Chapter 1

THE EQUATION OF MOTION

1.1 The asteroid multi-body problem

Given the point masses[1] m_i located at positions $\mathbf{x}_i$, with velocities $\dot{\mathbf{x}}_i$, for $i = 0, 1, \ldots, S$, the accelerations $\ddot{\mathbf{x}}_i$ due to the gravitational attraction between all bodies are given by the **equation of motion**

$$m_i\, \ddot{\mathbf{x}}_i = \sum_{j \neq i, j=0}^{S} \frac{G\, m_j\, m_i}{x_{ij}^3} \, (\mathbf{x}_j - \mathbf{x}_i) \tag{1.1}$$

where $x_{ij} = |\mathbf{x}_j - \mathbf{x}_i|$. Let $\mu_i = G\, m_i$ be the gravitational masses, which are the only ones considered in Celestial Mechanics[2], then we can solve for the accelerations:

$$\ddot{\mathbf{x}}_i = \sum_{j \neq i, j=0}^{S} \frac{\mu_j}{r_{ij}^3} \, (\mathbf{x}_j - \mathbf{x}_i). \tag{1.2}$$

In our solar system, if the index 0 stands for the Sun, the first N positive indexes are for the major planets, the last M for asteroids and other minor bodies (with $S = N+M$), and we have that $\mu_0 >> \mu_j \; (j = 1, N) >> \mu_k \; (k = N+1, S)$. In fact $\mu_j < 10^{-3}\, \mu_0$ and $\mu_k < 5\times 10^{-10}\, \mu_0$, thus the ratios μ_i/μ_0 are **small parameters**. As a consequence, it is convenient to use **heliocentric coordinates**

$$\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_0 \quad , \quad r_i = |\mathbf{r}_i| \quad , \quad \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = \mathbf{x}_j - \mathbf{x}_i \quad , \quad r_{ij} = |\mathbf{r}_{ij}| = x_{ij}$$

then, assuming the masses of the asteroids are negligible to the point that they do not count as sources of gravitational attraction, the equation of motion (1.2) becomes

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i, j=0}^{N} \frac{\mu_j}{r_{ij}^3}\, \mathbf{r}_{ij} - \sum_{k=1}^{N} \frac{\mu_k}{r_k^3}\, \mathbf{r}_k \ .$$

[1]Point masses are approximations for the gravity field of an extended body, represented by the attraction of the total mass of the body concentrated in its center of mass. It can be shown that this approximation is good enough for long range perturbations such as the ones acting on asteroids, see Section ??

[2]With the exception of tests for the violation of the equivalence principle (cite iau261).

for $i = 1, S$. By isolating in the right hand side the largest term, the one with μ_0, we get

$$\ddot{\mathbf{r}}_i = -\frac{\mu_0}{r_i^3}\,\mathbf{r}_i + \sum_{j\neq i, j=1}^{N} \frac{\mu_j}{r_{ij}^3}\,\mathbf{r}_{ij} - \sum_{k=1}^{N} \frac{\mu_k}{r_k^3}\,\mathbf{r}_k \ . \tag{1.3}$$

The three terms are the **unperturbed 2-body acceleration**, the **direct perturbation** and the **indirect perturbation**. If the body i is a planet, the last sum contains also a $k = i$ term; it accounts for the fact that a heliocentric reference system is not an inertial one, that is the Sun is accelerated by all the planets (but not by the asteroids, that is the corresponding acceleration is neglected).

1.2 Equation of motion for the restricted problem

The above equation refers to the hypothesis that the orbits of N planets and M asteroids have to be computed at once, which may indeed be the case in a numerical integration. However, since the orbit of each asteroid does not depend at all upon where the other asteroids are, we can develop the theory of a single asteroid perturbed by the planets (hence the title of this book :-). If we are interested in computing only the orbit of the asteroid $\mathbf{r} = \mathbf{r}_{N+1}$, then $S = N + 1$, and we can indeed (because of the level of accuracy required) ignore the attraction from the other asteroids[3], the **restricted problem** has equation of motion

$$\ddot{\mathbf{r}} = -\frac{\mu_0}{r^3}\,\mathbf{r} + \sum_{i=1}^{N} \frac{\mu_i}{|\mathbf{r}_i - \mathbf{r}|^3}\,(\mathbf{r}_i - \mathbf{r}) - \sum_{i=1}^{N} \frac{\mu_i}{r_i^3}\,\mathbf{r}_i \ , \tag{1.4}$$

where $r = |\mathbf{r}|$: the direct and the indirect perturbations have sums of terms with the same indices, one for each planet.

The restricted problem is a good approximation because the asteroid mass is small, however by removing the terms with μ_{N+1} in the equations of motion for the planets, the action-reaction law by Newton is violated. In this way the asteroid does not contribute to the 10 classical integrals of motion (energy, angular momentum, linear momentum and center of mass, see Section 1.3) and the equation (1.4) has no exact integral.

