Università di Pisa

Dipartimento di Matematica
Corso di Laurea in Matematica

Tesi di Laurea Triennale

A class of one-dimensional free discontinuity problems

Data
15/02/2019

Candidato:
Marco Inversi

Relatore:
Prof. Massimo Gobbino

Anno Accademico 2017/2018

Alla mia famiglia.

O luce etterna che sola in te sidi,
sola t'intendi, e da te intelletta
e intendente te ami e arridi!

Contents

Notation

- $\mathbb{N} = \{0; 1; 2; \dots\}$ is the set of the natural numbers;
- $\mathbb{N}^* = \{1; 2; \dots\}$;
- $\mathbb{R}$ is the set of the real numbers;
- $\mathscr{L}$ is the one-dimensional Lebesgue measure in $[0, 1]$;
- L^2 stands for $L^2((0, 1))$, unless otherwise specified;
- L^1 stands for $L^1((0, 1))$, unless otherwise specified;
- $W^{1;1}$ stands for $W^{1;1}((0, 1))$, unless otherwise specified;
- C^0 stands for $C^0([0, 1])$, unless otherwise specified;
- $C_c^\infty((0, 1))$ is the set of the smooth function supported in a compact subset in $(0, 1)$;
- card is the cardinality of a set.

Introduction and motivations

The Mumford-Shah functional is the prototype of free discontinuity problem. It was introduced by David Mumford and Jayant Shah in 1989 to face up to the problem of the image segmentation. However, the one-dimensional version of the Mumford-Shah functional models the problem of the signal segmentation. Let h be a signal; we are looking for a signal u which is a "regular approximation" of h.

Figure 1: Example of the original signal and the segmented signal

We can identify signals with real-valued functions in $[0,1]$. Let $h : [0,1] \to \mathbb{R}$ be a function; we can introduce the Mumford-Shah functional

$$\mathscr{E}(u) := \int_0^1 (u-h)^2 \, dx + \int_0^1 (\dot{u})^2 \, dx + \sum_{x \in \mathcal{S}(u)} 1,$$

where $\dot{u}$ is the derivative of u (opportunely defined) and $\mathcal{S}(u)$ is the set of the discontinuities of u.

- $\displaystyle\int_0^1 (h-u)^2 \, dx$ is the fidelity term: the less it is, the closer to h it is the approximation.
- $\displaystyle\int_0^1 (\dot{u})^2 \, dx$ is the volume term: the less is it, the more regular it is the approximation.

- $\sum_{x \in \mathcal{S}(u)} 1$ counts the jumps of the function u.

Obviously, each term has minimum 0. However, we are interested in minimizing their sum; in other words, we are looking for the best compromise among the three terms, namely a function u that is quite "close" to h, enough "regular" and with a reasonable quantity of discontinuity points.

This thesis deals with a more general problem. Let φ, ψ be nonnegative functions in $\mathbb{R}$. Similarly, we define the generalized Mumford-Shah functional

$$\mathscr{E}_{\varphi;\psi}(u) := \int_0^1 (u-h)^2 \, dx + \int_0^1 \varphi(\dot{u}) \, dx + \sum_{x \in \mathcal{S}(u)} \psi(\Delta u\,(x)) \tag{0.1}$$

where $\Delta u\,(x)$ is the height of the jump of u in x. What makes this functional more general is that the function ψ gives each jump a weight depending on its height. We suppose that φ is convex, even, lower semicontinuous and $\varphi(0) = 0$; we assume that ψ is even, lower semicontinuous, globally subadditive and $\psi(0) = 0$. We also assume that

$$\liminf_{\theta \to 0} \frac{\psi(\theta)}{|\theta|} = \liminf_{\theta \to +\infty} \frac{\varphi(\theta)}{|\theta|} = +\infty.$$

The first part of this thesis is dedicated to well define the functional $\mathscr{E}_{\varphi;\psi}$ and show that it admits minimum. We point out in which sense u is "regular" and it is a "suitable approximation" of h. Formally, we introduce the space of the Special Functions of Bounded Variation ($\mathcal{SBV}$) as a subset of L^2. Although this definition is exquisitely one-dimensional, a function u in $\mathcal{SBV}$ can be decomposed in $u = w + v$, where w is in $W^{1;1}$ (it is the absolutely continuous part of u) and v is such that

$$v := \sum_{x \in \mathcal{S}(u)} \Delta u\,(x)\, \mathbb{1}_{[x,1]}$$

(it is the jump part). We assume that the $\mathcal{S}\,(u)$ is a set in $(0, 1)$ at most countable and it is formed by pairwise disjoint points. We also assume that the series that defines u is totally convergent. So, if h is a function in L^2, we can well define $\mathscr{E}_{\varphi;\psi}$ as in (0.1).

The existence of the minimum for the generalized Mumford-Shah functional can be obtained as a consequence of the direct method. So, we need compactness and lower semicontinuity theorems for $\mathscr{E}_{\varphi;\psi}$. Since the generalized Mumford-Shah functional is not convex, these results do not follow immediately from the classical calculus of variations.

In the second part of the thesis we show how to approximate the minimum of $\mathscr{E}_{\varphi;\psi}$. We introduce a discrete approximation of $\mathscr{E}_{\varphi;\psi}$ in the sense of the Γ-convergence and we conclude that the minimum of $\mathscr{E}_{\varphi;\psi}$ can be obtained as limit of the sequence of the minimizers of the approximating problems. Since $\mathscr{E}_{\varphi;\psi}$ is defined in an infinite-dimensional space and the approximating problems are defined in $\mathbb{R}^n$ (n is growing), the simplification is absolutely relevant.

The third part of the thesis is dedicated to the regularity of the principal part of the generalized Mumford-Shah functional. We define $\mathcal{MS}_{\varphi;\psi} : \mathcal{SBV} \to [0, +\infty]$ such that

$$\mathcal{MS}_{\varphi;\psi}(u) := \int_0^1 \varphi(\dot{u}) \, dx + \sum_{x \in \mathcal{S}(u)} \psi(\Delta u\,(x))$$

We introduce the descending metric slope of a functional: in some sense, it is very similar to the norm of the gradient for a differentiable function defined in $\mathbb{R}^n$. If $(\mathbb{X}; d)$ is a metric space, $F : \mathbb{X} \to [0, +\infty]$ is a functional and x_0 is a point in $\mathbb{X}$ such that $F(x_0)$ is a real number, the descending metric slope of F in x_0 measures how much it is possible to decrease the value of the functional with respect to the distance from x_0. We want to compute the descending metric slope of the functional $\mathcal{MS}_{\varphi;\psi}$. We consider a function u_0 in $\mathcal{SBV}$ such that the slope of $\mathcal{MS}_{\varphi;\psi}$ in u_0 is finite. We find out that the set of the discontinuities of u_0 is finite, u_0 is quite regular and there are Neumann boundary conditions that force the values of the height of the jump to be in a very special set. We also give a lower bound for the slope.

Surprisingly enough, these conditions turn out to be sufficient. If u_0 is a function in $\mathcal{SBV}$ with all the properties described, then the slope of $\mathcal{MS}_{\varphi;\psi}$ in u_0 is finite. We give an upper bound for the slope that coincides with the lower bound in most cases.

Chapter 1

A preparatory problem

We define the generalized Dirichlet functional. So, we introduce the definition of convex conjugate and we make a step toward the definition of the Orlicz spaces. These notions turn out to be very useful to show that the generalized Dirichlet functional is lower semicontinuous and there are some properties of compactness.

1.1 Convex conjugate

Definition 1.1.1 (Young function).
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a superlinear function, i. e.

$$\liminf_{|x| \to +\infty} \frac{\varphi(x)}{|x|} = +\infty.$$

Let us assume that φ is even, convex and such that $\varphi(0) = 0$. We say that φ is a Young function.

Definition 1.1.2 (Convex conjugate).
Let φ be a Young function as in definition 1.1.1. For all real numbers m, q we denote $r_{m;q}$ the straight line of equation $y = mx - q$. If m is any real number, we define

$$\mathscr{A}_\varphi(m) := \{q \in \mathbb{R} \mid \forall x \in \mathbb{R} \ r_{m,q}(x) \leq \varphi(x)\}.$$

We also define $\varphi^*(m) := \inf \mathscr{A}_\varphi(m)$. The function φ^* is also known as the convex conjugate of φ.

Remark 1.1.3. Thanks to the growth properties of φ, if m is any real number, then $\mathscr{A}_\varphi(m)$ is non-empty. Hence, the function φ^* is well defined.

Remark 1.1.4. We remark that the definitions of Young function and convex conjugate can be given in a more general context (see [7]). However, definitions 1.1.1 and 1.1.2 are sufficient for our purpose.

Lemma 1.1.5. *Let φ be a Young function as in definition 1.1.1; let m be a real number. If we define $\varphi^*(m)$ as in 1.1.2, then $\varphi^*(m)$ is in $\mathscr{A}_\varphi(m)$; in particular, for all x, m in $\mathbb{R}$ the following inequality hold true:*

$$\varphi(x) + \varphi^*(m) \geq mx. \tag{1.1}$$

Proof. It's easy to see that

$$(\varphi^*(m), +\infty) \subseteq \mathscr{A}_\varphi(m) \subseteq [\varphi^*(m), +\infty).$$

Let us assume that there exists x in $\mathbb{R}$ such that $\varphi(x) < r_{m;\varphi^*(m)}$; then, there exists a positive real number ε such that

$$\varphi(x) + \varepsilon < r_{m;\varphi^*(m)}(x).$$

In other words, we have that

$$\varphi(x) < r_{m;\varphi^*(m)+\varepsilon}(x).$$

So, $\varphi^*(m) + \varepsilon$ does not belong to $\mathscr{A}_\varphi(m)$. In particular, we obtain that $\mathscr{A}_\varphi(m)$ is completely contained in $[\varphi^*(m) + \varepsilon, +\infty)$, that is absurd because $\varphi^*(m)$ is the infimum of $\mathscr{A}_\varphi(m)$. □

Theorem 1.1.6. *Let φ be a Young function as in 1.1.1. Let φ^* be the convex conjugate as in 1.1.2. Then φ^* is a Young function.*

Proof. Since φ is even, it's easy to see that φ^* is even and $\varphi^*(0) = 0$.

