The domination monoid in henselian valued fields

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joint work with Martin Hils

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Agrume meeting
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Motivation and overview

T complete, $\mathfrak{U}$ a $\kappa(\mathfrak{U})$-monster, $\kappa(\mathfrak{U}) > \beth_\omega(|T|)$ strong limit of cofinality $> |T|$. *Small* = *of size* $< \kappa(\mathfrak{U})$.

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In ACVF ($k :=$ residue field, $\Gamma :=$ value group)

$$\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(k) \times \widetilde{\operatorname{Inv}}(\Gamma) \cong (\mathbb{N}, +) \times (\mathscr{P}_{\mathrm{fin}}(X), \cup)$$

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(to be precise, they use $\overline{\mathrm{Inv}}(\mathfrak{U})$; in ACVF they are equal, in general $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is nicer)

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Canonical extension and product

Definition ($p \in S(\mathfrak{U})$, $A \subseteq \mathfrak{U}$ small)

p A-*invariant* $:=$ whether $p(x) \vdash \varphi(x; d)$ depends only on $\varphi(x; w)$ and $\operatorname{tp}(d/A)$.

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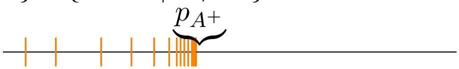

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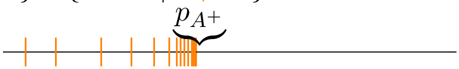

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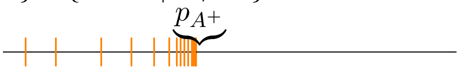

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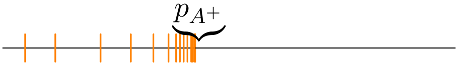

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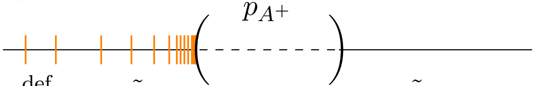

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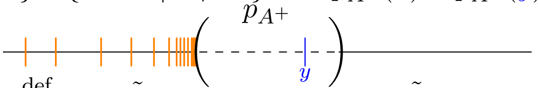

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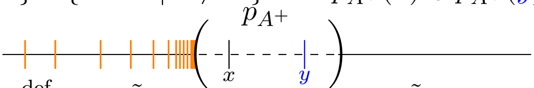

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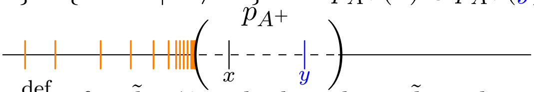

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Fact

$\otimes$ is associative. $\otimes$ commutative $\Leftrightarrow$ T stable (in which case $ab \vDash p \otimes q \iff a \vDash p, b \vDash q, a \mathop{\downarrow}_{\mathfrak{U}} b$).

Domination

Definition (Domination preorder on $S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})$; generalises Rudin–Keisler)

$p_x \geq_{\mathrm{D}} q_y$ iff there are a small $A \subset \mathfrak{U}$ and $r \in S_{xy}(A)$ such that:

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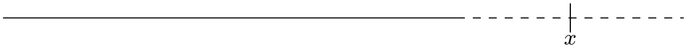
$\mathrm{tp}(x > \mathfrak{U})$



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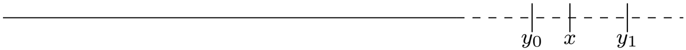

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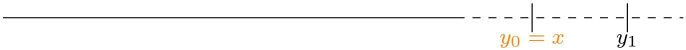

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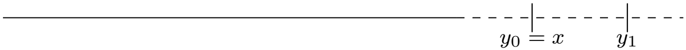

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$p \geq_{\mathrm{D}} q \iff p \supseteq q$ after renaming/duplicating variables and ignoring realised ones.

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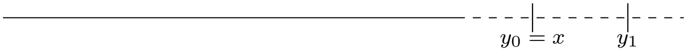

Example (Random Graph, or a set with no structure (*degenerate domination*))

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The domination monoid

Let $\widetilde{\mathrm{Inv}}(\mathfrak{U}) := S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})/\sim_{\mathrm{D}}$.

Fact

If $\geq_{\mathrm{D}}$ is compatible with $\otimes$, then

- $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}})$ is a partially ordered monoid, the *domination monoid*;
- the neutral element (and minimum) is the (unique) class of realised types; and
- nothing else is invertible ($p \otimes q$ realised $\implies p, q$ both realised!).

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There are some conditions (here) ensuring compatibility.
In certain concrete cases (e.g. ACVF) one shows compatibility directly, as a corollary of a computation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. (more on this later)

Examples

(In all of these $\geq_{\mathrm{D}}$ and $\otimes$ are compatible)

T strongly minimal (see here)

$$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathbb{N}, +, \leq).$$

For T stable, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \Leftrightarrow T$ is *unidimensional*, e.g. countable and $\aleph_1$-categorical, or $\mathrm{Th}(\mathbb{Z}, +)$.

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T superstable (*thin* is enough)

By classical results $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \bigoplus_{i<\lambda}(\mathbb{N}, +, \leq)$, for some $\lambda = \lambda(\mathfrak{U})$.

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For T stable, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \Leftrightarrow T$ is *unidimensional*, e.g. countable and $\aleph_1$-categorical, or $\mathrm{Th}(\mathbb{Z}, +)$.

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Invariant cut = small cofinality on exactly one side.

Examples

(In all of these $\geq_{\mathrm{D}}$ and $\otimes$ are compatible)

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Random Graph (see here)

$\sim_{\mathrm{D}}$ is degenerate, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ resembles $(S^{\mathrm{inv}}_{<\omega}(\mathfrak{U}), \otimes)$, e.g. it is noncommutative.