The equation (1.4) of the restricted problem can be derived from the Lagrange formalism: let the **kinetic energy** T, the **gravitational potential** U and the **Lagrange function** L be

$$T = \frac{1}{2}|\dot{\mathbf{r}}|^2 \quad , \quad U_0 = \frac{\mu_0}{r} \tag{1.5}$$

$$U_{DIR} = \sum_{j=1}^{N} \frac{\mu_j}{|\mathbf{r}_j - \mathbf{r}|^3} \quad , \quad U_{IND} = -\sum_{j=1}^{N} \frac{\mu_j}{r_j}\,\mathbf{r}_j \cdot \mathbf{r} \tag{1.6}$$

$$L(\mathbf{r}, \dot{\mathbf{r}}) = T + U = T + U_0 + U_{DIR} + U_{IND} \tag{1.7}$$

Then the restricted Lagrange equations are defined by the conjugate momentum vector $\mathbf{p}$

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{\partial T}{\partial \dot{\mathbf{r}}} = \dot{\mathbf{r}} \quad , \quad \ddot{\mathbf{r}} = \dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}} \ . \tag{1.8}$$

[3]This assumption may not be applicable in some extreme accuracy computation, such as the ones about predictions of impacts of an asteroid with a planet.

The **Hamilton function** is defined by the **Legendre transform** (cite?)

$$H(\mathbf{p},\mathbf{r}) = \mathbf{p}\cdot\dot{\mathbf{r}} - L = \frac{1}{2}\,|\mathbf{p}|^2 - U$$

and can be decomposed into an unperturbed portion H_0 and a perturbation part $\epsilon\, H_1$ with small parameter $\epsilon \simeq Max_i(\mu_i/\mu_0)$

$$H = H_0 + \epsilon H_1 \quad , \quad H_0 = \frac{1}{2}\,|\mathbf{p}|^2 - U_0 \quad , \quad H_1 = -\frac{U_{DIR} + U_{IND}}{\epsilon}\ . \tag{1.9}$$

Thus equation (1.4), with its Lagrange and Hamilton equivalent, is the basic equation of motion we are discussing in this book. However, we need to assume that also the equation of motion for the planets, that is (1.3) for $i = 1, N$ has been solved and the solution is available as a function of time t. Since this is by no means a trivial assumption, we need first to discuss the orbits of the planets. Thus in the following of this section we will give a general discussion of the complete equation (1.3) for $i = 1, N$, that is for the planets, which is an **autonomous equation**, that is it does not contain explicitly the time. Once the solution of the planetary motions is substituted in (1.4), the equation is not autonomous any more; if the two equations are considered together, they are autonomous.

1.3 First integrals for the planetary problem

To discuss the motion of the planets we have to return to the equations of motion in an inertial reference system (1.1), restricted to $N + 1$ bodies (with $S = N$) and to find the corresponding Lagrange function $\mathcal{L}$, with kinetic energy $\mathcal{T}$ and gravitational potential $\mathcal{U}$

$$\mathcal{T} = \frac{1}{2}\sum_{i=0}^{N} m_i|\dot{\mathbf{x}}_i|^2 \quad , \quad \mathcal{U} = \sum_{0\leq i<j\leq N} \frac{G\, m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \quad , \quad \mathcal{L} = \mathcal{T} + \mathcal{U}\ , \tag{1.10}$$

where $\mathcal{L}$ is a function of all the positions $\mathbf{x}_j$ and all the velocities $\dot{\mathbf{x}}_j$. The momenta vectors and Lagrange equations are

$$\mathbf{p}_j = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}_j} = m_j\dot{\mathbf{x}}_j \quad , \quad m_j\ddot{\mathbf{x}}_j = \dot{\mathbf{p}}_j = \frac{\partial\mathcal{U}}{\partial\mathbf{x}_j}\ . \tag{1.11}$$

It is easy to check that this Lagrangian is invariant with respect to a group of symmetries, namely the tranformations of the 3 dimensional space of each $\mathbf{x}_i$ that are **isometries**. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map $\mathbf{x} \mapsto A\,\mathbf{x} + \mathbf{d}$, with A a 3×3 matrix in the group $O(3)$ of orthogonal tranformations, that is $A^{-1} = A^T$, and $\mathbf{d}$ a constant vector. If this transformation is applied to the positions of all the $N + 1$ bodies, then all the distances r_{ij} are conserved, and the length of the velocity vectors $\dot{\mathbf{r}}_i$ are conserved too; this implies that also the Lagrange function is invariant

$$\mathcal{L}(\mathbf{x}_0, \ldots, \mathbf{x}_N, \dot{\mathbf{x}}_0, \ldots, \dot{\mathbf{x}}_N) = \mathcal{L}(A\,\mathbf{x}_0 + \mathbf{d}, \ldots, A\,\mathbf{x}_n + \mathbf{d}, A\,\dot{\mathbf{x}}_0, \ldots, A\,\dot{\mathbf{x}}_N)\ .$$

