with the Haar measure, this yields what is known as *compact domination*; see e.g. [HP11]. The meagre ideal has also been used, e.g. in [CS18].

7.1.3 Product

If we try to use (7.1) for types of a theory, we incur the problem that, even if A is definable, $\{b \mid \{a \mid (a,b) \in A\} \in p\}$ need not be. In fact, every such set is definable precisely when p is definable in the sense of Definition 2.3.2. This was observed by Lascar in [Las71, Las76], who defined the product on definable types only. I do not know where this was first generalised to invariant types, but of course it does appear in [HHM08].

As I was saying above, when dealing with ultrafilters, at present the definition of $p \otimes q$ is usually taken to be (7.2), rather then (7.1). I ignore when and why the order was swapped, or if [Boo70] just adopted a nonstandard convention. In [Las76], Lascar references [Boo70], and takes (7.1) as a definition of $p \otimes q$. The reader can check that, for a definable p in an arbitrary theory, this coincides with Definition 2.1.5. In the stable context of course it does not matter whether we write $p \otimes q$ or $q \otimes p$ but, in general, the two are different.

In these pages, I have adopted what currently seems to be the dominant convention in model-theoretic literature, i.e. to take (7.1), or rather its generalisation to invariant types, as a definition of $p\otimes q$. One of the reasons was to keep the same notation as in [Men20]. With that said, this convention might be susceptible to change, and for example [Tan15] writes products in the other order. According to [Tan15, p. 310], this was suggested by Newelski in order to have, when defining Morley sequences, $(a^0, \dots, a^{n-1}) \vDash p(x^0) \otimes \dots \otimes p(x^{n-1})$, while in the current notation we need to take $(a^0, \dots, a^{n-1}) \vDash p(x^{n-1}) \otimes \dots \otimes p(x^0)$. I will not hide the fact that I do agree with Newelski in this respect, and I would have liked to use this other convention here. I also have to admit that, by the time I started considering this seriously, too much of this document had already been written, and I have not had the courage to reverse every product in this thesis with the prospect of proofreading it while still being accustomed to the old conventions.

7.2 Further directions

As witnessed by the several questions I have left open in the previous chapters, the study of the domination monoid is everything but concluded. In this section we will briefly talk about some more possible lines of enquiry.

7.2.1 General theory

By Theorem 5.3.8, $\sim_{\mathrm{D}}$ need not be a congruence with respect to $\otimes$. Clearly, no one prevents us from considering the smallest congruence generated by domination-equivalence. It would be interesting to know if this has a nice model-theoretic description/meaning.

Problem 7.2.1. Characterise the smallest $\otimes$-congruence extending $\sim_{\mathrm{D}}$.

Another question that makes sense even in theories where $\otimes$ does not respect $\geq_{\mathrm{D}}$ is the following.

Question 7.2.2. Is it possible to have $p \not\geq_{\mathrm{D}} p^{(2)} \sim_{\mathrm{D}} p^{(3)}$?

In the results on real closed fields, and in those on valued fields, we were enlarging bases in order for them to be sufficiently nice, e.g. maximally complete with respect to some valuation. Therefore, it might be too optimistic to expect every theory to have local domination. On the other hand, in both cases there is a bound on how larger a "sufficiently nice" base needs to be, and I do not know of any example where the extension map $\mathfrak{e}$ is not injective.

Question 7.2.3. Is there a function F on infinite cardinals such that, for every theory T and every cardinal λ, there is $\mathfrak{U} \vDash T$ such that $\kappa(\mathfrak{U}) > \lambda$ and $\mathfrak{U}$ has F-local domination?

A broad problem is to investigate the interaction of domination with the generalised notions of regularity from Definition 2.3.19. More generally, in view of the existing minimality results for regular types, we can ask the question below. Note that it might be unreasonable to expect this to happen in full generality, since the existence of Rudin–Keisler-minimal ultrafilters on ω depends on set-theoretic hypotheses: they are the same as Ramsey ultrafilters.

Question 7.2.4. In which theories, for every nonrealised p, is there a nonrealised $\geq_{\mathrm{D}}$-minimal q such that $p \geq_{\mathrm{D}} q$?