Regular ordered abelian groups

from now on, joint work with M.Hils

In DOAG, by [HHM08], $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{\mathrm{fin}}(\{\text{invariant convex subgroups of } \mathfrak{U}\})$. This can be "lifted" to Presburger Arithmetic along the map $\mathfrak{U} \to \mathfrak{U}/\mathbb{Z}$. We can say more.

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The "hole" depends on the lack of an hyperimaginary sort for $\varprojlim \mathfrak{U}/n\mathfrak{U}$. This does not seem to work in general (consider the Fraïssé limit of two linear orders).

Pure short exact sequences of abelian groups

Consider a s.e.s. $0 \to A \to B \to C \to 0$ where $A \to B$ is pure (e.g. C torsion-free).
A, C may carry extra structure (individually).

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- "Every A/nA finite" may be dropped passing to $\widetilde{\mathrm{Inv}}_\omega(\mathfrak{U})$ plus sorts A/nA.
- More generally: for pure s.e.s. of *L-abelian structures*, even with A and C expanded, we get $\widetilde{\mathrm{Inv}}_{|L|}(A_{\mathcal{F}}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_{|L|}(C(\mathfrak{U}))$. ($A_{\mathcal{F}} = A$ plus certain imaginaries)

Benign valued fields

Let K be an henselian valued field of characteristic $(0,0)$ or of characteristic (p,p) algebraically maximal Kaplansky. Recall the leading term structure

$$\mathcal{RV} := 1 \to k^\times \to K^\times/(1+\mathfrak{m}) \to \Gamma \to 0$$

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General technique to show transfer of compatibility from $\mathcal{A}(\mathfrak{U})$ to $\mathfrak{U}$: find a family of definable functions τ to $\mathcal{A}$ such that $\tau_*^p p \sim_{\mathrm{D}} p$ and $p \otimes q \sim_{\mathrm{D}} \tau_*^p p \otimes \tau_*^q q$.

Putting things together

The s.e.s. $\mathcal{RV}$ is pure. Combining the results we obtain e.g.:

Theorem (Hils, M.)

Let $\mathfrak{U}$ be a benign valued field, with residue field k eliminating imaginaries, or such that every $(k^\times)/(k^\times)^n$ is finite. Then $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(k(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$.

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$\widetilde{\mathrm{Inv}}_\omega(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\omega(k(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\omega(\Gamma(\mathfrak{U})) \cong \hat{\omega} \times \mathscr{P}_{\leq\omega}(X)$

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with $\hat{\omega}$ the set of countable cardinals with cardinal sum.

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  and compatibility of $\otimes$ with $\geq_{\mathrm{D}}$ holds.
- The reduction to $\mathcal{RV}$ also holds for σ-henselian valued difference fields of residue characteristic 0. In the isometric and multiplicative (e.g. contractive) cases, the reduction to k, Γ holds in the model companions.

Where next?

- Non-regular oags?
  - *Polyregular* oags may be dealt with by using the material on s.e.s.
  - By Gurevich–Schmitt/Cluckers–Halupczok, oags eliminate quantifiers in a language with certain sorts parameterising definable convex subgroups.
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- Adding imaginaries?
  - Regular oags: the A/nA suffice. Pleasant side-effect: they fill "finitary holes".
  - [Vic21] allows to deal with polyregular oags.
  - ACVF and RCVF: $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ does not change ([HHM08, EHM19]).
  - In general, it may depend on which kind of resolutions are available.

More open questions

1. Can one bound the size of a witness of $p \geq_{\mathrm{D}} q$ in terms of the size of invariance bases for p, q? (This would imply that for $\mathfrak{U} \prec^+ \mathfrak{U}_1$ the natural map $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \to \widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$ is injective.)
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Slides

Thanks for listening!

Preprint

Bibliography

this is not a proper bibliography, it's just a list of the sources mentioned in these slides

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More examples: Branches

Example

Let T be the theory in the language $\{P_\sigma \mid \sigma \in 2^{<\omega}\}$ asserting that every point belongs to every $P_{\eta\restriction n}$ for exactly one $\eta \in 2^\omega$. Then $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \bigoplus_{2^{\aleph_0}} \mathbb{N}$.

Basically, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ here is counting how many new points are in a "branch".

More Examples: Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes).
A set of generators for $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

- a single $\sim_{\mathrm{D}}$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E(x,a)\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash E(a,b)$; corresponds to new points in an existing equivalence class
- a single $\sim_{\mathrm{D}}$-class $[\![p_g]\!]$, where $p_g := \{\neg E(x,a) \mid a \in \mathfrak{U}\}$; corresponds to new equivalence classes.

The product adds new points/new classes. So, if $\mathfrak{U}$ has κ equivalence classes,

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

More Examples: Cross-cutting Equivalence Relations

$T_n :=$ n generic equivalence relations E_i; intersection of classes of different E_i always infinite. Here $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is generated by:

- a single $\sim_{\mathrm{D}}$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E_i(x,a) \mid i < n\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash \bigwedge_{i<n} E_i(a,b)$; corresponds to new points in E_i-relation with a for all i
- For each $i < n$, a class $[\![p_i]\!]$ saying x is in a new E_i class, but in existing E_j-classes for $j \neq i$ (does not matter which)

So

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \prod_{i<n} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Why $\prod$ instead of $\bigoplus$? If we allow, say, $\aleph_0$ equivalence relations, then

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \prod_{i<\aleph_0}^{\mathrm{bdd}} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Back

Other Notions

One can define a finer equivalence relation:

Definition

$p \equiv_{\mathrm{D}} q$ is defined as $p \sim_{\mathrm{D}} q$, but by asking the same r to work in both directions: $p \cup r \vdash q$ and $q \cup r \vdash p$.

Another notion classically studied is:

Definition

$p \geq_{\mathrm{RK}} q$ iff every model realising p realises q.