We claim that φ^* is a convex function. Let m_1, m_2 be real numbers; let t be in $[0, 1]$. Thanks to lemma 1.1.5, for all x in $\mathbb{R}$ the following inequalities hold true:

$$tm_1x - t\varphi^*(m_1) \leq t\varphi(x),$$

$$(1-t)m_2x - (1-t)\varphi^*(m_2) \leq (1-t)\varphi(x).$$

Joining the inequalities, we have that

$$x[tm_1 + (1-t)m_2] - [t\varphi^*(m_1) + (1-t)\varphi^*(m_2)] \leq \varphi(x).$$

By definition 1.1.2, $t\varphi^*(m_1) + (1-t)\varphi^*(m_2)$ is in $\mathscr{A}_\varphi(tm_1 + (1-t)m_2)$; in particular, we have that

$$t\varphi^*(m_1) + (1-t)\varphi^*(m_2) \geq \varphi^*(tm_1 + (1-t)m_2).$$

We claim that φ^* is a superlinear function. Let x be a positive real number; thanks to (1.1), if m is a positive real number, we have that

$$\frac{\varphi^*(m)}{m} \geq x - \frac{\varphi(x)}{m}.$$

In particular, for all x in $(0, +\infty)$ we have that

$$\liminf_{m\to+\infty} \frac{\varphi^*(m)}{m} \geq x - \limsup_{m\to+\infty} \frac{\varphi(x)}{m} = x.$$

As φ^* is even, we have that

$$\liminf_{|m|\to+\infty} \frac{\varphi^*(m)}{|m|} = +\infty.$$

□

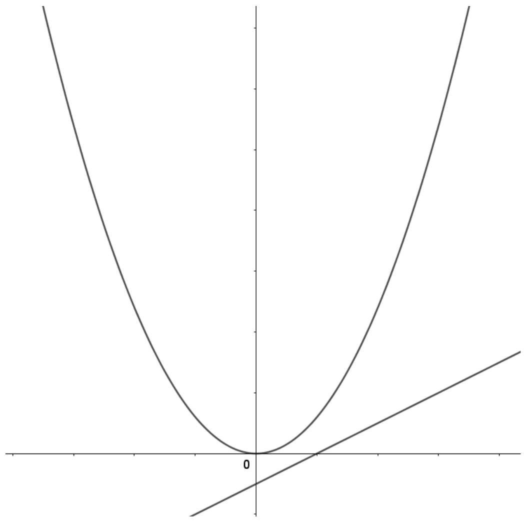

Figure 1.1: The Young function $\varphi(x) = \frac{3}{10}x^2$ and the straight line $y = \frac{1}{2}x - 1$

Example 1.1.7. Let p be a real number in $(1, +\infty)$; let us consider the Young function

$$\varphi_p(x) \coloneqq \frac{|x|^p}{p}.$$

Let q be the conjugate index of p, i. e. q is in $(1, +\infty)$ and such that

$$\frac{1}{q} = 1 - \frac{1}{p}.$$

We claim that φ_p^* is equal to φ_q. We can assume x, m are positive real number. If we ask the straight line of equation $y = mx - \varphi_p^*(m)$ to be tangent to $\varphi(x)$, we have to solve the following system:

$$\begin{cases} mx - \varphi_p^*(m) = \dfrac{x^p}{p}; \\ \dfrac{px^{p-1}}{p} = m. \end{cases}$$

Hence, if m is a positive real number, we obtain that

$$\varphi_p^*(m) = \frac{m^q}{q}.$$

We have shown that $\varphi_p^* = \varphi_q$. If we rewrite (1.1), we notice that for all x, m in $\mathbb{R}$ it holds that

$$mx \leq \frac{|x|^p}{p} + \frac{|m|^q}{q},$$

which is the classical Young's inequality.

Remark 1.1.8. The example 1.1.7 explains why we refer to (1.1) as generalized Young's inequality.

1.2 Luxemburg norm

Definition 1.2.1 (Luxemburg norm)**.**
Let φ be a Young function as in 1.1.1. Let $\mathcal{M}$ be a positive real number. For all measurable function $f : [0,1] \to \mathbb{R}$ we define

$$\|f\|_{\varphi;\mathcal{M}} \coloneqq \inf\left\{ b > 0 \;\middle|\; \int_0^1 \varphi\left(\frac{f(x)}{b}\right) dx \le \mathcal{M} \right\},$$

assuming that $\inf\{\emptyset\}$ is equal to $+\infty$.

Example 1.2.2. Let p be a real number in $(1, +\infty)$; let us fix $\mathcal{M} = 1$. We show that the Young function φ_p introduced in (1.1.7) defines the L^p norm. For all measurable function $f : [0,1] \to \mathbb{R}$ the following identities hold true:

$$\begin{aligned} p^{\frac{1}{p}} \|f\|_{\varphi_p;1} &= p^{\frac{1}{p}} \inf\left\{ b > 0 \;\middle|\; \int_0^1 \frac{1}{p}\left|\frac{f(x)}{b}\right|^p dx \le 1 \right\} \\ &= p^{\frac{1}{p}} \inf\left\{ b > 0 \;\middle|\; \frac{1}{p}\int_0^1 |f(x)|^p \, dx \le b^p \right\} \\ &= \begin{cases} \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } f \in L^p, \\ p^{\frac{1}{p}} \inf\{\emptyset\} = +\infty & \text{if } f \notin L^p. \end{cases} \end{aligned}$$

Remark 1.2.3. It's easy to see that if $f, g : [0,1] \to \mathbb{R}$ are measurable functions coinciding almost everywhere, then $\|f\|_{\varphi;\mathcal{M}} = \|g\|_{\varphi,\mathcal{M}}$. In particular, it is well defined $\|[f]\|_{\varphi;M}$, where $[f]$ is a class of functions coinciding almost everywhere. It can be shown that $\|\cdot\|_{\varphi;\mathcal{M}}$ is a norm on the set of the measurable functions with the relation that identifies functions coinciding almost everywhere. In particular, $\|\cdot\|_{\varphi;\mathcal{M}}$ is homogeneous and triangular inequality holds. Indeed, we won't show these facts: the interested reader can see [7].

Lemma 1.2.4. *Let φ be a Young function as in 1.1.1; let $\mathcal{M}$ be a positive real number. Let $f : [0,1] \to \mathbb{R}$ be a measurable function. Then $\|f\|_{\varphi;\mathcal{M}} = 0$ if and only if $f(x) = 0$ for almost every x in $[0,1]$.*

Proof. Let us assume that $f(x) = 0$ for almost every x in $[0,1]$. If b is any positive real number, we have that

$$\int_0^1 \varphi\left(\frac{f(x)}{b}\right) dx = 0 \le \mathcal{M}.$$

By definition 1.2.1, it holds that $\|f\|_{\varphi;\mathcal{M}} \le b$; so we can conclude that $\|f\|_{\varphi;\mathcal{M}} = 0$.

By definition 1.2.1, it holds that $\|f\|_{\varphi;\mathcal{M}} = 0$ if and only if for all positive real number a it holds that

$$\int_0^1 \varphi\left(af(x)\right) dx \le \mathcal{M}.$$

For all positive integer n, we define the measurable set

$$\mathscr{B}_n \coloneqq f^{-1}\left(\left(-\infty, -\frac{1}{n_0}\right] \cup \left[\frac{1}{n_0}, +\infty\right)\right).$$

If we show that for all n in $\mathbb{N}$ it holds that $\mathscr{L}(\mathscr{B}_n) = 0$, then the conclusion is immediate. By contradiction, let us assume that there exists a natural number n_0 such that $\mathscr{L}(\mathscr{B}_{n_0}) > 0$. Under the growth hypothesis of φ, there exists a positive real number a_0 such that for all a greater than a_0 it holds that

$$\varphi\left(\frac{a}{n_0}\right) > \frac{M}{\mathscr{L}(\mathscr{B}_{n_0})}.$$

Hence, if a is greater than a_0, then

$$\int_0^1 \varphi(af(x))\,dx \geq \int_0^1 \varphi\left(\frac{a}{n_0}\right) \mathbb{1}_{\mathscr{B}_{n_0}}(x)\;dx = \mathscr{L}(\mathscr{B}_{n_0})\varphi\left(\frac{a}{n_0}\right) > \mathcal{M}.$$

So, the absurd follows immediately. □

Lemma 1.2.5. *Let φ be a Young function as in 1.1.1; let $\mathcal{M}$ be a positive real number. Let $f : [0,1] \to \mathbb{R}$ be a measurable function such that $\|f\|_{\varphi;\mathcal{M}}$ is in $(0, +\infty)$. Then, the infimum in the definition 1.2.1 is actually a minimum.*

Proof. Obviously, it is enough to show that

$$\int_0^1 \varphi\left(\frac{f(x)}{\|f\|_{\varphi;\mathcal{M}}}\right) dx \leq \mathcal{M}.$$

Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive real number with the following properties:

- if n is any natural number, then $\displaystyle\int_0^1 \varphi\left(\frac{f(x)}{b_n}\right) dx \leq \mathcal{M}$;
- the sequence is monotonically decreasing and the infimum is $\|f\|_{\varphi;\mathcal{M}}$.