A 1–parameter **group of symmetries** of the Lagrange function L is a diffeomorphism F^s of the positions $\mathbf{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_N)$ depending (in a differentiable way) upon a parameter $s \in \mathbb{R}$ so that $F^s \circ F^z = F^{s+z}$ and the Lagrange function is invariant:

$$L\left(F^s(\mathbf{X}), \frac{d}{dt}F^s(\mathbf{X})\right) = L\left(F^s(\mathbf{X}), \frac{\partial F^s}{\partial \mathbf{X}}\dot{\mathbf{X}}\right) = L(\mathbf{X}, \dot{\mathbf{X}}) \ .$$

F^0 is the identity transformation; we also assume the mixed derivatives $\partial^2 F^s/\partial \mathbf{X} \partial s$ are continuous. A *local 1–parameter group of symmetries* of the Lagrange function is defined by the same properties for s in a neighborhood of 0. The main result we need is the **Noether theorem**, stating that if the Lagrange function L admits a local 1–parameter group of symmetries F^s then

$$I(\mathbf{X}, \dot{\mathbf{X}}) = \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0} \tag{1.12}$$

is a first integral of the Lagrange equation (1.11).

To prove this, let us compute the change in L because of F^s by a Taylor series expansion in s

$$L(F^s(\mathbf{X}), \frac{d}{dt}F^s(\mathbf{X})) - L(\mathbf{X}, \dot{\mathbf{X}}) = s \left[\frac{\partial L}{\partial \mathbf{X}} \cdot \left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \frac{\partial}{\partial s}\frac{d}{dt} \left.F^s(\mathbf{X})\right|_{s=0}\right] + \mathcal{O}(s^2) \ .$$

Since this change in L is identically zero by hypothesis, the first order (in s) term must be zero: by exchanging the derivatives d/dt and $\partial/\partial s$

$$\begin{aligned} 0 \ &= \ \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \frac{d}{dt}\left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \mathbf{X}} \cdot \left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0} = \quad \text{(by the Lagrange equation)} \\ &= \ \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \frac{d}{dt}\left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0} + \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{X}}} \left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot \left.\frac{\partial F^s(\mathbf{X})}{\partial s}\right|_{s=0}\right] \end{aligned}$$

and the function defined in Eq. 1.12 is an integral.

Therefore Noether theorem applies to all the one parameter subgroups of the group of linear isometries. The simplest case is that of the one parameter groups of translations, e.g. the translations along one coordinate axis: $F^s(\mathbf{x}) = \mathbf{x} + s\,\mathbf{e}_h$, with $\mathbf{e}_h$ the unit vector along the axis $x_h, h = 1, 3$. If equal translations are applied to all bodies, then the **first integral** described by Neother's theorem is

$$P_h = \sum_{j=0}^{N} \left.\frac{\partial F^s(\mathbf{x}_j)}{\partial s}\right|_{s=0} \cdot \mathbf{p}_j = \hat{\mathbf{e}}_h \cdot \sum_{j=0}^{N} m_j\,\dot{\mathbf{x}}_j = \hat{\mathbf{e}}_h \cdot \mathbf{P} \ ,$$

that is the component along the axis h of the **total linear momentum P**. Thus **P** is a 3-vector integral, and the **center of mass $\mathbf{b}_0$**

$$\mathbf{b}_0 = \frac{1}{M_0}\sum_{j=0}^{N} m_j \mathbf{x}_j \quad , \quad M_0 = \sum_{j=0}^{N} m_j \tag{1.13}$$

moves with uniform velocity:

$$\dot{\mathbf{b}}_0 = \frac{1}{M_0}\mathbf{P} \ . \tag{1.14}$$

This leads to 3 scalar first integrals independent from time (the coordinates of $\dot{\mathbf{b}}_0$), plus 3 integrals dependent from time (the coordinates of $\mathbf{b}_0$ at some epoch).

Other one parameter subgroups of the group of isometries are the groups of rotations around a fixed axis. If F^s is the rotation by an angle of s radians around the unit vector $\mathbf{v}$

$$\frac{\partial F^s(\mathbf{x})}{\partial s}\Big|_{s=0} = \mathbf{v}\times\mathbf{x}$$

and the corresponding integral is:

$$c_h = \sum_{j=1}^{N} \frac{\partial F^s(\mathbf{x}_j)}{\partial s}\Big|_{s=0} \cdot \mathbf{p}_j = \sum_{j=1}^{N} (\mathbf{v}\times\mathbf{x}_j)\cdot\mathbf{p}_j = \mathbf{v}\cdot\sum_{j=1}^{N} \mathbf{x}_j\times\mathbf{p}_j = \mathbf{v}\cdot\sum_{j=1}^{N} m_j(\mathbf{x}_j\times\dot{\mathbf{x}}_j)\ ,$$

namely, the component along $\mathbf{v}$ of the total angular momentum

$$\mathbf{c} = \sum_{j=1}^{N} \mathbf{x}_j\times\mathbf{p}_j\ , \tag{1.15}$$

which is also a 3-vector integral, that is another 3 scalar integrals, for a total of 9 integrals deduced from the simmetry group of isometries.