This behaves best in totally transcendental theories (because of prime models). It corresponds to $p(x) \cup \{\varphi(x, y)\} \vdash q(y)$.
But even there, modulo $\sim_{\mathrm{RK}}$ it is *not* true that every type decomposes as a product of $\geq_{\mathrm{RK}}$-minimal types (but in non-multidimensional totally transcendental theories every type decomposes as a product of strongly regular types).
A classical example where $\geq_{\mathrm{D}}$ differs from $\geq_{\mathrm{RK}}$: generic equivalence relation with a bijection s such that $\forall x\, E(x, s(x))$. Back

Hrushovski's Counterexample

Example (Hrushovski)

In DLO plus a dense-codense predicate P, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not commutative.

Proof idea.

Let $p(x) := \{P(x)\} \cup \{x > \mathfrak{U}\}$ and $q(y) := \{\neg P(x)\} \cup \{y > \mathfrak{U}\}$. Then p, q do not commute, even modulo $\equiv_{\mathrm{D}}$ (but they do modulo $\sim_{\mathrm{D}}$).
The predicate P forbids to "glue" variables. One will be "left behind": e.g. if $r \vdash x_0 < y_0 < y_1 < x_1$, knowing that $y_1 > \mathfrak{U}$ does not imply $x_0 > \mathfrak{U}$. □

In this case, for each cut C there are generators $[\![p_{C,P}]\!]$ and $[\![p_{C,\neg P}]\!]$, with relations

- $[\![p_{C,P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,\neg P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,P}]\!]$
- (same relations swapping P and $\neg P$)
- $[\![p_{C_0,-}]\!] \otimes [\![p_{C_1,-}]\!] = [\![p_{C_1,-}]\!] \otimes [\![p_{C_0,-}]\!]$ whenever $C_0 \neq C_1$.

Back

Stable Case

In a stable theory, $\leq_{\mathrm{D}}$, $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ can be expressed in terms of forking:

Definition

$a \triangleright_E b$ iff, for all c,

$$a \underset{E}{\downarrow} c \Longrightarrow b \underset{E}{\downarrow} c$$

$p \triangleright_E q$ (p *dominates* q over E) iff there are $a \models p$ and $b \models q$ such that $a \triangleright_E b$

$p \bowtie_E q$ (p and q are *domination equivalent*) iff $p \triangleright_E q \triangleright_E p$, i.e. there are

$$\underbrace{a}_{\models p} \triangleright_E \underbrace{b}_{\models q} \triangleright_E \underbrace{c}_{\models p}$$

$p \doteq_E q$ (p and q are *equidominant* over E) iff there are $a \models p$ and $b \models q$ such that $a \triangleright_E b \triangleright_E a$

These are well-behaved with non-forking extensions: we can drop $_E$.

Comparison

Proposition (T stable)

The previous definitions of $\leq_{\mathrm{D}} = \triangleleft$, $\sim_{\mathrm{D}} = \bowtie$ and $\equiv_{\mathrm{D}} = \doteq$.

Remark

The proof uses crucially stationarity of types over models.

In almost all examples we saw before, $\sim_{\mathrm{D}}$ coincides with $\equiv_{\mathrm{D}}$.

Exception: in DLO with a predicate, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is not commutative, while $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is (in fact, it is the same as in DLO).

Fact

Even in the stable case, $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ are generally different.

Classical Results

In the thin case (generalises superstable), this is classical:

Theorem (T thin)

$\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is a direct sum of copies of $\mathbb{N}$.
If T is moreover superstable, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Superstability (even just thinness) implies that $\equiv_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$ coincide.

The behaviour of $\geq_{\mathrm{D}}$ in general seems related to the existence of some kind of prime models (in the stable case, "prime a-models" are the way to go).
Also, some suitable generalisation of the Omitting Types Theorem would help.

Back

(Non-multi)Dimensionality

At least in the superstable case, independence of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ on $\mathfrak{U}$ already had a name:

Definition

T is *(non-multi)dimensional* iff no type is orthogonal to (every type that does not fork over) $\emptyset$.

If $\mathfrak{U}_0 \prec^+ \mathfrak{U}_1$ one has a map $\mathfrak{e}\colon \widetilde{\operatorname{Inv}}(\mathfrak{U}_0) \to \widetilde{\operatorname{Inv}}(\mathfrak{U}_1)$.

Proposition (T thin)

$\mathfrak{e}$ surjective $\iff$ T dimensional.

Question

Is this true under stability? It boils down to the image of $\mathfrak{e}$ being downward closed. I suspect this should follow from classical results. Back

Generically Stable Part

Proposition

$q \leq_{\mathrm{D}} p$ definable/finitely satisfiable/generically stable $\implies$ so is q.

As generically stable types commute with everything, in any theory the monoid generated by their classes is well-defined. (Warning: p generically stable $\not\Rightarrow$ $p \otimes p$ generically stable)

Hope

At least in special cases, get decompositions similar to $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \overbrace{\widetilde{\mathrm{Inv}}(k)}^{\text{g.s. part}} \times \widetilde{\mathrm{Inv}}(\Gamma)$.

Probably one should really work in T^{eq}:

Example

In $T = \mathsf{DLO}$+equivalence relation with (no finite classes and infinitely many) dense classes, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ grows when passing to T^{eq}, which has more generically stable types.

Question

How can the generically stable part look like?

Interaction with Weak Orthogonality

Definition

$p(x)$ is *weakly orthogonal* to $q(y)$ iff $p \cup q$ is complete.

Remark

Weakly orthogonal types commute.

Proposition

Weak orthogonality strongly negates domination: $q \perp^{\mathrm{w}} p_0 \geq_{\mathrm{D}} p_1 \implies q \perp^{\mathrm{w}} p_1$.
In particular if $q \perp^{\mathrm{w}} p \geq_{\mathrm{D}} q$ then q is realised.

Question

Under which conditions if $p \not\perp^{\mathrm{w}} q$ then they dominate a common nonzero class?