Problem 7.2.5. Assume that $\otimes$ respects $\geq_{\mathrm{D}}$. Characterise the centraliser of $[\![p]\!]$ in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$.

Yet another direction is to study variants of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, e.g. by fixing the invariance base or by focusing on special classes of invariant types, such as definable or finitely satisfiable ones. Or we could look at analogous objects for Keisler measures, or in continuous logic, or in positive logic. We could also take some sort of compactification of $S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})$ and hope to be able to carry out a topological-dynamical study, but a clear obstruction in this direction is that $S^{\mathrm{inv}}_x(\mathfrak{U})$ is dense in $S_x(\mathfrak{U})$, and exhausts it if and only if T is stable. Here the compact space $S^{\mathrm{inv}}_\omega(\mathfrak{U}, M)$ may be a better candidate.

Finally, let me point out that domination is clearly related to prime models and omitting types. Since certain proofs of the Omitting Types Theorem for countable theories use Baire category (see e.g. [Poi00, Theorem 10.3]), it would be interesting to see if the recent developments in Generalised Descriptive Set Theory can be used in this fashion, e.g. to improve the results on generalised prime models from [She90, Chapter IV].

7.2.2 The o-minimal case

Most of the material in Section 4.1 works under the sole assumption of o-minimality, and I do not know of any o-minimal theory not satisfying Assumption 4.1.19. I have not seen a proof that this always holds either and, while this is not the only possible approach to the problem, the reduction to Assumption 4.1.28 is a step towards such a proof.

Question 7.2.6. Does Assumption 4.1.28 hold in every o-minimal theory, or at least in o-minimal expansions of DOAG? More generally, in the setting of Proposition 4.1.31, does $\pi_M(x) \vdash p(x)$?

We saw that in o-minimal groups and fields with no extra structure, generators of the domination monoid correspond to invariant convex subgroups and subrings. Of course the particular description of a set of generators will depend on the particular theory at hand, so we state the following problem for a particular structure, and we phrase it in a way that makes sense even if Assumption 4.1.19 turns out to fail.

Problem 7.2.7. Identify a nice maximal set of pairwise weakly orthogonal invariant 1-types in monster models of the theory of $\mathbb{R}_{\mathrm{exp}}$.

7.2.3 Stability

Question 2.3.18 asks whether $\perp^{\mathrm{w}}$ can be defined internally to $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. In a sense, this question is not new. In [Her91], Hernandez lists as open whether for every $p \not\perp q$ there is a nonrealised r such that $p \geq_{\mathrm{D}} r$ and $q \geq_{\mathrm{D}} r$. I do not know whether this has been already answered,[2] but Hernandez did prove an infinitary version of this property.

Theorem 7.2.8 (Hernandez, "existence of components")**.** If T is a countable stable theory, then $p_0 \not\perp p_1$ if and only if there is a nonrealised q such that $p_0^\omega \geq_{\mathrm{D}} q^\omega$ and $p_1^\omega \geq_{\mathrm{D}} q^\omega$.

This allows for the development of a deep theory of *locally Boolean spaces* of (classes of) types, which are objects very similar to $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}, \perp)$, but based on types of the form p^ω. In general, [Her91] contains several results and questions that are very close to the topic of this thesis (in the stable case) and,

[2] [Her91] is not too recent, and for instance also lists as open whether equidominance coincides with domination-equivalence.

while I will not go into details, I would like to at least highlight a variation of what Hernandez calls the problem of *unboundedness of types*. He asks this in the countable stable case and with $p \geq_{\mathrm{D}} q$ replaced by $p^\omega \geq_{\mathrm{D}} q^\omega$, but we formulate the following question in a more general context.

Question 7.2.9. Are there a theory T and a global invariant type p such that p dominates unboundedly many domination-equivalence classes? More precisely, is there an invariant type p such that there is no bound on the number of domination-equivalence classes dominated by some $p \mid \mathfrak{U}_1$?