The 10-th integral is the **energy integral** which can be computed as Hamilton function

$$\mathcal{H}(\mathbf{p}_0,\ldots,\mathbf{p}_N,\mathbf{x}_0,\ldots,\mathbf{x}_N) = \sum_{j=0}^{N} \mathbf{p}_j\cdot\dot{\mathbf{x}}_j - L = \frac{1}{2}\sum_{j=0}^{N}\frac{|\mathbf{p}_j|^2}{m_j} - U(\mathbf{x}_0,\ldots,\mathbf{x}_N) = E\ . \tag{1.16}$$

It is well known (cite Poincaré) that besides these 10 integrals the $N+1$ body problem, as defined by either (1.10) or (1.16), has no other integrals.

Of course the Hamilton function defines the Hamilton equations, which are the equations of motion as a function of the time variable (taken as independent variable); this can be described by the expression H, t are **conjugated variables**. Similarly, other integrals can be taken as Hamilton functions, and provide with the corresponding Hamilton equations the motion under the action of one-parameter simmetry groups, e.g., c_h is the Hamiltonian of the rotation around the $\hat{\mathbf{e}}_h$ axis for $h=1,3$, with as independent variable the rotation angle s (in radians), that is c_h, s are also conjugated variables; also $P_h, \mathbf{b}_0\cdot\hat{\mathbf{e}}_h$ for $h=1,3$.

1.4 The 2-body problem

As the simplest example of the use of the first integrals to reduce the order of the equations (1.11), and also for later reference, let us consider the 2-body problem with Lagrangian

$$\mathcal{L} = \frac{1}{2}m_0\,|\dot{\mathbf{x}}_0|^2 + \frac{1}{2}m_1\,|\dot{\mathbf{x}}_1|^2 + \frac{G\,m_0\,m_1}{|\mathbf{x}_1-\mathbf{x}_0|}\ .$$

We can change coordinates by using, in place of $\mathbf{x}_0, \mathbf{x}_1$, the coordinates of the center of mass and the relative position of $\mathbf{x}_1$ with respect to $\mathbf{x}_0$

$$\mathbf{b}_0 = \epsilon_1\,\mathbf{x}_1 + (1-\epsilon_1)\mathbf{x}_0\ ,\quad \epsilon_1 = \frac{m_1}{m_0+m_1}\quad ,\quad \mathbf{b}_1 = \mathbf{x}_1-\mathbf{x}_0\ . \tag{1.17}$$

Then $\mathcal{U}(\mathbf{b}_1) = Gm_0m_1/b_1$, with $b_1 = |\mathbf{b}_1|$; to write $\mathcal{L}$ as a function of $\mathbf{b}_0, \mathbf{b}_1$ we express $\dot{\mathbf{x}}_0$ and $\dot{\mathbf{x}}_1$ as a function of $\dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1$ and substitute in $\mathcal{T}$

$$\dot{\mathbf{x}}_0 = \dot{\mathbf{b}}_0 - \epsilon_1 \dot{\mathbf{b}}_1 \quad , \quad \dot{\mathbf{x}}_1 = \dot{\mathbf{b}}_0 + (1-\epsilon_1)\dot{\mathbf{b}}_1$$

$$2\mathcal{T} = m_0\,|\dot{\mathbf{x}}_0|^2 + m_1\,|\dot{\mathbf{x}}_1|^2 = (m_0+m_1)\,|\dot{\mathbf{b}}_0|^2 + \frac{m_0m_1}{m_0+m_1}\,|\dot{\mathbf{b}}_1|^2$$

the mixed terms canceling. The Lagrange function as a function of the new coordinates is

$$\mathcal{L}(\mathbf{b}_0, \mathbf{b}_1, \dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1) = \frac{1}{2}\,M_0\,|\dot{\mathbf{b}}_0|^2 + \frac{1}{2}\,M_1\,|\dot{\mathbf{b}}_1|^2 + \frac{G\,M_0\,M_1}{b_1}$$

with $M_0 = m_0 + m_1$ the total mass and M_1 the **reduced mass** (harmonic mean):

$$M_1 = \frac{m_0m_1}{m_0+m_1} \Longleftrightarrow \frac{1}{M_1} = \frac{1}{m_0} + \frac{1}{m_1}\ . \tag{1.18}$$

Then the Lagrange function $\mathcal{L}$ can be decomposed as the sum of two Lagrange functions $\mathcal{L} = M_0\,L_0(\dot{\mathbf{b}}_0) + M_1\,L_1(\mathbf{b}_1, \dot{\mathbf{b}}_1)$, one containing only $\mathbf{b}_0$, the other containing only $\mathbf{b}_1$, and the Lagrange equations decouple:

$$M_0\,\ddot{\mathbf{b}}_0 = 0 \quad , \quad M_1\,\ddot{\mathbf{b}}_1 = \frac{\partial \mathcal{U}(\mathbf{b}_1)}{\partial \mathbf{b}_1}\ .$$

The first equation states that the center of mass moves with constant velocity along a straight line, the second equation is the **Kepler problem**, with a particle of mass M_1 attracted by a fixed center of mass M_0.