Known:

- Superstable (or *thin*) is enough. See here
- Fails in the Random Graph.

Action on Type Space

$f \in \mathrm{Aut}(\mathfrak{U})$ acts on $p \in S(\mathfrak{U})$ by changing parameters in formulas:

$$f \cdot p \coloneqq \{\varphi(x, f(d)) \mid \varphi(x, d) \in p\}$$

Consider this action restricted to $\mathrm{Aut}(\mathfrak{U}/A)$.

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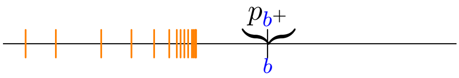
Example

$T = \mathsf{DLO}$, consider $p_{b^+}(x) \coloneqq \{x < d \mid d > b\} \cup \{x > d \mid d \leq b\}$



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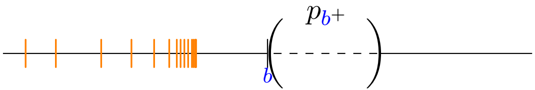

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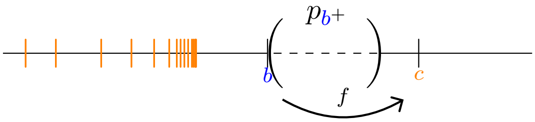

Action on Type Space

$f \in \mathrm{Aut}(\mathfrak{U})$ acts on $p \in S(\mathfrak{U})$ by changing parameters in formulas:

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Consider this action restricted to $\mathrm{Aut}(\mathfrak{U}/A)$.

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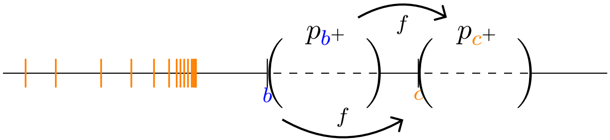
$T = \mathsf{DLO}$, consider $p_{b^+}(x) := \{x < d \mid d > b\} \cup \{x > d \mid d \leq b\}$ and let $f \in \mathrm{Aut}(\mathfrak{U}/A)$ be such that $f(b) = c$. Then $f \cdot p_{b^+} = p_{c^+}$.



Back

Invariant Extension

How to canonically extend an invariant type to bigger sets

Recall: $p \in S_x^{\mathrm{inv}}(\mathfrak{U}, A) \iff$ whether $p(x) \vdash \varphi(x; d)$ or not depends only on $\mathrm{tp}(d/A)$

Fact (B arbitrary, A small)

Every $p \in S_x^{\mathrm{inv}}(\mathfrak{U}, A)$ has a unique extension $(p \mid \mathfrak{U}B) \in S_x^{\mathrm{inv}}(\mathfrak{U}B, A)$

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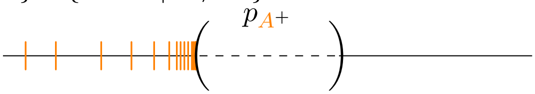

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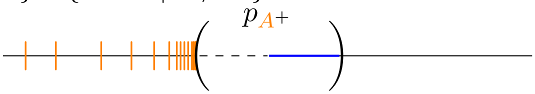

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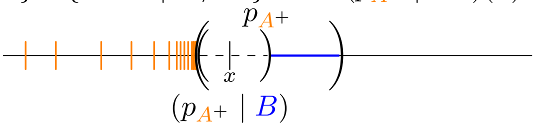
$p_{A^+}(x) \coloneqq \{x < d \mid d > A\} \cup \{x > d \mid d \not> A\}$ " = " $(p_{A^+} \mid \mathfrak{U}B)(x)$ (now $d \in \mathfrak{U}B$)



Product of Invariant Types

Definition (p invariant)

$$\varphi(x, y; d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x; b, d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

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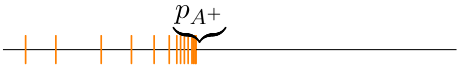
Example

$$(p_{A^+}(x) \coloneqq \{x < d \mid d > A\} \cup \{x > d \mid d \not> A\}) \qquad p_{A^+}(x) \otimes p_{A^+}(y)$$



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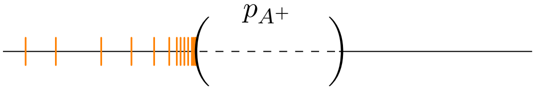

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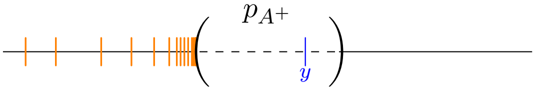

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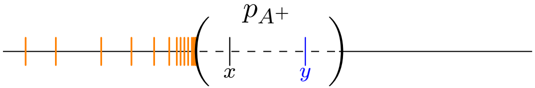
$(p_{A^+}(x) \coloneqq \{x < d \mid d > A\} \cup \{x > d \mid d \not> A\})$ $\qquad p_{A^+}(x) \otimes p_{A^+}(y) \vdash x < y$



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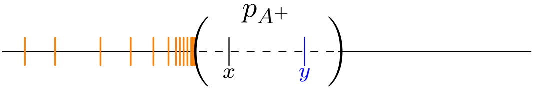
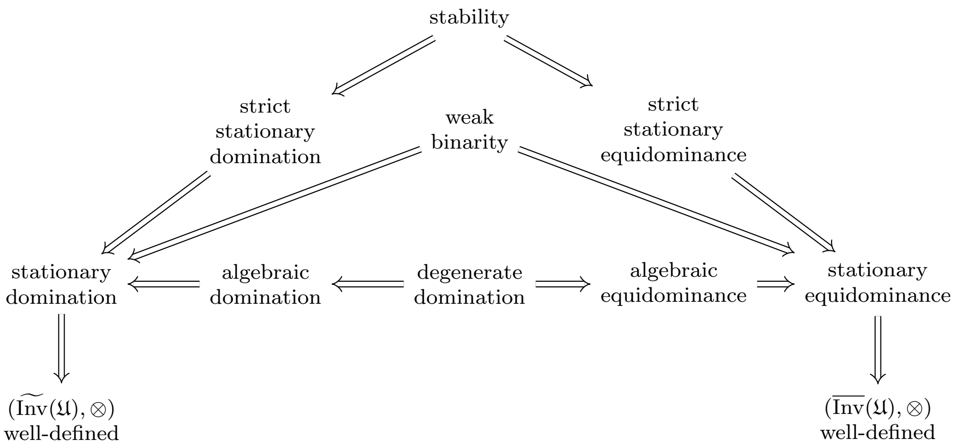


Fact

$\otimes$ is associative. It is commutative if and only if T is stable.