We notice that $\left\{\varphi\left(\frac{f}{b_n}\right)\right\}_{n\in\mathbb{N}}$ is a sequence of measurable nonnegative functions that converges toward $\varphi\left(\frac{f}{\|f\|_{\varphi;\mathcal{M}}}\right)$ pointwise for almost every x in $[0,1]$. Under our assumption on $\{b_n\}_{n\in\mathbb{N}}$, we notice that for almost every x in $[0,1]$ for all n in $\mathbb{N}$ it holds that

$$\varphi\left(\frac{f(x)}{b_n}\right) \leq \varphi\left(\frac{f(x)}{b_{n+1}}\right).$$

Thanks to Beppo Levi's theorem, we have that

$$\int_0^1 \varphi\left(\frac{f}{\|f\|_{\varphi;M}}\right) dx = \lim_{n\to+\infty} \int_0^1 \varphi\left(\frac{f}{b_n}\right) dx \leq \mathcal{M}.$$

□

Let φ be a Young function as in 1.1.1; let φ^* be the convex conjugate as in 1.1.2. We have shown in 1.1.6 that φ^* is a Young function. Hence, for all positive real number $\mathcal{M}$, we can consider $\|\cdot\|_{\varphi;\mathcal{M}}$ and $\|\cdot\|_{\varphi^*;\mathcal{M}}$: the next statement ties them up.

Proposition 1.2.6 (Generalized Hölder's inequality)**.**
Let φ be a Young function as in 1.1.1; let φ^* be the convex conjugate as in 1.1.2; let $\mathcal{M}$ be a positive real number. Let $f, g : [0,1] \to \mathbb{R}$ be measurable functions such that $\|f\|_{\varphi;\mathcal{M}}$ and $\|g\|_{\varphi^*;\mathcal{M}}$ are real numbers. The following inequality holds true:

$$\int_0^1 |f(x)g(x)|\, dx \le 2\mathcal{M}\, \|f\|_{\varphi;\mathcal{M}}\, \|g\|_{\varphi^*;\mathcal{M}}\,.$$

Proof. If $\|f\|_{\varphi;\mathcal{M}} = 0$ or $\|g\|_{\varphi^*;\mathcal{M}} = 0$ the conclusion is an immediate consequence of lemma 1.2.4.

Let us assume that both $\|f\|_{\varphi;\mathcal{M}}$ and $\|g\|_{\varphi^*;\mathcal{M}}$ are positive real numbers. Thanks to 1.1, for almost every x in $[0,1]$ it holds that

$$\frac{|f(x)|}{\|f\|_{\varphi;\mathcal{M}}}\frac{|g(x)|}{\|g\|_{\varphi^*;\mathcal{M}}} \le \varphi\left(\frac{|f(x)|}{\|f\|_{\varphi;\mathcal{M}}}\right) + \varphi^*\left(\frac{|g(x)|}{\|g\|_{\varphi^*;\mathcal{M}}}\right).$$

We integrate and we have that

$$\int_0^1 \frac{|f(x)|}{\|f\|_{\varphi;\mathcal{M}}}\frac{|g(x)|}{\|g\|_{\varphi^*;\mathcal{M}}}\, dx \le \int_0^1 \varphi\left(\frac{|f(x)|}{\|f\|_{\varphi;\mathcal{M}}}\right) dx + \int_0^1 \varphi^*\left(\frac{|g(x)|}{\|g\|_{\varphi^*;\mathcal{M}}}\right) dx.$$

Thanks to lemma 1.2.5, the right hand side is lower that $2\mathcal{M}$ and the proposition is completely proved. □

1.3 Generalized Dirichlet functional in $W^{1;1}$

Definition 1.3.1 (Generalized Dirichlet functional in $W^{1;1}$)**.**
Let φ be a Young function as in 1.1.1. We define $\mathcal{D}_\varphi : L^2 \to [0,+\infty]$ such that

$$\mathcal{D}_\varphi(u) := \begin{cases} \displaystyle\int_0^1 \varphi(\dot{u})\; dx & \text{if } u \in W^{1;1}; \\ +\infty & \text{if } u \in L^2 \setminus W^{1;1}. \end{cases}$$

We refer to $\mathcal{D}_\varphi$ as the generalized Dirichlet functional in $W^{1;1}$.

Proposition 1.3.2. *Let φ be a Young function as in 1.1.1. Let us define $\mathcal{D}_\varphi$ as in 1.3.1. Let α, β be real numbers. Let us denote*

$$\mathbb{X}_{\alpha;\beta} := \left\{u \in W^{1;1} \mid u(0) = \alpha,\ u(1) = \beta\right\}.$$

Then, the straight line that joins $(0;\alpha), (1;\beta)$ is a minimum point for $\mathcal{D}_\varphi$ in $\mathbb{X}_{\alpha;\beta}$. We say that it minimizes the functional with Dirichlet boundary conditions.

Proof. Let u be any function in $\mathbb{X}_{\alpha;\beta}$. We denote as u_0 the straight line that joins $(0;\alpha), (1;\beta)$. Let us denote $v := u - u_0$. We notice that $\dot{u}_0(x) = \beta - \alpha$ for all x in $[0,1]$. Since φ is a convex function and $\dot{u}_0$ is constant, there exists a real number μ such that for almost every x in $[0,1]$ it holds that

$$\varphi(\dot{u}_0(x) + \dot{v}(x)) \ge \mu\dot{v}(x) + \varphi(\dot{u}_0(x)).$$

If we integrate, we obtain that

$$\mathcal{D}_\varphi(w) = \int_0^1 \varphi(\dot{u}_0(x) + \dot{v}(x))\ dx \geq \mu \int_0^1 \dot{v}(x)\ dx + \int_0^1 \varphi(\dot{u}_0(x))\ dx.$$

Since v is in $W^{1;1}$ and $v(0) = v(1) = 0$, it's easy to see that

$$\int_0^1 \dot{v}(x)\ dx = v(1) - v(0) = 0.$$

So, we have that

$$\mathcal{D}_\varphi(w) \geq \int_0^1 \varphi(\dot{u}_0(x))\ dx = \mathcal{D}_\varphi(u_0).$$

□

1.3.1 Compactness of the generalized Dirichlet functional

We state a compactness theorem for the generalized Dirichlet functional. The proof is a consequence of the theory developed in the previous section and the Dunford-Pettis theorem.

Lemma 1.3.3. *Let φ be a Young function as in 1.1.1; let $\mathcal{M}$ be any positive real number. For all ε in $(0, +\infty)$ there exists a positive real number δ with the following property: if C is a measurable set in $[0,1]$ such that $\mathscr{L}(C) < \delta$, then $\|\mathbb{1}_C\|_{\varphi;\mathcal{M}} \leq \varepsilon$.*

Proof. Let ε be a positive real number. It's easy to see that there exists a real number k greater that 1 such that $\varphi\left(\frac{k}{\varepsilon}\right)$ is greater than 0. If we define

$$\delta := \frac{\mathcal{M}}{\varphi\left(\frac{k}{\varepsilon}\right)},$$

we claim that δ satisfies all the requests. Let C be a measurable set in $[0,1]$ such that $\mathscr{L}(C) < \delta$. If $\mathscr{L}(C)$ is equal to 0 the conclusion is trivial; hence, we can assume that $\mathscr{L}(C)$ is in $(0, \delta)$. By definition of δ, k, C, we notice that

$$\varphi\left(\frac{k}{\varepsilon}\right) = \frac{\mathcal{M}}{\delta} \leq \frac{\mathcal{M}}{\mathscr{L}(C)}.$$

By definition 1.2.1, we have that

$$\begin{aligned}
\|\mathbb{1}_C\|_{\varphi;\mathcal{M}} &= \inf\left\{b > 0 \;\middle|\; \int_0^1 \varphi\left(\frac{\mathbb{1}_C(t)}{b}\right) dt \leq \mathcal{M}\right\} \\
&= \inf\left\{b > 0 \;\middle|\; \varphi\left(\frac{1}{b}\right) \mathscr{L}(C) \leq \mathcal{M}\right\} \\
&= \inf\left\{b > 0 \;\middle|\; \varphi\left(\frac{1}{b}\right) \leq \frac{\mathcal{M}}{\mathscr{L}(C)}\right\} \\
&\leq \frac{\varepsilon}{k} \leq \varepsilon.
\end{aligned}$$

□

Theorem 1.3.4. *Let φ be a Young function as in 1.1.1. Let us define $\mathcal{D}_\varphi$ as in 1.3.1. Let $\mathcal{M}, \varepsilon$ be positive real numbers. There exists a positive real number δ with the following property: if x, y are in $[0,1]$ and $|x-y| \le \delta$, if w is a real-valued function that belongs to $W^{1,1}$ such that $\mathcal{D}_\varphi(w) \le \mathcal{M}$, if we consider the continuous representative, it holds that $|w(x) - w(y)| < \varepsilon$.*

Proof. Let $\mathcal{M}, \varepsilon$ be positive real numbers. Let us define the convex conjugate φ^* as in 1.1.2. We recall that φ^* is a Young function (see 1.1.6). Thanks to lemma 1.3.3, there exists a positive real number δ such that if x, y are in $[0,1]$ and $|x-y| \le \delta$, then it holds that

$$\left\|\mathbb{1}_{(x,y)}\right\|_{\varphi^*;\mathcal{M}} \le \frac{\varepsilon}{2\mathcal{M}}. \tag{1.2}$$