By repeating the same computations done for $\mathcal{T}$, we find that also the angular momentum has a simple expression in the $(\mathbf{b}_0, \mathbf{b}_1)$ coordinates:

$$\mathbf{c} = m_0\,\mathbf{x}_0 \times \dot{\mathbf{x}}_0 + m_1\,\mathbf{x}_1 \times \dot{\mathbf{x}}_1 = M_0\,\mathbf{b}_0 \times \dot{\mathbf{b}}_0 + M_1\,\mathbf{b}_1 \times \dot{\mathbf{b}}_1\ .$$

When $\mathbf{b}_0(t) = \dot{\mathbf{b}}_0\,t + \mathbf{b}_0(0)$ from eq. (1.14) is substituted, the $\mathbf{b}_0$ contribution is constant

$$\mathbf{c}_0 = \mathbf{b}_0 \times \dot{\mathbf{b}}_0 = \frac{1}{M_0}\,\mathbf{b}_0(0) \times \mathbf{P} \quad , \quad \mathbf{c} = M_0\,\mathbf{c}_0 + M_1\,\mathbf{c}_1$$

and the contribution from $\mathbf{b}_1$ is $\mathbf{c}_1 = \mathbf{b}_1 \times \dot{\mathbf{b}}_1$, the angular momentum per unit (reduced) mass of $\mathbf{x}_1$ with respect to the center $\mathbf{x}_0$; $\mathbf{c}_1$ is also a vector first integral, thus $\mathbf{b}_1, \dot{\mathbf{b}}_1$ will lie for each t in the orbital plane normal to $\mathbf{c}_1$.

The Laplace-Lenz vector and the energy integral

The 2-body problem has another vector integral, not occurring in the $N+1 \geq 3$-body problem: the **Laplace-Lenz vector**

$$\mathbf{e} = \frac{1}{G\,M_0}\,\dot{\mathbf{b}}_1 \times \mathbf{c}_1 - \frac{1}{b_1}\,\mathbf{b}_1\ . \tag{1.19}$$

This can be shown by using a reference frame formed by three orthogonal unit vectors, $\mathbf{v}_z = \mathbf{c}_1/c_1$ (with $c_1 = |\mathbf{c}_1|$), $\mathbf{v}_r = \mathbf{b}_1/b_1$, and $\mathbf{v}_\theta$ such that $\mathbf{v}_r \times \mathbf{v}_\theta = \mathbf{v}_z$. If θ is the angle between the vector $\mathbf{v}_r$ and a fixed direction in the orbital plane, and $r = b_1$, we have $\mathbf{c}_1 = r^2\,\dot{\theta}\,\mathbf{v}_z$, and

$$G\,M_0\,\mathbf{e} = -r^2\,\dot{r}\,\dot{\theta}\,\mathbf{v}_\theta + (r^3\,\dot{\theta}^2 - G\,M_0)\,\mathbf{v}_r\ . \tag{1.20}$$

Along the solutions we have the equations for the tangential and the radial acceleration

$$\dot{\mathbf{c}}_1 = 0 \Longrightarrow 2\dot{r}\dot{\theta} + r|\dot{\theta}|^2 = 0 \quad , \quad \ddot{r} = -\frac{GM_0}{r^2} + \frac{c_1^2}{r^3} \, ,$$

so that

$$G\, M_0\, \dot{\mathbf{e}} = \ddot{\mathbf{b}}_1 \times \mathbf{c}_1 - G\, M_0\, \dot{\theta}\, \mathbf{v}_\theta = -G\, M_0\, \dot{\theta}\, (\mathbf{v}_r \times \mathbf{v}_z + \mathbf{v}_\theta) = \mathbf{0} \, .$$

Thus $\mathbf{e}$ contains two integrals (not all independent from $\mathbf{c}_1$ because $\mathbf{e}\cdot\mathbf{c}_1 = 0$). We define the **true anomaly** f as the angle between $\mathbf{e}$ and $\mathbf{v}_r$ on the orbital plane, that is

$$e\cos f = \mathbf{e}\cdot\mathbf{v}_r = \frac{r^3\,\dot{\theta}^2}{G\,M_0} - 1 = \frac{c_1^2}{G\,M_0\,r} - 1$$

where $r^2\dot{\theta} = c_1$ is the (scalar) angular momentum of $\mathbf{b}_1$ and is constant. From this the familiar formula of a conic section

$$r = \frac{c_1^2/G\,M_0}{1 + e\cos f}$$

and the interpretation of the two additional **two-body integrals** as **eccentricity** $e = |\mathbf{e}|$ and **argument of pericenter** ω, that is the angle of $\mathbf{e}$ (direction of pericenter) with a fixed direction in the orbital plane, in such a way that $\theta = v + \omega$.