Map of Sufficient Conditions



Back

Sufficient Conditions

Proposition

$q_0 \geq_{\mathrm{D}} q_1 \implies p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ is implied by any of the following:

- q_1 algebraic over q_0: every $c \vDash q_1$ is algebraic over some $b \vDash q_0$. E.g. $q_1 = f_* q_0$ for some definable function f. Reason: $\{c \mid (b, c) \vDash r\}$ does not grow with $\mathfrak{U}$.
- Or even *weakly binary*: $\operatorname{tp}(a/\mathfrak{U}) \cup \operatorname{tp}(b/\mathfrak{U}) \cup \operatorname{tp}(ab/M) \vDash \operatorname{tp}(ab/\mathfrak{U})$: few questions about $a \vDash p$ and $c \vDash q_1$.
- T is stable.

Any condition in the Proposition implies that if there is some $r \in S_{yz}(M)$ witnessing $q_0(y) \geq_{\mathrm{D}} q_1(z)$, then there is one such that, in addition, if

- $b, c \in \mathfrak{U}_1 \ {}^+\!\succ \mathfrak{U}$ are such that $(b, c) \vDash q_0 \cup r$,
- $p \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $a \vDash p(x) \mid \mathfrak{U}_1$,
- $r[p] := \operatorname{tp}_{xyz}(abc/M) \cup \{x = w\}$.

then $p \otimes q_0 \cup r[p] \vdash p \otimes q_1$. We call this *stationary domination*.

Dense Meet-trees and Expansions

Theorem (M.)

Let $L_0 = \{<, \sqcap\}$ and $L = L_0 \cup \{R_j^{(2)}, P_{j'}^{(1)} \mid j \in J, j' \in J'\}$. Let T be a completion in L of the theory of dense meet-trees with quantifier elimination and such that:

1. $R_j(x, y) \to x \parallel y$.
2. If $x \parallel y$, $x \sqcap x' > x \sqcap y$, and $y \sqcap y' > x \sqcap y$, then $R_j(x, y) \leftrightarrow R_j(x', y')$.

Then T is weakly binary.

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E.g.: $L = L_0 \cup \{R\}$, where $R(x, y)$ induces a Random Graph on each set of open cones above a point.

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2. If $x \parallel y$, $x \sqcap x' > x \sqcap y$, and $y \sqcap y' > x \sqcap y$, then $R_j(x, y) \leftrightarrow R_j(x', y')$.

Then T is weakly binary.

E.g.: $L = L_0 \cup \{R\}$, where $R(x, y)$ induces a Random Graph on each set of open cones above a point.

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In T as above with no unary predicates there is $X = X(\mathfrak{U})$ such that $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{\mathrm{fin}}(X) \times \bigoplus_{g \in \mathfrak{U}} \widetilde{\mathrm{Inv}}(O_g)$, where O_g is the structure induced on the open cones above g.

Dense Meet-trees and Expansions

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For pure dense meet-trees $\forall g\, O_g \cong \mathbb{N}$. Back

Counterexamples

Theorem (M.)

There is a ternary, ω-categorical, supersimple theory of SU-rank 2 with degenerate algebraic closure in which neither $\sim_{\mathrm{D}}$ nor $\equiv_{\mathrm{D}}$ are congruences with respect to $\otimes$. In the same theory, $\geq_{\mathrm{D}}$ and domination in the sense of forking differ. More

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Moreover, examples of theories where

1. $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is not commutative (see here),
2. $p \perp^{\mathrm{w}} q$ but $p \otimes p \not\perp^{\mathrm{w}} q$,
3. if $p_0 \geq_{\mathrm{D}} q$ and $p_1 \geq_{\mathrm{D}} q$ then q is realised, but $p \not\perp^{\mathrm{w}} q$ (even under NIP),
4. Being *generically* NIP is not preserved by $\geq_{\mathrm{D}}$.
5. $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \neq \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$,
6. $\geq_{\mathrm{D}}$ is different from $\mathrm{F}^{\mathrm{s}}_{\kappa(\mathfrak{U})}$-isolation à la Shelah.

A Counterexample

(with SOP and IP_2)

Idea: DLO

Back

A Counterexample

(with SOP and IP_2)

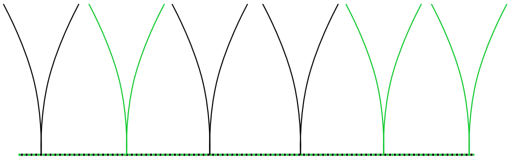
Idea: 2-coloured DLO

Back

A Counterexample

(with SOP and IP_2)

Idea: fiber over a 2-coloured DLO



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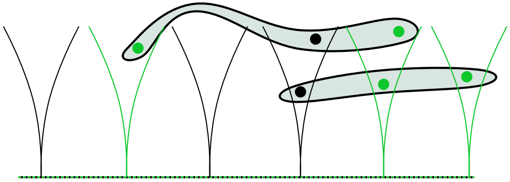
Idea: fiber over a 2-coloured DLO; put a generic tripartite 3-hypergraph on triples of fibers:



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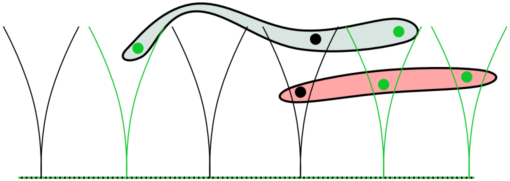

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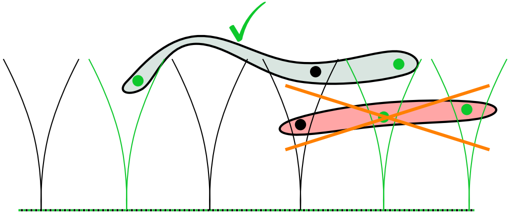

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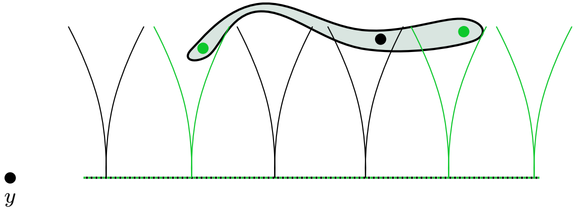

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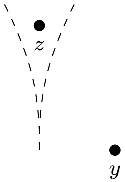

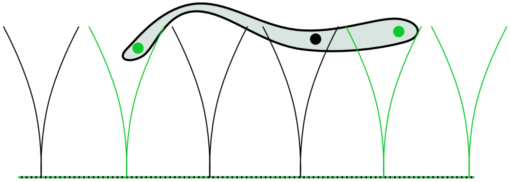

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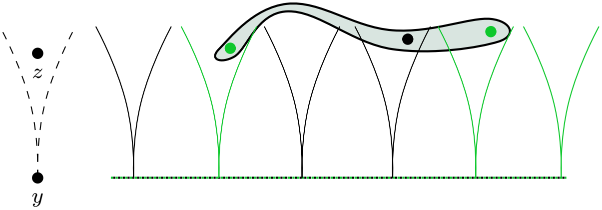

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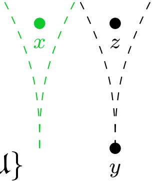

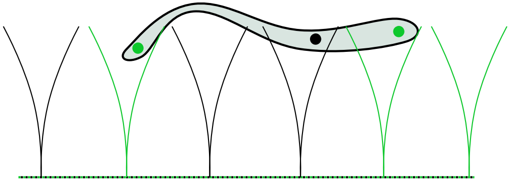

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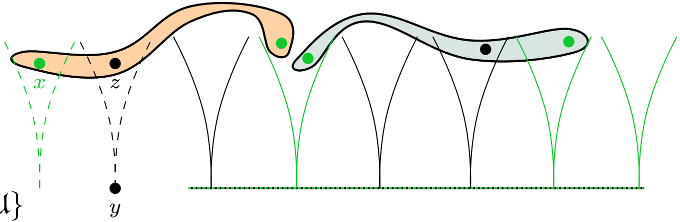

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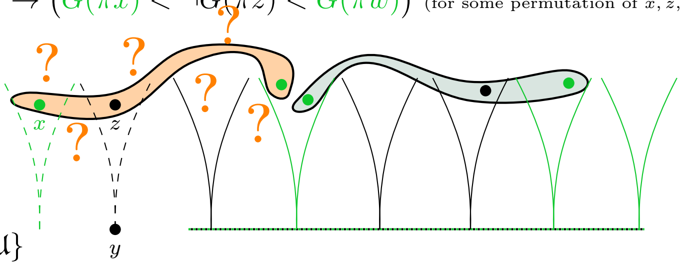

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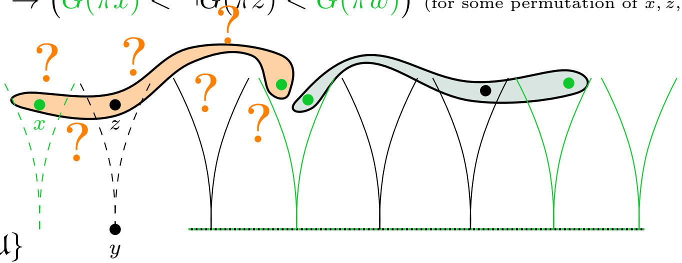

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Supersimple version here. Also works for a number of variations of $\sim_{\mathrm{D}}$.

Back

Another Counterexample

Ternary, supersimple, ω-categorical, can be tweaked to have degenerate algebraic closure

Replacing the densely coloured DLO with a random graph R_2 yields a supersimple counterexample of SU-rank 2; forking is $a \underset{C}{\downarrow} b \iff (a \cap b \subseteq C) \wedge (\pi a \cap \pi b \subseteq \pi C)$.

$$R_3(x_0, x_1, x_2) \to \bigvee_{\sigma \in S_3} \left(R_2(\pi x_{\sigma 0}, \pi x_{\sigma 1}) \wedge R_2(\pi x_{\sigma 0}, \pi x_{\sigma 2}) \wedge \neg R_2(\pi x_{\sigma 1}, \pi x_{\sigma 2})\right)$$

(exactly two edges between $\pi x_0, \pi x_1, \pi x_2$)

$$q_0(y) := \{\neg R_2(y, a) \mid a \in \mathfrak{U}\}$$
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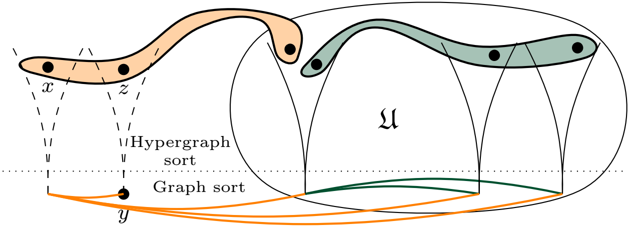

$q_0 \cup r \vdash q_1$: no hyperedges to decide. Same problem: $p \otimes q_0(x, y) \not\geq_{\mathrm{D}} p \otimes q_1(t, z)$.

