Involutions on Zilber fields

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Outline

Exponential fields

Definition

An *exponential field*, or *E-field*, is a structure

$$(K, 0, 1, +, \cdot, E)$$

where $(K, 0, 1, +, \cdot)$ is a field, and the following equation holds

$$E(x + y) = E(x) \cdot E(y).$$

- $\mathbb{R}_{\exp}$ (*o*-minimal, model complete, decidable if Schanuel's Conjecture is true).
- $\mathbb{C}_{\exp}$ (undecidable, interprets Peano's Arithmetic).

Schanuel's Conjecture

A special role in the model-theoretic study is played by a long standing conjecture in transcendental number theory.

Conjecture (Schanuel)

For any $z_1, \ldots, z_n \in \mathbb{C}$ *linearly independent over* $\mathbb{Q}$*,*

$$\mathrm{tr.deg.}_{\mathbb{Q}}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n.$$

If Schanuel's Conjecture holds at least for $z_1, \ldots, z_n \in \mathbb{R}$, then the first order theory of $\mathbb{R}_{\exp}$ is decidable [1].

On the other hand, $\mathbb{C}_{\exp}$ defines $(\mathbb{Z}, +, \cdot)$, hence it is always undecidable. First order theory may not be sufficient.

Conjectural, but categorical axioms for $\mathbb{C}_{\exp}$ in $\mathcal{L}_{\omega_1,\omega}(Q)$

Zilber looked for (uncountably) *categorical* axioms in $\mathcal{L}_{\omega_1,\omega}(Q)$.

Properties of $\mathbb{C}_{\exp}$:

(ACF$_0$) $\mathbb{C}$ is an algebraically closed field of characteristic 0.

(E) exp is a homomorphism $\exp : (\mathbb{C}, +) \to (\mathbb{C}^\times, \cdot)$.

(LOG) exp is surjective.

(STD) $\ker(\exp) = 2\pi i\mathbb{Z}$ (needs $\mathcal{L}_{\omega_1,\omega}$).

Conjectures on $\mathbb{C}_{\exp}$:

(SP) $\mathrm{tr.deg.}_{\mathbb{Q}}(\bar{z}, \exp(\bar{z})) \geq \mathrm{lin.d.}_{\mathbb{Q}}(\bar{z})$ (Schanuel's Property).

(SEC) every "rotund" variety contains a generic solution $(\bar{z}, \exp(\bar{z}))$.

Another property of $\mathbb{C}_{\exp}$:

(CCP) every "rotund" variety of "depth 0" contains at most countably many generic solutions $(\bar{z}, \exp(\bar{z}))$ (needs Q).

Zilber's categoricity result

Theorem (Zilber, 2005 [2])

The axioms are uncountably categorical.

We call "Zilber field", or $\mathbb{B}_E$, the unique model of cardinality $2^{\aleph_0}$.

The conjecture becomes the following.

Conjecture (Zilber, 2005 [2])

$\mathbb{C}_{\exp}$ *is isomorphic to* $\mathbb{B}_E$.

Automorphisms

Definition

An *involution* of K_E is an automorphism $\sigma : K_E \to K_E$ s.t. $\sigma^2 = \mathrm{Id}$.

$\mathbb{C}_{\exp}$ has one involution, complex conjugation.

- It is the unique known automorphism of $\mathbb{C}_{\exp}$.
- exp is continuous in the induced topology.
- exp is the unique continuous exponential (up to constants).

If $\mathbb{B}_E \cong \mathbb{C}_{\exp}$, $\mathbb{B}_E$ would have an involution as well.

Theorem (M., 2011)

1. *There is an involution σ on $\mathbb{B}_E$ (such that $\mathbb{B}^\sigma \cong \mathbb{R}$).*
2. *There are $2^{2^{\aleph_0}}$ non-conjugate involutions on $\mathbb{B}_E$.*

Problems in our proof

Unfortunately, what we found is different from complex conjugation.

- the solutions $(\bar{z}, E(\bar{z}))$ of rotund varieties are *dense*;
- hence, E is not continuous;
- moreover, the restriction $E_{\restriction \mathbb{B}^\sigma}$ is not increasing.

This is also in contrast with the fact that on $\mathbb{C}_{\mathrm{exp}}$ the solutions $(\bar{z}, \exp(\bar{z}))$ of rotund varieties of "depth 0" are isolated.

Remark. We are not refuting Zilber's conjecture: *other* involutions can still be such that E is continuous.

The construction

We start from K and $\sigma : K \to K$, and we build E.
For instance, $K = \mathbb{C}$ and σ the complex conjugation.

For any E, we know that $\sigma \circ E = E \circ \sigma$ if and only if

1. $E(\mathbb{R}) \subset \mathbb{R}_{>0}$;
2. $E(i\mathbb{R}) \subset \mathbb{S}^1(\mathbb{C})$.

Hence, we build E on $\mathbb{C}$ by 'back-and-forth', while respecting the restrictions 1, 2. We can easily obtain an E satisfying all of the axioms except (CCP).

In order to build E with (CCP), we add dense sets of solutions to rotund varieties (destroying continuity).

Summary

Zilber produced a sentence ψ in $\mathcal{L}_{\omega_1,\omega}(Q)$ which is uncountably categorical, and conjecturally an axiomatization of $\mathbb{C}_{\mathrm{exp}}$. Its unique model in cardinality $2^{\aleph_0}$ is called $\mathbb{B}_E$.

Looking for an analogue of complex conjugation, we found that

- There are $2^{2^{\aleph_0}}$ involutions on $\mathbb{B}_E$.
- One of them is such that $\mathbb{B}^\sigma \cong \mathbb{R}$.
- However, E is not continuous w.r.t. them.

Thanks for your attention!

Bibliography I

Angus Macintyre and A. J. Wilkie.
On the decidability of the real exponential field.
In *Kreiseliana*, pages 441–467. A K Peters, Wellesley, MA, 1996.

Boris Zilber.
Pseudo-exponentiation on algebraically closed fields of characteristic zero.
Annals of Pure and Applied Logic, 132(1):67–95, 2005.
doi:10.1016/j.apal.2004.07.001.