We claim that δ satisfies all the request. Let w be a function in $W^{1;1}$ such that $\mathcal{D}_\varphi(w) < \mathcal{M}$. Let x, y be in $[0,1]$ such that $|x-y| < \delta$. It is not restrictive to assume that $x < y$; we also denote as w the continuous representative. Since $\mathcal{D}_\varphi(w)$ is lower than $\mathcal{M}$, by definition 1.2.1 it immediately follows that

$$\|\dot{w}\|_{\varphi;\mathcal{M}} \le 1. \tag{1.3}$$

If we join 1.2.6, (1.2) and (1.3), we obtain that

$$\begin{aligned} |w(x) - w(y)| &\le \int_x^y |\dot{w}(t)|\, dt = \int_0^1 |\dot{w}(t)|\, \mathbb{1}_{(x,y)}(t)\ dt \\ &\le 2\mathcal{M}\, \|\dot{w}\|_{\varphi;\mathcal{M}} \left\|\mathbb{1}_{(x,y)}\right\|_{\varphi^*;\mathcal{M}} \le \varepsilon. \end{aligned}$$

□

Corollary 1.3.5. *Let φ be a Young function as in 1.1.1; let us define the generalized Dirichlet functional as in 1.3.1. Let $\mathcal{M}$ be a positive real number; let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $W^{1,1} \cap C^0$ such that $\mathcal{D}_\varphi(w_n) \le \mathcal{M}$ for all n in $\mathbb{N}$. Then, $\{w_n\}_{n\in\mathbb{N}}$ is a equi-uniformly continuous sequence of functions. Moreover, let us assume that there exists a positive real number R such that for all n in $\mathbb{N}$ there exists x_n in $[0,1]$ such that $|w_n(x_n)| \le R$; then $\{w_n\}_{n\in\mathbb{N}}$ is a equi-bounded sequence of functions. In particular, there exists w_∞ in C^0 and a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly toward w_∞.*

Proof. The first statement is an immediate consequence of theorem 1.3.4.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]$. Let $R, \mathcal{M}$ be positive real numbers as in the hypothesis; let ε be equal to 1. Let $\delta > 0$ be given by the theorem 1.3.4, with $\varepsilon, \mathcal{M}$ as declared. There exists a finite sequence

$$0 = y_0 < y_1 < \cdots < y_p < y_{p+1} = 1$$

such that for all integer i in $\{0; \ldots; p\}$ we have that $|y_i - y_{i+1}| < \delta$. Let n be a natural number: there exists i_n in $\{0; \ldots; p\}$ such that x_n is in $[y_{i_n}, y_{i_n+1}]$. Let x be any point in $[0,1]$: there exists i_x in $\{0; \ldots; p\}$ such that x is in $[y_{i_x}, y_{i_x+1}]$. Without loss of generality, we can assume that $i_x \le i_n$; thanks to triangular inequality, we have that

$$\begin{aligned} |w_n(x)| &\le |w_n(x) - w_n(x_n)| + |w_n(x_n)| \\ &\le |w_n(x_n)| + |w_n(x) - w_n(y_{i_x})| + \sum_{j=i_x}^{i_n-1} |w_n(y_j) - w_n(y_{j+1})| \\ &\le \mathcal{M} + p + 2. \end{aligned}$$

As for the last statement, it is an immediate consequence of the Ascoli-Arzelà's theorem. □

Definition 1.3.6 (Uniformly integration)**.**
Let $\mathscr{F}$ be a set in L^1. We say that $\mathscr{F}$ is uniformly integrable if to ε in $(0, +\infty)$ there corresponds a positive real number δ with the following property: if C is any measurable set in $[0, 1]$ such that $\mathscr{L}(C) \leq \delta$, for all f in $\mathscr{F}$ it holds that

$$\int_C |f(x)|\, dx \leq \varepsilon.$$

Theorem 1.3.7 (Dunford-Pettis theorem in $(0, 1)$)**.**
Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of functions in L^1. Let us denote $\mathscr{F} := \{w_n \mid n \in \mathbb{N}\}$. Those facts are equivalent:

- *$\mathscr{F}$ is uniformly integrable as in 1.3.6;*
- *there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ and a function w_∞ such that $\{w_{n_k}\}_{k\in\mathbb{N}}$ converges L^1-weakly toward w_∞.*

Proof. As for the proof, see [2]. □

Proposition 1.3.8. *Let φ be a Young function as in 1.1.1. Let us define $\mathcal{D}_\varphi$ as in 1.3.1. Let $\mathcal{M}$ be a positive real number; let us consider a sequence of functions $\{w_n\}_{n\in\mathbb{N}}$ in $W^{1;1}$ such that for all n in $\mathbb{N}$ it holds that $\mathcal{D}_\varphi(w_n) \leq \mathcal{M}$. If we define $\mathscr{F} := \{\dot{w}_n \mid n \in \mathbb{N}\}$, then $\mathscr{F}$ is uniformly integrable as in 1.3.6.*

Proof. Thanks to the growth hypothesis on φ, there exists a positive real number B such that for all x in $\mathbb{R}$ it holds that $\varphi(x) \geq |x| - B$. Then, for all n in $\mathbb{N}$ for almost every x in $[0, 1]$ it holds that

$$|\dot{w}_n(x)| \leq |\varphi(\dot{w}_n(x))| + B. \tag{1.4}$$

Let ε be a positive real number. Thanks to lemma 1.3.3, there exists δ in $(0, +\infty)$ with the following property: if C is measurable set in $[0, 1]$ such that $\mathscr{L}(C) < \delta$, then it holds that

$$\|1_C\|_{\varphi^*;\mathcal{M}} < \frac{\varepsilon}{4\mathcal{M}}. \tag{1.5}$$

We immediately notice that it is not restrictive to assume that $\delta < \frac{\varepsilon}{2B}$. By definition 1.2.1, for all n in $\mathbb{N}$ we have that

$$\|\dot{w}_n\|_{\varphi;\mathcal{M}} \leq 1. \tag{1.6}$$

If we join the generalized Hölder's inequality (see 1.2.6), (1.4), (1.5) and (1.6), for all n in $\mathbb{N}$ the following inequalities hold true:

$$\begin{aligned}
\int_C |\dot{w}_n(x)|\, dx &\leq \int_C [\varphi(\dot{w}_n(x)) + B]\, dx \\
&= \int_C \varphi(\dot{w}_n(x))\, dx + B\mathscr{L}(C) \\
&\leq \int_0^1 \varphi(\dot{w}_n(x))\mathbb{1}_C(x)\, dx + \frac{\varepsilon}{2} \\
&\leq 2\mathcal{M}\, \|\dot{w}_n\|_{\varphi;\mathcal{M}} \|\mathbb{1}_C\|_{\varphi^*;\mathcal{M}} + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

Theorem 1.3.9 (Compactness theorem for the generalized Dirichlet functional)**.**
Let φ be a Young function as in 1.1.1; let us define $\mathcal{D}_\varphi$ as in 1.3.1. Let $\mathcal{M}$ be a positive real number. Let us consider a sequence of functions $\{w_n\}_{n\in\mathbb{N}}$ in $W^{1;1} \cap C^0([0,1])$ such that for all n in $\mathbb{N}$ it holds that $\mathcal{D}_\varphi(w_n) \leq \mathcal{M}$. Let us assume that there exists a real number R and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$ such that $|w_n(x_n)| \leq R$ for all n in $\mathbb{N}$. Then, there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ and a function w_∞ with the following properties:

- *$\{w_{n_k}\}_{k\in\mathbb{N}}$ converges uniformly toward w_∞;*
- *w_∞ is in $W^{1;1}$ and $\{\dot{w}_{n_k}\}_{k\in\mathbb{N}}$ converges L^1-weakly toward $\dot{w}_\infty$.*

Proof. Thanks to the Dunford-Pettis theorem (see 1.3.7), there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ and a function v_∞ in L^1 such that $\{\dot{w}_{n_k}\}_{k\in\mathbb{N}}$ converges L^1-weakly toward v_∞. Thanks to corollary 1.3.4, up to further subsequences, not relabelled, we can assume that there exists a function w_∞ such that $\{w_{n_k}\}_{k\in\mathbb{N}}$ converges uniformly in $[0,1]$ toward w_∞. We claim that w_∞ is in $W^{1;1}$ and $\dot{w}_\infty$ is equal to v_∞. By definition of weak derivative, for all k in $\mathbb{N}$ for all ρ in $C_c^\infty((0,1))$, it holds that

$$\int_0^1 w_{n_k}(x)\varphi'(x)\, dx = -\int_0^1 \dot{w}_{n_k}(x)\varphi(x)\, dx.$$

Thanks to the uniform convergence, we have that

$$\lim_{k\to+\infty} \int_0^1 w_{n_k}(x)\varphi'(x)\, dx = \int_0^1 w_\infty(x)\varphi'(x)\, dx.$$

By definition of L^1-weak convergence, we have that

$$\lim_{k\to+\infty} \int_0^1 \dot{w}_{n_k}(x)\varphi(x)\, dx = \int_0^1 v_\infty(x)\varphi'(x)\, dx.$$

This is enough to state that w_∞ is in $W^{1;1}((0,1)) \cap C^0$ and $\dot{w}_\infty = v_\infty$. □

1.3.2 Lower semicontinuity of the generalized Dirichlet functional

We state a lower semicontinuity theorem for the generalized Dirichlet functional. The proof is a consequence of the Hahn-Banach separation theorem.