◂ Back

Strongly Minimal Theories

$(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes)$ well-defined by stability

Example

If T is strongly minimal, $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathbb{N}, +, \leq)$.

(for T stable, $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \mathbb{N} \Leftrightarrow T$ is *unidimensional*, e.g. countable and $\aleph_1$-categorical, or $\operatorname{Th}(\mathbb{Z}, +)$)

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In this case, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is basically "counting the dimension". E.g.: in ACF_0 we have $p(x_1, \ldots, x_n) \sim_{\mathrm{D}} q(y_1, \ldots, y_m) \iff \operatorname{tr}\deg(x/\mathfrak{U}) = \operatorname{tr}\deg(y/\mathfrak{U})$.
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Taking products corresponds to adding dimensions: if $(a, b) \vDash p \otimes q$, then $\dim(a/\mathfrak{U}b) = \dim(a/\mathfrak{U})$, and in strongly minimal theories

$$\dim(ab/\mathfrak{U}) = \dim(b/\mathfrak{U}) + \dim(a/\mathfrak{U}b)$$

More generally, in superstable theories (or even *thin* theories), by classical results $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \bigoplus_{i<\lambda}(\mathbb{N}, +, \leq)$, for some λ.

◂ Back

Dense Linear Orders

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ **well-defined by binarity**

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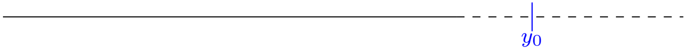

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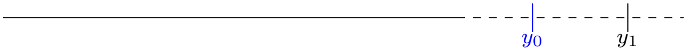

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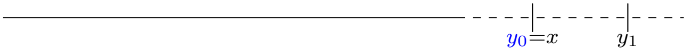

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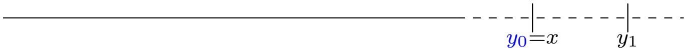

$\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is the free idempotent commutative monoid generated by the invariant cuts:

$$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathscr{P}_{\mathrm{fin}}(\{\text{invariant cuts}\}), \cup, \subseteq)$$

Back

Random Graph

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ **well-defined by binarity**

In the Random Graph, $\sim_{\mathrm{D}}$ is degenerate and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ resembles closely $(S^{\mathrm{inv}}_{<\omega}(\mathfrak{U}), \otimes)$. For instance, it is not commutative:

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Example (All types ∅-invariant)

These types do not commute, even modulo $\sim_{\text{D}}$:

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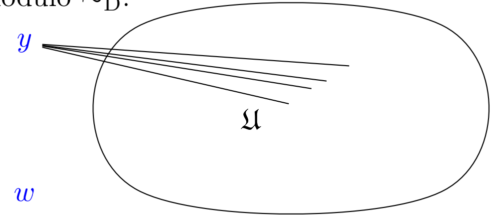

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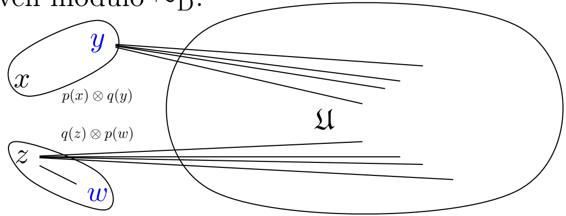

Proof Idea.

As $p_x \otimes q_y \vdash \neg E(x,y)$ and $q_z \otimes p_w \vdash E(z,w)$, gluing cannot work. But in the random graph domination is degenerate and there is not much more one can do. □

More examples here · Back to examples

Weak orthogonality

Definition

$p(x)$ is *weakly orthogonal* to $q(y)$ iff $p(x) \cup q(y)$ is complete. Write $p \perp^{\mathrm{w}} q$.

Why "weak"? Because in general it need not pass to invariant extensions.

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Example

In any o-minimal T with $0 \in L$, these two are $\emptyset$-invariant 1-types:

$$p(x) := \operatorname{tp}(+\infty/\mathfrak{U}) := \{x > d \mid \in \mathfrak{U}\} \qquad q(y) := \operatorname{tp}(0^+/\mathfrak{U}) := \{0 < y < d \mid d \in \mathfrak{U}, d > 0\}$$

In DOAG, $p \perp^{\mathrm{w}} q$, but in RCF $p \not\perp^{\mathrm{w}} q$. Reason: "$\operatorname{dcl}(p) \cap q \neq \emptyset$": is $x \geq 1/y$?

Weak orthogonality

Definition

$p(x)$ is *weakly orthogonal* to $q(y)$ iff $p(x) \cup q(y)$ is complete. Write $p \perp^{\mathrm{w}} q$.

Why "weak"? Because in general it need not pass to invariant extensions.

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- Since $q \perp^{\mathrm{w}} p_0 \geq_{\mathrm{D}} p_1 \implies q \perp^{\mathrm{w}} p_1$, we may expand to $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}}, \perp^{\mathrm{w}})$.
- In particular if $q \perp^{\mathrm{w}} p \geq_{\mathrm{D}} q$ then q is realised.

Reduction to generation by 1-types

Theorem (M., T o-minimal)

If every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ is $\sim_{\mathrm{D}}$ to a product of 1-types, then $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined, and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}, \perp^{\mathrm{w}}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup, \supseteq, D)$

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Hence, given an o-minimal T, to conclude the study of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ it is enough to:

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Sufficient condition for 1: if c is a $\mathfrak{U}$-independent tuple, then

$$\bigcup_{f \in \mathcal{F}_T^{|x|,1}} \mathrm{tp}_{w_f}(f(c)/\mathfrak{U}) \cup \left\{ w_f = f(x) \;\middle|\; f \in \mathcal{F}_T^{|x|,1} \right\} \vdash \mathrm{tp}_x(c/\mathfrak{U}) \qquad (\dagger)$$

$\mathcal{F}_T^{|x|,1} :=$ set of $\emptyset$-definable functions of T with domain $\mathfrak{U}^{|x|}$ and codomain $\mathfrak{U}^1$.

