Automorphisms of ordered abelian groups, the Amalgamation Property, and dependent positive theories

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joint work with Jan Dobrowolski

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In this talk

Aut, pec, NIP

- Generic automorphisms
- Dependent positive theories

oags, AP

- Automorphisms of oags
- Proof ideas and byproducts

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Main Result (Dobrowolski–M.)

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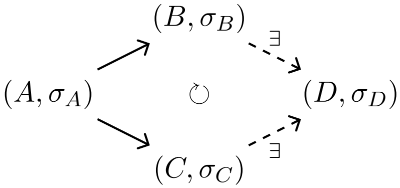

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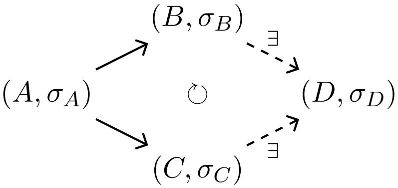

Main Corollary

Their positive theory is NIP.
(more precisely, their *h-inductive* theory)

(using a suitable notion of NIP)

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- The h-inductive theory of $M = (\omega, \leq, 0, 1, 2, \dots)$ has 2 pec models: M and $\omega + 1$.
- There are theories with only *bounded* pec models which are not finite. Types?
- Why bother? Hyperimaginaries, neostability... More

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T h-inductive with JCP. A positive $\varphi(x,y)$ has IP iff there are a positive $\psi(x,y)$, some $M \vDash T$, and tuples $(a_i)_{i\in\omega}$, $(b_W)_{W\in\mathscr{P}(\omega)}$ in M such that

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Example (de Aldama Sánchez/Dobrowolski–M.)

The positive theory of DLO's with a G-action by automorphisms is NIP. Proof

Pec OAGAs: basic properties

$\mathbb{Q}-\mathsf{OVSA} \coloneqq (L_{\text{oag}} \cup \{\sigma, \sigma^{-1}, q \cdot - \mid q \in \mathbb{Q}\})$–theory of $\mathbb{Q}-\mathsf{OVS}$'s with auto.

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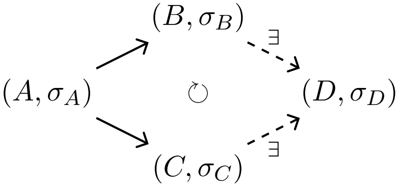

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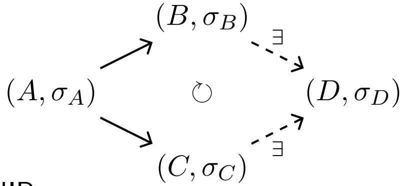

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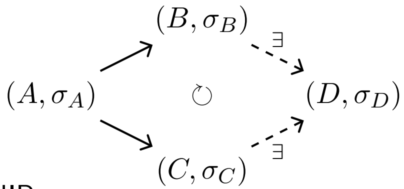

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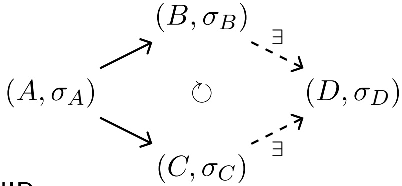

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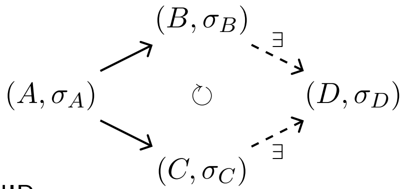

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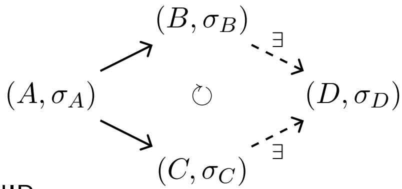

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- Q.f. formulas are easily shown to be NIP. □

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So does every $\min_{j\leq k} f_j(x)$, with $f_j(x) = \sum \lambda_i \sigma^i(x) + d_j$.

These turned out to be trickier than expected. Why?

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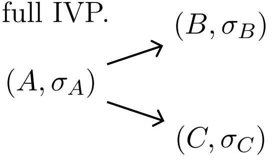

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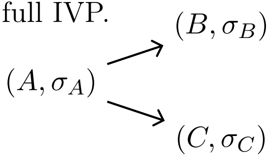

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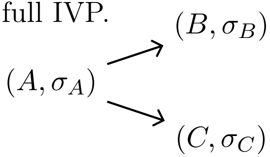

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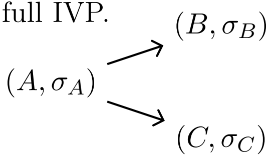
  Idea: A closed in pec B implies $A_{>0}$ coinitial in $B_{>0}$ and belonging to an open cell can be written as $\min_i f_i(x) > 0$.
- This + compactness $\implies$ amalgamate σ-transcendental $b \in B$.



Thanks for listening!

Preprint: `arxiv.org/abs/2209.03944` or scan the QR code:

Existentially closed structures

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K a class of L-structures. Recall:

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$M \in K$ is *existentially closed* (ec) in K iff, for every existential formula $\exists y\, \varphi(x, y)$ and $a \in M$, if there is an embedding $M \to N \in K$ with $N \models \exists y\, \varphi(a, y)$, then $M \models \exists y\, \varphi(a, y)$.

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T_1, T companions := each $M \vDash T$ embeds in a $M_1 \vDash T_1$ and conversely; model companion := model-complete companion.

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Therefore DLOA does not exists: in an ω-saturated M, on one hand

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but if $\sigma(a) \neq a$, by cptns+saturation there is $b \in M$ satisfying RHS but not LHS.

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Theorem (Pal)

Let $L = L_{\text{oag}} \cup \{\sigma\}$, and let MODAG be the theory of difference oags together with, for every $L \in \mathbb{Z}[\sigma]$, the axiom

$$(\forall x > 0\ L(x) > 0) \vee (\forall x > 0\ L(x) = 0) \vee (\forall x > 0\ L(x) < 0)$$

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This approach has been useful in the context of valued difference fields (e.g. isometric, contractive); see Azgın–van den Dries, Chernikov–Hils, Scanlon,...

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2a. The first topology restricted to maximal types is not always compact. In that topology, they are generic points of irreducible components.

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One reason: can develop stability and generalisations on pec structures.

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- Ben Yaacov: simplicity in general setting (and more).
- Haykazyan–Kirby: ec exponential fields are TP_2 and NSOP_1.
- d'Elbée–Kaplan–Neuhauser: ec fields with an R-submodule are TP_2 and NSOP_1.
- Dobrowolski–Kamsma: development of NSOP_1.

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- These go through for NIP: phrasing in terms of alternation on an indiscernible sequence, being NIP is preserved by $\vee, \wedge, ^{\mathrm{op}}$, can be checked in one variable, some version of Borel definability…

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- Assumptions that are sometimes required/useful: being *semi-Hausdorff* (equality of types is type-definable), being *thick* (indiscernibility is type-definable).

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From this, one deduces that the trimmed sequence is b-indiscernible.

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- In general, IVP functions are not closed under sum (example just above!).

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