Theorem 1.3.10. *Let $\mathbb{V}$ be a normed vector space. Let $\Upsilon : \mathbb{V} \to [0,+\infty]$ be a convex, lower semicontinuous map. Let w_∞ be in $\mathbb{V}$; let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{V}$ that converges weakly in $\mathbb{V}$ toward w_∞. Then, it holds that*

$$\liminf_{n\to+\infty} \Upsilon(w_n) \geq \Upsilon(w_\infty).$$

Proof. As for the proof, it is a consequence of the Hahn-Banach separation theorem (see [2]). □

Theorem 1.3.11. *Let φ be a Young function as in 1.1.1. Let us define the generalized Dirichlet functional $\mathcal{D}_\varphi$ as in 1.3.1. Then, $\mathcal{D}_\varphi$ is lower semicontinuous.*

Proof. Let us define $\Upsilon : L^1 \to [0, +\infty]$ such that

$$\Upsilon(w) \coloneqq \int_0^1 \varphi(w)\ dx.$$

Since φ is a convex function, it's immediate to see that Υ is a convex functional. We claim that it is lower semicontinuous. Let w_∞ be a function in L^1; let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence in L^1 that converges toward w_∞ with respect to L^1 norm. Up to subsequences, not relabelled, we can assume that $\{w_n\}_{n\in\mathbb{N}}$ converges pointwise toward w_∞ for almost every x in $(0, 1)$. Since φ is a continuous nonnegative function, the following inequality is a consequence of the Fatou's lemma:

$$\liminf_{n\to+\infty} \int_0^1 \varphi(w_n)\ dx \geq \int_0^1 \varphi(w_\infty)\ dx.$$

Thanks to theorem 1.3.10, if w_∞ is a function in $W^{1;1}$ and $\{w_n\}_{n\in\mathbb{N}}$ is a sequence in $W^{1;1}$ such that $\{\dot{w}_n\}_{n\in\mathbb{N}}$ converges L^1-weakly toward $\dot{w}_\infty$, it holds that

$$\liminf_{n\to+\infty} \int_0^1 \varphi(\dot{w}_n)\ dx \geq \int_0^1 \varphi(\dot{w}_\infty)\ dx.$$

In conclusion, let us consider a function w_∞ in L^2 and a sequence $\{w_n\}_{n\in\mathbb{N}}$ in L^2 that converges toward w_∞ with respect to L^2 norm. We claim that

$$\liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n) \geq \mathcal{D}_\varphi(w_\infty).$$

If the left hand side is $+\infty$, the conclusion is trivial. Let us assume that there exists a real number $\mathcal{M}$ such that the left hand side is equal to $\mathcal{M}$; up to subsequences, not relabelled, we can assume that

- the inferior limit is actually a limit;
- $\mathcal{D}_\varphi(w_n) < \mathcal{M} + 1$ for all n in $\mathbb{N}$;
- the sequence $\{w_n\}_{n\in\mathbb{N}}$ converges pointwise toward w_∞ for almost every x in $(0, 1)$.

Thanks to the compactness theorem for the generalized Dirichlet functional (see 1.3.9), there exists another subsequence, not relabelled, such that $\{\dot{w}_n\}$ converges L^1-weakly toward $\dot{w}_\infty$. Therefore, we can conclude that

$$\mathcal{D}_\varphi(w_\infty) \leq \liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n).$$

□

Chapter 2

An example of free discontinuity problem

2.1 Generalized Mumford-Shah functional in $\mathcal{PJ}$

We introduce the space of the Pure Jump Functions ($\mathcal{PJ}$); we define the generalized Mumford-Shah functional in $\mathcal{PJ}$; we state and prove a lower semicontinuity theorem and some compactness result.

2.1.1 The space of functions $\mathcal{PJ}$

Definition 2.1.1 (Limit in measure theory).
Let $\mathscr{F}$ be a class of functions in L^2 coinciding almost everywhere; let x_0 be any point in $[0,1)$. Let g be any representative of $\mathscr{F}$. Let us assume that there exists a real number l with the following property: to a positive real number ε there corresponds a positive real number δ such that $(x_0, x_0+\delta)$ is contained in $[0,1]$ and it holds that

$$\mathscr{L}\left(\{x \in (x_0, x_0+\delta) \mid |g(x) - l| \leq \varepsilon\}\right) = \delta.$$

We say that l is the right limit for $\mathscr{F}$ as x approaches x_0; it is denoted as $\mathscr{F}(x_0)^+$.

Similarly, if x_0 is in $(0,1]$, we define the left limit for $\mathscr{F}$ as x approaches x_0; it is denoted as $\mathscr{F}(x_0)^-$.

Remark 2.1.2. It's easy to see that definition 2.1.1 does not depend on the specific representative of $\mathscr{F}$ chosen. Hence, it is well posed.

Remark 2.1.3. By definition 2.1.1, it immediately follows that the algebraic properties and the uniqueness of measure theory right limit and left limit still hold true.

Definition 2.1.4 (Jump in measure theory).
Let $\mathscr{F}$ be a class of functions in L^2 coinciding almost everywhere. Let x_0 be any point in $(0,1)$. Let us assume that the right limit $\mathscr{F}(x_0)^+$ and the left limit $\mathscr{F}(x_0)^-$ are well defined as in 2.1.1. We define the jump of $\mathscr{F}$ in x_0 as

$$\Delta\mathscr{F}(x_0) := \mathscr{F}(x_0)^+ - \mathscr{F}(x_0)^-.$$

Definition 2.1.5 (Essential discontinuity).
Let $\mathscr{F}$ be a class of functions in L^2 coinciding almost everywhere such that $\Delta\mathscr{F}(x)$ is well defined for all x in $(0,1)$ as in 2.1.4. Let x_0 be a point in $(0,1)$. We say that

$\mathscr{F}$ is continuous in x_0 if and only if $\Delta\mathscr{F}(x_0) = 0$. We define the set of the essential discontinuities as

$$\mathcal{S}(\mathscr{F}) \coloneqq \{x \in (0,1) \mid \Delta\mathscr{F}(x) \neq 0\}.$$

Definition 2.1.6 ($\mathcal{PJ}$)**.**
Let $\mathscr{F}$ be a class of function in L^2 coinciding almost everywhere such that $\Delta\mathscr{F}(x)$ is well defined for all x in $(0,1)$ as in 2.1.4. We define $\mathcal{S}(\mathscr{F})$ as in 2.1.5. Let us assume that

- $\mathcal{S}(\mathscr{F})$ is at most countable;
- $\displaystyle\sum_{x\in\mathcal{S}(\mathscr{F})} |\Delta\mathscr{F}(x)| < +\infty$;
- $\mathscr{F}(0)^+$ is well defined as in 2.1.1 and it is a real number.

Hence, we can well define $f : [0,1] \to \mathbb{R}$ as follows:

$$f \coloneqq \mathscr{F}(0)^+ + \sum_{y\in\mathcal{S}(\mathscr{F})} \Delta\mathscr{F}(y)\,\mathbb{1}_{[y,1]}.$$

We say that $\mathscr{F}$ is in $\mathcal{PJ}$ (Pure Jump Functions) if and only if f belongs to $\mathscr{F}$. We refer to f as the canonical representative of $\mathscr{F}$.

Remark 2.1.7. By definition 2.1.6, it's immediate to see that $\mathcal{PJ}$ is a vector space. Let $\mathscr{F}, \mathscr{G}$ be in $\mathcal{PJ}$. We denote as f, g the canonical representatives respectively of $\mathscr{F}$ and $\mathscr{G}$ as in 2.1.6, i. e.

$$f \coloneqq \mathscr{F}(0)^+ + \sum_{y\in\mathcal{S}(\mathscr{F})} \Delta\mathscr{F}(y)\,\mathbb{1}_{[y,1]},$$

$$g \coloneqq \mathscr{G}(0)^+ + \sum_{y\in\mathcal{S}(\mathscr{G})} \Delta\mathscr{G}(y)\,\mathbb{1}_{[y,1]}.$$

Obviously, $f + g$ is in $\mathscr{F} + \mathscr{G}$. We also notice that

$$\mathcal{S}(\mathscr{F} + \mathscr{G}) \subseteq \mathcal{S}(\mathscr{F}) \cup \mathcal{S}(\mathscr{G})$$

and the inclusion can be strict. We also notice that $\mathscr{F}$ is completely determined by the continuous representative f. In fact, if f and g coincides for almost every x in $[0,1]$, then $\mathscr{F}$ and $\mathscr{G}$ coincide in L^2; if we assume that $\mathscr{F} = \mathscr{G}$ in L^2, by definition 2.1.6, we have that $f(x) = g(x)$ for all x in $[0,1]$. Having said that, we can identify $\mathscr{F}$ with its canonical representative f; we denote $\mathscr{F}(x_0)^+$ as $f(x_0)^+$, $\mathscr{F}(x_0)^-$ as $f(x_0)^-$, $\Delta\mathscr{F}(x_0)$ as $\Delta f(x_0)$ and $\mathcal{S}(\mathscr{F})$ as $\mathcal{S}(f)$.

We introduce the following decomposition, that will be very useful later.