The eccentricity e is an integral depending upon angular momentum and energy. The energy integral of the 2-body problem in $(\mathbf{b}_0, \mathbf{b}_1)$ coordinates is

$$E(\mathbf{b}_0, \mathbf{b}_1, \dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1) = M_0\, E_0 + M_1\, E_1 \quad , \quad E_0 = \frac{1}{2}\,|\dot{\mathbf{b}}_0|^2 \quad , \quad E_1 = \frac{1}{2}\,|\dot{\mathbf{b}}_1|^2 - \frac{G\,M_0}{|\mathbf{b}_1|}$$

and the eccentricity squared, computed from eq. (1.20), is

$$e^2 = \mathbf{e}\cdot\mathbf{e} = \frac{r^4\,\dot{r}^2\,\dot{\theta}^2 + \left(r^3\,\dot{\theta}^2 - G\,M_0\right)^2}{G^2\,M_0^2} = 1 + \frac{2\,E_1\,c_1^2}{G^2\,M_0^2} \, .$$

If the energy of the relative motion E_1 is negative, then $e < 1$ and the trajectory of $\mathbf{b}_1$ is an ellipse with semimajor axis a; its relation with energy and angular momentum can be derived from the equation above:

$$a = \frac{q+Q}{2} = \frac{1}{2}\left[\frac{c_1^2/G\,M_0}{1+e} + \frac{c_1^2/G\,M_0}{1-e}\right] = \frac{GM_0}{-2\,E_1} \, , \tag{1.21}$$

where q, Q are the pericenter and apocenter distances, and the scalar angular momentum per unit mass of the relative motion is

$$c_1 = \sqrt{G\,M_0\;a\,(1-e^2)} \, . \tag{1.22}$$

1.5 Barycentric coordinates

The set of positions of the $N+1$ bodies can be represented in different coordinates; we are interested in the linear coordinate changes of the form

$$\mathbf{b}_i = \sum_{j=0}^{N} a_{ij}\,\mathbf{x}_j \quad , \quad A = (a_{ij}),\; i,j = 0, N \tag{1.23}$$

where the matrix A is a function of the masses only. The purpose is to exploit the integrals of the center of mass to reduce the number of equations, generalizing the results of the 2-body case. A natural choice is to use the center of mass as $\mathbf{b}_0$, thus by (1.13) the first row of the matrix A is

$$a_{0i} = \frac{m_i}{M_0} \quad , \quad i = 0, N \ . \tag{1.24}$$

The choice of the other $\mathbf{b}_i, i = 1, N$, is not as simple as in the 2-body case. Different choices have different advantages, and can be used for different purposes. We shall review in this and in the next section(s) the coordinate systems useful for the $(N+1)$-body problem.

The **barycentric coordinate** system uses the fact that a reference system with a constant velocity translation with respect to an inertial system is also inertial. Thus a reference system with $\mathbf{b}_0 = \mathbf{0}$ as origin and barycentric positions $\mathbf{b}_i = \mathbf{x}_i - \mathbf{b}_0 = \mathbf{x}_i$ for $i = 1, N$ is inertial; the equation of motion is the same as eq. (1.1). However, in this approach the barycentric coordinates of mass index 0 (i.e. the Sun) are not dynamical variables, but are deduced from the coordinates of the other bodies and $\mathbf{b}_0$, by eq. (1.13):

$$\mathbf{s} = \mathbf{s}_B(\mathbf{b}_1, \ldots, \mathbf{b}_N) = \mathbf{x}_0 - \mathbf{b}_0 = -\sum_{i=1}^{N} \frac{m_i}{m_0}\, \mathbf{b}_i \ . \tag{1.25}$$

The change to barycentric is not just a change of coordinates, but also a reduction of the dimension of the problem: we write 3 differential equations less. The reduced equation of motion is

$$m_i\, \ddot{\mathbf{b}}_i = \frac{G\, m_0\, m_i}{|\mathbf{b}_i - \mathbf{s}|^3}\,(\mathbf{s} - \mathbf{b}_i) + \sum_{j\neq i, j=1}^{N} \frac{G\, m_j\, m_i}{|\mathbf{b}_j - \mathbf{b}_i|^3}\,(\mathbf{b}_j - \mathbf{b}_i) \qquad i = 1, \ldots, N \tag{1.26}$$

and can be in conservative form

$$m_j\, \ddot{\mathbf{b}}_j = \frac{\partial \mathcal{U}(\mathbf{s}, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N)}{\partial \mathbf{b}_i} \quad , \quad j = 1, N \, ,$$

where the partial derivatives of the potential $\mathcal{U}$ have to be computed before substituting $\mathbf{s} = \mathbf{s}_B(\mathbf{b}_1, \ldots, \mathbf{b}_N)$. The integrals of energy and angular momentum have a less simple expression, including the contributions from $\dot{\mathbf{s}}$.

Barycentric coordinates are efficient to be used for numerical integrations[4]: only the $3N$ equations (1.26) have to be integrated, and the only additional computation to be performed at each step is $\mathbf{s}$ according to (1.25). On the other hand, barycentric coordinates are seldom used in analytical developments and in theoretical discussions, because of the lack of symmetry of the equation and of the less simple expressions for the classical integrals. This is not a problem because the computed orbit does not need to be used in barycentric coordinates: to change back the output to heliocentric coordinates is the normal procedure.