It would be nice to compute more examples of exotic $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in strictly stable theories, where with "exotic" I mean "not of the form in Theorem 3.1.24". One related question concerns idempotent elements. A positive answer would provide a converse to Theorem 3.1.24, assuming stability.

Question 7.2.10. Let T be stable and suppose that p has infinite weight with respect to itself. Is $[\![p]\!]_{\sim_{\mathrm{D}}}$ idempotent?

Another question, for which I would like to thank Yatir Halevi and Ludomir Newelski, is whether domination can be defined for φ-types and the corresponding monoid internalised.

Question 7.2.11. Let T be stable. Is it possible to define "local" objects $\widetilde{\mathrm{Inv}}(\mathfrak{U})_\varphi$ and, in this case, "internalise" them in the fashion of [New14, Hal18]?

In [OU11b], Onshuus and Usvyatsov study domination in the sense of forking outside stable theories, but with a focus on stable types. An interesting line of enquiry would be to compare this with $\geq_{\mathrm{D}}$, and see which results transfer. Notably, if the parallel is nice, it should be possible to use [OU11b, Corollary 6.7] to generalise Theorem 3.1.24.

7.2.4 The domination monoid under NIP

Stepping away from stability, there is heuristic evidence indicating that an enquiry on the properties of domination in NIP theories could be fruitful. To begin with, most of the counterexamples we encountered in the previous pages had IP, and the most vicious ones IP_2. If I were allowed to only leave one open problem, it would be the study of the conjecture below.

Conjecture 7.2.12. If T is NIP, then $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined and commutative.

The conjecture above is just the first step towards understanding domination in NIP theories. In what follows, we assume it has a positive answer. If that turns out not to be true, the questions below are to be intended for theories where $\otimes$ respects $\geq_{\mathrm{D}}$.

In all the NIP examples computed to date, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is built from copies of $\mathbb{N}$ and the finite powerset of some set (not necessarily as a direct sum; see Proposition 3.2.6) or at least embeds in such an object (recall Subsection 5.1.3). While Counterexample 5.1.14 implies that Theorem 7.2.8 does not even generalise to the case of finite dp-rank, it would still be interesting to see how much of Hernandez's machinery on locally Boolean spaces can be adapted under NIP, or at least in the dp-minimal case. In this regard, note that what Proposition 4.1.23 uses is the fact that $\otimes$ preserves $\geq_{\mathrm{D}}$ and $\perp$, along with the existence of a family of pairwise orthogonal generators, and with idempotency. Dropping the latter and allowing for copies of $\mathbb{N}$ might yield a similar result.

Question 7.2.13. Assume $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined and generated by a family of pairwise weakly orthogonal classes. Does it follow that it is of the form $\mathscr{P}_{\mathrm{fin}}(X) \oplus \bigoplus_\lambda \mathbb{N}$? Is this the case in dp-minimal theories?

A dp-minimal theory of interest is that of p-adic numbers.

Problem 7.2.14. Compute $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in the theory of the p-adic field $\mathbb{Q}_p$.

Question 7.2.15. If T is NIP, is every invariant type domination-equivalent to a product of invariant 1-types?

Beside heuristics, there are already some results about NIP theories that are likely to be relevant, e.g. the good behaviour of externally definable sets, which are intimately linked to $\otimes$, Borel-definability of invariant types, and Fact 4.1.18. We refer the interested reader to [Sim15].

An important theme in the general study of NIP theories is that of *type decompositions*; see e.g. [Sim20]. From this perspective, I would like to ask the following question, which also makes sense without NIP. Note that, in the case of ACVF, this holds as a consequence of the Ax–Kochen–Eršov behaviour observed in [HHM08]. Of course, analogous questions may be raised for submonoids generated by classes of types with other properties, and are possibly more interesting where such a property is preserved by domination and products, e.g. finite satisfiability, or definability.

Question 7.2.16. Assume that $\otimes$ respects $\geq_{\mathrm{D}}$, and possibly that T is NIP. Is the monoid $\widetilde{\mathrm{Inv}}_{\mathrm{gs}}(\mathfrak{U})$ a direct summand of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$?

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