Definition 2.1.8 (i-jump set and band)**.**
Let f be in $\mathcal{PJ}$ as specified in 2.1.7; let x be in $\mathcal{S}(f)$. For all positive integer i we define the i-jump set as follows:

$$\mathcal{S}(f)^i \coloneqq \left\{ x \in \mathcal{S}(f) \;\middle|\; |\Delta f(x)| \in \left(\frac{1}{i}, \frac{1}{i-1}\right] \right\},$$

assuming that $\frac{1}{0} = +\infty$. We also define the i-band of f as follows:

$$f^i := \sum_{x \in \mathcal{S}(f)^i} \Delta f\,(x)\,\mathbb{1}_{[x,1]},$$

assuming that $f^i := 0$ if $\mathcal{S}\,(f)^i = \emptyset$. In particular, it holds that

$$f = f(0) + \left(\sum_{i \geq 1} f^i \right). \tag{2.1}$$

Remark 2.1.9. Let f be in $\mathcal{PJ}$. By definition 2.1.6, the series that defines f converges totally; hence, for all i in $\mathbb{N}^*$ we have that $\mathcal{S}\,(f)^i$ is a finite set.

2.1.2 Weak formulation in $\mathcal{PJ}$

Definition 2.1.10 (Incremental ratio in 0)**.**
Let $\psi : \mathbb{R} \to \mathbb{R}$ be an even function; let θ be a positive real number. We define

$$\Gamma_\psi(\theta) := \frac{\psi(\theta)}{|\theta|}.$$

Definition 2.1.11 (Weight function)**.**
Let $\psi : \mathbb{R} \to [0, +\infty)$ be an even, lower semicontinuous function with the following properties:

- $\psi(\theta) = 0$ if and only if $\theta = 0$;
- $\liminf_{\theta \to 0^+} \Gamma_\psi(\theta) = +\infty$;
- it is globally subadditive, namely if a, b are real numbers then

$$\psi(a + b) \leq \psi(a) + \psi(b);$$

- $\liminf_{\theta \to +\infty} \psi(\theta) > 0$.

We say that ψ is a weight function.

Definition 2.1.12. Let ψ be a weight function as in definition 2.1.11; let a be a positive real number. We define

$$\mathcal{I}_\psi(a) := \inf \{\psi(x) \mid x \in [a, +\infty)\}.$$

Remark 2.1.13. If ψ is weight function as in definition 2.1.11 and a is a positive real number, then $\mathcal{I}_\psi(a)$ is greater than 0. By contradiction, let us assume that there exists a positive real number a_0 such that $\mathcal{I}_\psi(a_0) = 0$; by definition of infimum, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[a_0, +\infty)$ such that

$$\lim_{n \to +\infty} \psi(x_n) = \mathcal{I}_\psi(a_0) = 0.$$

Up to subsequences, not relabelled, we can assume that there exists x_0 in $[a_0, +\infty]$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges toward x_0. Let us assume that x_0 is a real number; since ψ is lower semicontinuous, we have that

$$\psi(x_0) \leq \liminf_{n\to+\infty} \psi(x_n) = 0,$$

that is against the fact that $\psi(t) = 0$ if and only if $t = 0$. If x_0 is equal to $+\infty$, we have that

$$\liminf_{t\to+\infty} \psi(t) = 0,$$

that contradicts the definition of ψ.

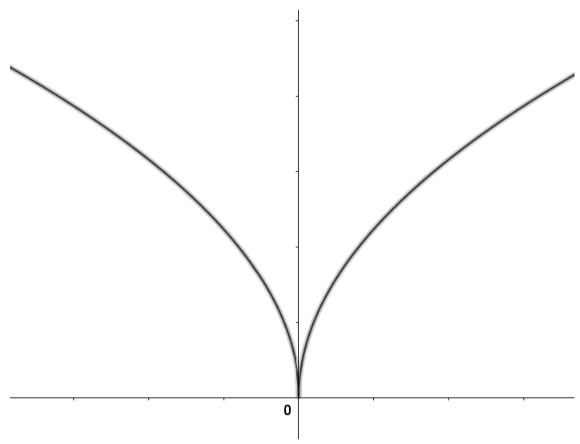

Figure 2.1: The weight function $\psi(x) = \sqrt{|x|}$

Definition 2.1.14 (Generalized Mumford-Shah functional in $\mathcal{PJ}$)**.**
Let ψ be a weight function as in 2.1.11. We define the generalized Mumford-Shah functional in $\mathcal{PJ}$ $\mathcal{MS}_\psi : L^2 \to [0, +\infty]$ such that

$$\mathcal{MS}_\psi(\mathscr{U}) := \begin{cases} \displaystyle\sum_{x\in S(\mathscr{U})} \psi(\Delta\mathscr{U}(x)) & \text{if } \mathscr{U} \in \mathcal{PJ}, \\ +\infty & \text{if } \mathscr{U} \in L^2 \setminus \mathcal{PJ}. \end{cases}$$

If the right hand side series does not converge, we put $\mathcal{MS}_\psi(\mathscr{U}) := +\infty$, obviously.

Lemma 2.1.15. *Let $\mathscr{U}$ be in $\mathcal{PJ}$; let u be its canonical representative as in definition 2.1.6, i. e.*

$$u := \left(\sum_{y\in\mathcal{S}(\mathscr{U})} \Delta\mathscr{U}(y)\, \mathbb{1}_{[y,1]} \right) + \mathscr{U}(0)^+.$$

Let $\mathscr{I}$ be a set at most countable; let $\{x^i\}_{i\in\mathscr{I}}$ be a sequence in $[0, 1]$; let $\{h^i\}_{i\in\mathscr{I}}$ be a sequence in $[0, +\infty)$ such that

$$\sum_{i\in\mathscr{I}} |h^i| < +\infty.$$

Then, we can well define the function

$$\omega := \sum_{i\in\mathscr{I}} h^i \mathbb{1}_{[x^i,1]}.$$

Let us assume that $\omega(x) = u(x)$ for almost every x in $(0,1)$. In particular, ω is a representative of u. Let ψ be a weight function as in 2.1.11; we define $\mathcal{MS}_\psi$ as in 2.1.14. We claim that

$$\mathcal{MS}_\psi(u) \leq \sum_{i\in\mathscr{I}} \psi(h^i).$$

In other words, the canonical representative is the minimal one with respect to the generalized Mumford-Shah functional among all the other representatives of $\mathscr{U}$.

Proof. **Step 1**: For all i, j in $\mathscr{I}$, we say that i is equivalent to j if and only $x^i = x^j$ and we write $i \sim j$. Since $\sim$ is an equivalence relation, it provides a set $\mathscr{J}$ at most countable and a partition of $\mathscr{I}$ into disjoint equivalence classes, namely

$$\{\mathscr{I}^j \mid j \in \mathscr{J}\}.$$

For all j in $\mathscr{J}$, we define

$$l^j := \sum_{i\in\mathscr{I}^j} h^i;$$

if i is any element in $\mathscr{I}$, we can also well define $y^j := x^i$. It's immediate to see that for all x in $(0,1)$ it holds that

$$\omega(x) = \sum_{j\in\mathscr{J}} l^j \mathbb{1}_{[y^j,1]}(x).$$

We notice that it's not restrictive to make the following assumption:

- $\mathscr{J}$ is contained in $\mathbb{N}$;
- $l^j \neq 0$ for all j in $\mathscr{J}$;
- $y^j \neq 1$ for all j in $\mathscr{J}$.

Step 2: Let x_0 be any point in $[0,1)$. We claim that

$$\omega(x_0) = \lim_{x\to x_0^+} \omega(x).$$

Let ε be any positive real number. For all n in $\mathbb{N}$ we define $\mathscr{J}_n := \mathscr{J} \cap (n, +\infty)$. Since $\sum_{j\in\mathscr{J}} \left|l^j\right| < +\infty$, we can well define

$$j_\varepsilon := \min\left\{ n_0 \in \mathscr{J} \;\middle|\; \sum_{j\in\mathscr{J}_{n_0}} \left|l^j\right| \leq \varepsilon \right\}.$$

There exists a positive real number δ such that $(x_0, x_0+\delta)$ is completely contained in $[0,1]$ and if j is an integer in $\mathscr{J} \cap \{1;\dots;j_\varepsilon\}$, then y^j does not belong to $(x_0, x_0+\delta)$.

Hence, if x is in $(x_0, x_0+\delta)$, the following inequalities hold true:

$$\begin{aligned}|\omega(x)-\omega(x_0)| &= \left|\left(\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{[y^j,1]}(x)\right) - \left(\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{[y^j,1]}(x_0)\right)\right| \\ &= \left|\sum_{j\in\mathscr{J}} l^j \left(\mathbb{1}_{[y^j,1]}(x) - \mathbb{1}_{[y^j,1]}(x_0)\right)\right| \\ &= \left|\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{(x_0,x]}(y^j)\right| \leq \sum_{j\in\mathscr{J}_{j_\varepsilon}} \left|l^j\right| \leq \varepsilon.\end{aligned}$$

Step 3: Let x_0 be any point in $(0,1]$ such that $x_0 \neq y^j$ for all j in $\mathscr{J}$. We claim that

$$\omega(x_0) = \lim_{x\to x_0^-} \omega(x).$$

Let ε be a positive real number. As defined in the previous step, we set j_ε and δ. Similarly, if x is in $(x_0-\delta, x_0)$, we have that

$$\begin{aligned}|\omega(x_0)-\omega(x)| &= \left|\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{(x,x_0]}(y^j)\right| \\ &= \left|\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{(x,x_0)}(y^j)\right| \leq \sum_{j\in\mathscr{J}_{j_\varepsilon}} \left|l^j\right| \leq \varepsilon.\end{aligned}$$