[4] As an alternative approach, in a numerical integration it is possible to compute the full solution of eq. (1.1), then use $\mathbf{b}_0 = \dot{\mathbf{b}}_0 = \mathbf{0}$ as accuracy check. Besides the small increase in efficiency, which is not important with current computers, there are advantages in describing the general relativistic effects in barycentric coordinates, although the very definition of barycenter has to be modified to remain an integral.

1.6 Heliocentric Canonical Coordinates

To derive equation (1.3) from a single Lagrange (or Hamilton) function is not immediate, mostly because of the asymmetric indirect term. To solve this, Poincaré invented the **heliocentric canonical coordinates** $(\mathbf{r}_i, m_i\dot{\mathbf{b}}_i)$, in which the positions are heliocentric and the linear momenta are barycentric (cite laskar89). To show their properties, let us use a linear coordinate change

$$\mathbf{r}_i = \sum_{j=0}^{N} a_{ij}\mathbf{x}_j \quad , \quad A = (a_{ij}), i, j = 0, N$$

such that $\mathbf{r}_0 = \mathbf{x}_0$, that is $a_{0j} = \delta_{0j}$ and the others are heliocentric vectors: $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_0$ for $i = 1, N$, that is $a_{ij} = \delta_{ij} - \delta_{0j}$ for $j = 0, N$ (the notation δ_{ij} stands for the Kronecker δ, $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise). To complete the transformation of the coordinates $\mathbf{x}_i$ with a linear change of the momenta $m_i\dot{\mathbf{x}}_i$ such that the new coordinates are canonic (see later in Section 2.1), we need to use the matrix B

$$\mathbf{p}_i = \sum_{j=0}^{N} b_{ij}m_j\dot{\mathbf{x}}_j \quad , \quad B = (b_{ij}), i, j = 0, N$$

such that $B = (A^{-1})^T$, that is $b_{0j} = 1$ and $b_{ij} = \delta_{ij}$. Then $\mathbf{p}_0 = \mathbf{P} = M_0\dot{\mathbf{b}}_0$ is the linear momentum integral. The other momentum vectors $\mathbf{p}_i = m_i\dot{\mathbf{x}}_i$, for $i = 1, N$, are barycentric.

To perform the reduction to $3N$ differential equation, we assume that the coordinates $\mathbf{x}_i$ had already been translated in such a way that $\mathbf{b}_0 = \mathbf{0}$ for all time t, thus also $\dot{\mathbf{b}}_0 = \mathbf{0} = \mathbf{p}_0$. Then the momentum vectors $\mathbf{p}_i = m_i\dot{\mathbf{x}}_i$, for $i = 1, N$, are barycentric, and $\mathbf{r}_0 = \mathbf{s} = \mathbf{x}_0$ is given by a formula similar, but not the same as (1.25), because it is a function of heliocentric position vectors:

$$\mathbf{s}_H(\mathbf{r}_1, \ldots, \mathbf{r}_N) = -\sum_{i=1}^{N} \frac{m_i}{M_0}\mathbf{r}_i \tag{1.27}$$

The Lagrange function $\mathcal{L} = \mathcal{T} + \mathcal{U}$ in the coordinates $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$ has to have the same value as the one in the $(\mathbf{x}_i, \dot{\mathbf{x}}_i)$ coordinates: for the kinetic energy

$$\mathcal{T} = \frac{1}{2}\sum_{i=0}^{N} m_i|\dot{\mathbf{x}}_i|^2 = \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{\mathbf{r}}_i + \dot{\mathbf{s}}|^2 + \frac{1}{2}m_0|\dot{\mathbf{s}}|^2 \ , \tag{1.28}$$

and by replacing $\dot{\mathbf{s}}$ with the value constrained by (1.27)

$$\dot{\mathbf{s}}_H(\dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N) = -\sum_{j=1}^{N} \frac{m_j}{M_0}\dot{\mathbf{r}}_j \tag{1.29}$$

we get $\mathcal{T} = \mathcal{T}(\dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N)$; we could check that

$$\mathbf{p}_i = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}_i} = m_i\left(\dot{\mathbf{r}}_i + \dot{\mathbf{s}}_H\right)$$

as claimed. $\mathcal{U}$ it has the same expression in the heliocentric coordinates, since $\mathbf{x}_i - \mathbf{x}_j = \mathbf{r}_i - \mathbf{r}_j$. Thus it is possible to derive the Lagrange equations and check that they are the same as (1.3):

by collecting together the direct attraction from the Sun and the indirect term from the same planet being attracted:

$$\ddot{\mathbf{r}}_i = -\frac{\mu_0+\mu_i}{r_i^3}\,\mathbf{r}_i + \sum_{j\neq i, j=1}^{N} \frac{\mu_j}{r_{ij}^3}\,\mathbf{r}_{ij} - \sum_{k\neq i,k=1}^{N} \frac{\mu_k}{r_k^3}\,\mathbf{r}_k\ . \tag{1.30}$$