Applications

Theorem ([HHM08])

In DOAG, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{\mathrm{fin}}(\{\text{invariant convex subgroups}\})$.

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Corollary

In RCVF, by [EHM19] $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(k) \times \widetilde{\mathrm{Inv}}(\Gamma)$. So $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{\mathrm{fin}}(X)$, where

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The Idempotency Lemma

Lemma (M., Idempotency Lemma, T o-minimal, $M \prec^+ N \prec^+ \mathfrak{U}$)

If $b \models p \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$ then $p(\mathrm{dcl}(Nb))$ is cofinal and coinitial in $p(\mathrm{dcl}(\mathfrak{U}b))$.

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If $b > \mathfrak{U} \models \mathsf{RCF}$, then $\{b, b^2, b^3, \dots\}$ is cofinal in $\mathrm{dcl}(\mathfrak{U}b)$.

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If T is o-minimal and $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ then $p(y) \otimes p(z) \sim_{\mathrm{D}} p(x)$.

Proof.

A small type is enough to say e.g. "$x = z$ and $y > p(\mathrm{dcl}(Nz))$". □

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Proof idea for the Lemma: use the Monotonicity Theorem to show that, otherwise, there is $d \in \mathfrak{U}$ such that $b, f(b, d), f(f(b, d), d), \dots$ is an infinite N-independent sequence. By Steinitz exchange this is nonsense: d depends on a long enough piece of the sequence. N is used to "copy" parameters of definable functions.

Back

A technical proposition

Let T be o-minimal. Let $p(x) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, let $c \vDash p$ be $\mathfrak{U}$-independent.

1. There is a tuple $b \in \operatorname{dcl}(\mathfrak{U}c)$ of maximal length among those satisfying a product of nonrealised invariant 1-types.
2. Let b be as above, and let $q \coloneqq \operatorname{tp}(b/\mathfrak{U}) = q_0 \otimes \ldots \otimes q_n$, where $q_i \in S_1^{\mathrm{inv}}(\mathfrak{U})$. Up to replacing q_i with $\tilde{q}_i \sim_{\mathrm{D}} q_i$, we may assume that either $q_i \perp^{\mathrm{w}} q_j$ or $q_i = q_j$.

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4. Let b, q be as above, $r \coloneqq \operatorname{tp}_{xy}(cb/N_1)$, and $\mathcal{F}^{m,1}_{T(M)}$ the set of $T(M)$-definable functions with domain $\mathfrak{U}^m$ and codomain $\mathfrak{U}^1$. Then $p(x) \cup r(x,y) \vdash q(y)$ and

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Using this and some valuation theory, in RCF, it can be shown that $q \cup r \vdash p$.
"Almost converse": $(\exists M' \; q \cup \mathrm{tp}(cb/M') \vdash p) \Rightarrow (\exists M' \; \pi_{M'} \vdash p)$.

◂ Back to o-minimal ◂ Back to questions

Distality and idempotency

Recall the following definition of *distal type*:

Definition

$p \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ is *distal over A* iff whenever $I \vDash p^{(\omega)} \upharpoonright Ab$ we have $(p \upharpoonright AI) \perp^{\mathrm{w}} \mathrm{tp}(b/AI)$.

By taking $b = \mathfrak{U}$ and some syntactical manipulations, this implies that $p^{(\omega)} \sim_{\mathrm{D}} p^{(\omega+1)}$ (witnessed over A).

Question

Let p be distal (and T dp-minimal?). Is it true that we can replace I with a single realisation of p, possibly after changing A?

A positive answer would imply that $p \sim_{\mathrm{D}} p^{(2)}$; recall that the latter holds for 1-types in o-minimal theories. Back

Properties preserved by domination

Domination equivalence is quite coarse; for instance it does not preserve Morley rank (generic equivalence relation), nor dp-rank (DLO) (but in stable T it preserves weight).

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- Finite satisfiability (in *some* small set, not necessarily the same as p)
- Generic stability (over *some* small set, not necessarily the same as p)
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Generic stability is particularly interesting:

- It is possible to have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \neq \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ (more g.s. types, e.g. DLO+dense eq. rel.).
- Strongly regular g.s. types are $\leq_{\mathrm{D}}$-minimal (among the nonrealised ones).
- $(\widetilde{\mathrm{Inv}}^{\mathrm{gs}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}})$ makes sense in any theory (can be trivial).

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Let D be a (fully) stably embedded definable set. Then $\widetilde{\mathrm{Inv}}(D(\mathfrak{U}))$ embeds in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. The embedding preserves $\geq_{\mathrm{D}}$, $\perp^{\mathrm{w}}$, $\not\perp^{\mathrm{w}}$ and, when it makes sense, $\otimes$.

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Some further facts: (a.k.a.: a shameless ad for my thesis)

- $\geq_{\mathrm{D}}$ is *not* $\mathrm{F}^{\mathrm{s}}_{\kappa}$-isolation. It is the semi-isolation version of that.
- $\geq_{\mathrm{D}}$ can be be viewed as being induced by a "partial quaternary independence relation". This is uncharted territory, that I know of.
- If $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ does not depend on $\mathfrak{U}$ then T is NIP. The converse is far from true, conjecturally this should be equivalent to T being stable nonmultidimensional.
- One can define a category $\mathrm{Inv}(\mathfrak{U})$ where morphisms are witnesses of domination. If T is stable, $\otimes$ makes it a strict symmetric monoidal category.