Step 4: Let i be in $\mathscr{J}$. We claim that

$$\omega(y^i) - l^i = \lim_{x\to y^{i-}} \omega(x).$$

Let ε be a positive real number. As defined in the second step, we set j_ε and δ. Similarly, if x is in $(y^i-\delta, y^i)$, we have that

$$\begin{aligned}\left|\omega(y^i) - l^i - \omega(x)\right| &= \left|\left(\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{(x,y^i]}(y^j)\right) - l^i\right| \\ &= \left|\left(l^i + \sum_{j\in\mathscr{J}} l^j \mathbb{1}_{(x,y^i)}(y^j)\right) - l^i\right| \\ &= \left|\sum_{j\in\mathscr{J}} l^j \mathbb{1}_{(x,y^i)}(y^j)\right| \leq \sum_{j\in\mathscr{J}_{j_\varepsilon}} \left|l^j\right| \leq \varepsilon.\end{aligned}$$

Step 5: Since ω is a representative of $\mathscr{U}$, for all x_0 in $(0,1)$ we have that

$$\Delta\mathscr{U}\,(x_0) = \lim_{x\to x_0^+} \omega(x) - \lim_{x\to x_0^-} \omega(x) = \begin{cases} 0 & \text{if } \forall j \in \mathscr{J} \,:\, x_0 \neq y^j; \\ l^j & \text{if } \exists j \in \mathscr{J} \,:\, x_0 = y^j. \end{cases}$$

Similarly, we have that

$$\mathscr{U}(0)^+ = \lim_{x\to 0^+} \omega(x) = \begin{cases} 0 & \text{if } \forall j \in \mathscr{J} : 0 \neq y^j; \\ l^j & \text{if } \exists j \in \mathscr{J} : 0 = y^j. \end{cases}$$

We recall that ψ is globally subadditive; hence, the following inequalities hold true:

$$\begin{aligned} \sum_{i\in\mathscr{I}} \psi(h^i) &= \sum_{j\in\mathscr{J}} \left(\sum_{i\in\mathscr{I}^j} \psi(h^i) \right) \\ &\geq \sum_{j\in\mathscr{J}} \left(\psi\left(\sum_{i\in\mathscr{I}^j} h^i \right) \right) \\ &= \sum_{j\in\mathscr{J}} \psi(l^j) \\ &\geq \mathcal{MS}_\psi(\mathscr{U}). \end{aligned}$$

□

Remark 2.1.16. Similarly, we can easily show that $\mathcal{MS}_\psi$ is a subadditive functional: if $\mathscr{U}_1, \mathscr{U}_2$ are in $\mathcal{PJ}$, then

$$\mathcal{MS}_\psi(\mathscr{U}_1) + \mathcal{MS}_\psi(\mathscr{U}_2) \geq \mathcal{MS}_\psi(\mathscr{U}_1 + \mathscr{U}_2).$$

If we denote $\mathcal{S} := \mathcal{S}(\mathscr{U}_1) \cup \mathcal{S}(\mathscr{U}_2)$, as shown in 2.1.7, it holds that

$$\mathcal{S}(\mathscr{U}_1 + \mathscr{U}_2) \subseteq \mathcal{S}.$$

Since $\psi(0) = 0$, the following inequality hold true:

$$\begin{aligned} \mathcal{MS}_\psi(\mathscr{U}_1) + \mathcal{MS}_\psi(\mathscr{U}_2) &= \sum_{x\in\mathcal{S}} \psi(\Delta\mathscr{U}_1(x)) + \psi(\Delta\mathscr{U}_2(x)) \qquad (2.2) \\ &\geq \sum_{x\in\mathcal{S}} \psi(\Delta\mathscr{U}_1(x) + \Delta\mathscr{U}_2(x)) \\ &= \sum_{x\in\mathcal{S}} \psi(\Delta(\mathscr{U}_1 + \mathscr{U}_2)(x)) \\ &\geq \mathcal{MS}_\psi(\mathscr{U}_1 + \mathscr{U}_2). \end{aligned}$$

In (2.2) we used the fact that $\psi(0)$ is globally subadditive.

2.1.3 Compactness and lower semicontinuity in $\mathcal{PJ}$

From now on, with a slight abuse of notation, we identify $\mathscr{U}$ in $\mathcal{PJ}$ with the corresponding canonical representative.

Lemma 2.1.17. *Let ψ be a weight function as in 2.1.11; let us define $\mathcal{MS}_\psi$ as in 2.1.14. Let $\mathcal{M}, \varepsilon$ be positive real numbers. There exists n_0 in $\mathbb{N}$ with the following property: if n is a natural number greater than or equal to n_0 and v is a function in $\mathcal{PJ}$ such that $\mathcal{MS}_\psi(v) \leq \mathcal{M}$, then*

$$\left\| v - \left(v(0) + \sum_{i\leq n} v^i \right) \right\|_\infty \leq \sum_{i>n} \left(\sum_{x\in\mathcal{S}(v_n)^i} |\Delta v_n(x)| \right) \leq \varepsilon.$$

Proof. Let $\mathcal{M}, \varepsilon$ be positive real numbers; let θ be a positive real number. We define

$$\Gamma_\psi(\theta) := \frac{\psi(\theta)}{|\theta|}$$

as in 2.1.10. By definition of weight function (see 2.1.11), it holds that

$$\lim_{\theta \to 0^+} \Gamma_\psi(\theta) = +\infty.$$

Hence, there exists a natural number n_0 such that if θ is in $\left(0, \frac{1}{n_0}\right]$ then

$$\Gamma_\psi(\theta) \geq \frac{\mathcal{M}}{\varepsilon}.$$

As declared in 2.1.8, if n is greater that n_0 and x is $\mathcal{S}(v)^n$, then

$$|\Delta v(x)| \leq \frac{1}{n_0}.$$

In particular, we can conclude that

$$\frac{\psi(|\Delta v(x)|)}{|\Delta v(x)|} \geq \frac{\mathcal{M}}{\varepsilon}.$$

If we denote

$$\mathcal{S} := \bigcup_{i > n_0} S(v)^i,$$

the following inequalities hold true:

$$\begin{aligned}
\left\| v - \left(v(0) + \sum_{i \leq n} v^i \right) \right\|_\infty &= \left\| \sum_{i > n} v^i \right\|_\infty \leq \sum_{i > n} \|v^i\|_\infty \\
&\leq \sum_{i > n_0} \|v^i\|_\infty \leq \sum_{i > n_0} \left(\sum_{x \in \mathcal{S}(v)^i} |\Delta v(x)| \right) \\
&= \sum_{x \in \mathcal{S}} |\Delta v(x)| \leq \frac{\varepsilon}{\mathcal{M}} \sum_{x \in \mathcal{S}} \psi(|\Delta v(x)|) \\
&\leq \varepsilon.
\end{aligned}$$

□

Proposition 2.1.18. *Let ψ be a weight function as in 2.1.11; let us define $\mathcal{MS}_\psi$ as in 2.1.14. Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{PJ}$. Let $\mathcal{M}$ be a positive real number. Let us assume that for all n in $\mathbb{N}$ it holds that*

- $|v_n(0)| \leq \mathcal{M}$;
- *if x is in $\mathcal{S}(v_n)$, then $|\Delta v_n(x)| \leq \mathcal{M}$;*
- $\mathcal{MS}_\psi(v_n) \leq \mathcal{M}$.

There exists a subsequence $\{v_{n_i}\}_{i \in \mathbb{N}}$ with the following properties:

- *there exists a sequence of nonnegative integers* $\{\beta^i\}_{i\in\mathbb{N}}$ *such that if* n *is a natural number greater than or equal to* i*, then* $\operatorname{card}\mathcal{S}\left(v_n\right)^i = \beta^i$*; in particular for all* i *in* $\mathbb{N}$ *for all* $n \geq i$ *we denote*

$$\mathcal{S}\left(v_n\right)^i := \left\{x_n^{i;1}; \dots ; x_n^{i;\beta^i}\right\},$$

with the assumption that

$$0 := x_n^{i;0} < x_n^{i;1} < \dots < x_n^{i;\beta^i} < x_n^{i;\beta^i+1} := 1.$$

We also denote

$$v_n^i := \sum_{t=1}^{\beta^i} \Delta_n^{i;t} \mathbb{1}_{\left[x_n^{i;t},1\right]}.$$

- *Let* i *be a positive integer; if* t *is any integer in* $\{1; \dots ; \beta^i\}$*, there exist* $x_\infty^{i;t}$ *in* $[0,1]$ *and* $\Delta_\infty^{i;t}$ *whose absolute value is in* $\left[\frac{1}{i}, \frac{1}{i-1}\right] \cap (0, \mathcal{M}]$*, such that*

$$\lim_{n\to+\infty} x_n^{i;t} = x_\infty^{i;t},$$

$$\lim_{n\to+\infty} \Delta_n^{i;t} = \Delta_\infty^{i;t}.$$

- *For all positive integer* i *we define*

$$v_\infty^i := \sum_{t=1}^{\beta^i} \Delta_\infty^{i;t} \mathbb{1}_{\left[x_\infty^{i;t},1\right]};$$

then $\{v_n^i\}_{n\in\mathbb{N}}$ *converges toward* v_∞^i *with respect to* L^2 *norm and pointwise for almost every* x *in* $[0,1]$*; moreover, if we define*

$$\mathcal{B}^i := \left\{t \in \{1; \dots ; \beta^i\} \mid x_\infty^{i;t} = 0\right\},$$

then

$$v_\infty^i(0) = \sum_{t\in\mathcal{B}^i} \Delta_\infty^{i;t}.$$

- $\left\{\sum_{t=1}^{i} v_\infty^t\right\}_{i\in\mathbb{N}}$ *is a Cauchy sequence with respect to* L^2 *norm; if we define*

$$v_\infty := \sum_{i\in\mathbb{N}^*} v_\infty^i,$$

then $\{v_{n_i}\}_{i\in\mathbb{N}}$ *converges toward* v_∞ *with respect to* L^2 *norm.*