[TBC: computations checking this from (laskar89 pages 7-8), but the notations are different]

The improvement (with respect to the conventional heliocentric variables) is in the Hamiltonian formulation. Since $\mathcal{T}$ is quadratic homogeneous in $\dot{\mathbf{r}}_i$, the Legendre transform is simply

$$\mathcal{H} = \sum_{i=1}^{N} \mathbf{p}_i \cdot \dot{\mathbf{r}}_i - \mathcal{L} = \mathcal{T} - \mathcal{U}$$

that is the value of the Hamiltonian is the total energy. To espress the quantity $\mathcal{T}$ as a function of the $\mathbf{p}_i$ we substitute in $\mathcal{T}$ given by (1.28) the relationships

$$\mathbf{p}_i = m_i(\dot{\mathbf{r}}_i + \dot{\mathbf{s}}_H) \quad , \quad \sum_{i=1}^{N} \mathbf{p}_i = -m_0\dot{\mathbf{s}}_H$$

(the first is a consequence of $\mathbf{p}_i = m_i\dot{\mathbf{x}}_i$, the second of the constraint $\dot{\mathbf{b}}_0 = \mathbf{0}$): we get

$$\mathcal{T}(\mathbf{p}_1,\ldots,\mathbf{p}_N) = \frac{1}{2}\sum_{i=1}^{N}\frac{|\mathbf{p}_i|^2}{m_i} + \frac{1}{2m_0}\left|\sum_{i=1}^{N}\mathbf{p}_i\right|^2 = \frac{1}{2}\sum_{i=1}^{N}|\mathbf{p}_i|^2\left[\frac{1}{m_i}+\frac{1}{m_0}\right] + \sum_{1\le i<j\le N}\frac{\mathbf{p}_i\cdot\mathbf{p}_j}{m_0}\ , \tag{1.31}$$

which is convenient because of the especially simple expression (just a sum of scalar products of the $\mathbf{p}_i$ vectors) for the indirect term, which has been moved in the $\mathcal{T}$ part. The Hamilton equations are

$$\dot{\mathbf{p}}_i = -\frac{\partial\mathcal{H}}{\partial\mathbf{r}_i} = -\frac{\partial\mathcal{U}}{\partial\mathbf{r}_i} \quad , \quad \dot{\mathbf{r}}_i = \frac{\partial\mathcal{H}}{\partial\mathbf{p}_i} = \frac{\partial\mathcal{T}}{\partial\mathbf{p}_i}\ ; \tag{1.32}$$

these equations can be shown to be again equivalent to the second order equation (1.3), with the indirect part arising from the kinetic energy rather than from the potential [TBC].

To decompose the Hamilton function into an unperturbed part $\mathcal{H}_0$ and a perturbation $\mathcal{H}_1$

$$\mathcal{H}_0 = \mathcal{T}_0 - \mathcal{U}_0 = \frac{1}{2}\sum_{i=1}^{N}|\mathbf{p}_i|^2\left[\frac{1}{m_i}+\frac{1}{m_0}\right] - \sum_{i=1}^{N}\frac{G(m_0+m_i)}{r_i} \tag{1.33}$$

$$\mathcal{H}_1 = \mathcal{T}_1 - \mathcal{U}_1 = \sum_{1\le i<j\le N}\frac{\mathbf{p}_i\cdot\mathbf{p}_j}{m_0} - \sum_{1\le i<j\le N}\frac{G\,m_i m_j}{|\mathbf{x}_i-\mathbf{x}_j|}\ . \tag{1.34}$$

The angular momentum integral

The heliocentric canonical coordinates have another advantage in an especially simple expression for the angular momentum integral: by starting from the expression of $\mathbf{c}$ in barycentric coordinates, then by using (1.27) and (1.29)

$$
\begin{aligned}
\mathbf{c} &= \sum_{i=0}^{N} \mathbf{x}_i \times m_i\,\dot{\mathbf{x}}_i = \mathbf{s}_H \times m_0\,\dot{\mathbf{s}}_H + \sum_{i=1}^{N} + \mathbf{s}_H \times m_i\,\dot{\mathbf{x}}_i + \sum_{i=1}^{N} \mathbf{r}_i \times m_i\,\dot{\mathbf{x}}_i \\
&= \mathbf{s}_H \times \left[\sum_{i=1}^{N} m_i\,\dot{\mathbf{x}}_i + m_0\,\dot{\mathbf{s}}_H \right] + \sum_{i=1}^{N} \mathbf{r}_i \times m_i\,\dot{\mathbf{x}}_i \ ,
\end{aligned}
$$

where the portion between square brackets is just $M_0\,\dot{\mathbf{b}}_0 = \mathbf{0}$, thus

$$
\mathbf{c} = \sum_{i=1}^{N} \mathbf{r}_i \times m_i\,\dot{\mathbf{x}}_i = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \ . \tag{1.35}
$$

Note that it would be the same if the sum was to include $i = 0$, since $\mathbf{p}_0 = \mathbf{0}$.