In particular, v_∞ *is in* $\mathcal{PJ}$ *and it holds that*

$$\liminf_{i\to+\infty} \mathcal{MS}_\psi\left(v_{n_i}\right) \geq \mathcal{MS}_\psi(v_\infty). \tag{2.3}$$

Proof. **Step 1**: Let i be a positive integer. Let Λ^i be an infinite subset in $\mathbb{N}$. We define

$$\mathcal{I}_\psi\left(\frac{1}{i}\right) := \inf\left\{\psi(x) \;\middle|\; x \in \left[\frac{1}{i}, +\infty\right)\right\}$$

as in 2.1.12 and we recall that it is a positive real number (see 2.1.13). Hence, if n is any integer in Λ^i, it holds that

$$\mathcal{M} \geq \mathcal{MS}_\psi(v_n^i) = \sum_{x\in\mathcal{S}(v_n)^i} \psi(\Delta v_n(x)) \geq \operatorname{card}\mathcal{S}\left(v_n^i\right)\cdot\mathcal{I}_\psi\left(\frac{1}{i}\right).$$

In particular, if n is any integer in Λ^i it holds that

$$\operatorname{card}\mathcal{S}\left(v_n\right)^i \geq \frac{M}{\mathcal{I}_\psi\left(\frac{1}{i}\right)}.$$

Therefore, we can find another infinite subset Λ^{i+1} completely contained in Λ^i and a positive integer β^i such that if k is an integer in Λ^{i+1}, then

$$\beta^i := \operatorname{card}\mathcal{S}\left(v_n\right)^i.$$

In other words, up to subsequence we can assume that the number of jumps whose height is in $\left(\frac{1}{i}, \frac{1}{i-1}\right]$ is equi-finite.

If we put $\Lambda^1 := \mathbb{N}$, by a diagonal procedure, we can find an infinite subset of natural numbers

$$\Lambda := \{\lambda_i \mid i \in \mathbb{N}\}$$

and a sequence of natural numbers $\{\beta^i\}_{i\in\mathbb{N}}$ with the following property: if i, m are positive integers such that $m \geq i$, then

$$\operatorname{card}\mathcal{S}\left(v_{\lambda_m}\right)^i = \beta^i.$$

We can also assume that the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ is monotonically increasing. Therefore, up to subsequences, not relabelled, we can assume that Λ is equal to $\mathbb{N}$.

Step 2: Let i be a positive integer. Let Θ^i be an infinite subset completely contained in $\mathbb{N}\cap[i, +\infty)$. For all n in Θ^i, we denote

$$\mathcal{S}\left(v_n\right)^i := \left\{x_n^{i;1}; \ldots; x_n^{i;\beta^i}\right\},$$

with the assumption that

$$0 := x_n^{i;0} < x_n^{i;1} < \cdots < x_n^{i;\beta^i} < x_n^{i;\beta^i+1} := 1.$$

We also denote

$$v_n^i := \sum_{t=1}^{\beta^i} \Delta_n^{i;t}\mathbb{1}_{\left[x_n^{i;t}, 1\right]}.$$

Then, we can identify the sequence $\{v_n^i\}_{n\in\Theta^i}$ with a sequence of vectors in $\mathbb{R}^{2\beta^i}$ as follows:

$$v_n^i \sim \left(x_n^{i;1}; \ldots; x_n^{i;\beta^i}; \Delta_n^{i;1}; \ldots; \Delta_n^{i;\beta^i}\right) := \nu_n^i.$$

Thanks to our hypothesis, $\{\nu_n^i\}_{n\in\Theta^i}$ is a sequence in $[0,1]^{\beta^i} \times [-\mathcal{M},\mathcal{M}]^{\beta^i}$. Since the closed balls are compact subsets in $\mathbb{R}^{2\beta^i}$, we can find another infinite subset Θ^{i+1} completely contained in Θ^i and two finite subsets

$$\left\{x_\infty^{i;1} \leq \cdots \leq x_\infty^{i;\beta^i}\right\} \subseteq [0,1],$$

$$\{\Delta_\infty^{i;1} \leq \cdots \leq \Delta_\infty^{i;B^i}\} \subseteq [-\mathcal{M},\mathcal{M}]$$

with the property the follows: if we define

$$\nu_\infty^i := \left(x_\infty^{i;1};\ldots;x_\infty^{i;\beta^i};\Delta_\infty^{i;1};\ldots;\Delta_\infty^{i;\beta^i}\right),$$

then $\{\nu_n^i\}_{n\in\Theta^{i+1}}$ converges toward ν_∞^i as vectors in $\mathbb{R}^{2\beta^i}$. Let us denote

$$v_\infty^i := \sum_{t=1}^{\beta^i} \Delta_\infty^{i;t} \mathbb{1}_{\left[x_\infty^{i;t},1\right]}.$$

We claim that the sequence $\{v_n^i\}_{n\in\Theta^{i+1}}$ converges toward v_∞^i with respect to L^2 norm. Since the finite sum is continuous with respect to L^2 norm, it is enough to show that if t is an integer in $\{1;\ldots;\beta^i\}$, then the sequence

$$\left\{\Delta_n^{i;t}\mathbb{1}_{\left[x_n^{i;t},1\right]}\right\}_{n\in\Theta^{i+1}}$$

converges toward $\Delta_\infty^{i;t}\mathbb{1}_{\left[x_\infty^{i;t},1\right]}$ with respect to the L^2 norm. Without loss of generality, we can assume that $x_n^{i;t} \leq x_\infty^{i;t}$ for all n in Θ^{i+1}. Therefore, the following inequalities hold true:

$$\begin{aligned}\left\|\Delta_n^{i;t}\mathbb{1}_{\left[x_n^{i;t},1\right]} - \Delta_\infty^{i;t}\mathbb{1}_{\left[x_\infty^{i;t},1\right]}\right\|_{L^2}^2 &= \int_0^1 \left(\Delta_n^{i;t}\mathbb{1}_{\left[x_n^{i;t},1\right]} - \Delta_\infty^{i;t}\mathbb{1}_{\left[x_\infty^{i;t},1\right]}\right)^2 dx \\ &= \int_{x_n^{i;t}}^{x_\infty^{i;t}} \left(\Delta_n^{i;t}\right)^2 dx + \int_{x_\infty^{i;t}}^1 \left(\Delta_n^{i;t} - \Delta_\infty^{i;t}\right)^2 dx \qquad (2.4) \\ &\leq \mathcal{M}^2\left|x_n^{i;t} - x_\infty^{i;t}\right| + \left(\Delta_n^{i;t} - \Delta_\infty^{i;t}\right)^2.\end{aligned}$$

In (2.4) we used the upper bound of the height of the jumps, as stated in hypothesis. So, we can take the limit as n in Θ^{i+1} approaches $+\infty$. It's easy to see that the convergence is also pointwise for almost every x in $[0,1]$. Moreover, if we define

$$\mathcal{B}^i := \left\{t \in \{1;\ldots;\beta^i\} \mid x_\infty^{i;t} = 0\right\},$$

then

$$v_\infty^i(0) = \sum_{t\in\mathcal{B}^i} \Delta_\infty^{i;t},$$

as immediately follows by definition of v_∞^i.

If we put $\Theta^1 := \mathbb{N}$, by a diagonal procedure, we can find an infinite subset Θ and a sequence $\{v_\infty^i\}_{i\in\mathbb{N}^*}$ with the following property: if i is in $\mathbb{N}^*$, then $\{v_n^i\}_{n\in\Theta}$ converges toward v_∞^i with respect to L^2 norm as n approaches $+\infty$ in Θ.

Moreover, we recall that $\{v_n(0)\}_{n\in\Theta}$ is a sequence in $[-\mathcal{M};\mathcal{M}]$. So, up to subsequences, not relabelled, we can assume that there exists a real number d in $[-\mathcal{M};\mathcal{M}]$

such that $\{v_n(0)\}_{n\in\Theta}$ converges toward d. Since Θ is infinite, up to subsequences, not relabelled, we can assume that it is equal to $\mathbb{N}$.

Step 3: Let us define

$$v_\infty \coloneqq d + \left(\sum_{i\geq 1} v_\infty^i\right) = d + \sum_{i\geq 1}\left(\sum_{t=1}^{\beta^i} \Delta_\infty^{i;t} \mathbb{1}_{\left[x_\infty^{i;t},1\right]}\right).$$

We have to show that v_∞ is well defined in $\mathcal{PJ}$, that is

$$\sum_{i\geq 1}\sum_{t=1}^{\beta^i} \left|\Delta_\infty^{i;t}\right| < \infty.$$

It is equivalent to require that the sequence of the partial sums is a Cauchy sequence. Let n be any integer. We define

$$w_n \coloneqq |v_n(0)| + \sum_{i\geq 1}\sum_{x\in\mathcal{S}(v_n)^i} \left|\Delta v_n\left(x\right)\right| \mathbb{1}_{[x,1]}.$$

Let i be a positive integer; we can define

$$w_n^i \coloneqq \sum_{x\in\mathcal{S}(v_n)^i} \left|\Delta v_n\left(x\right)\right| \mathbb{1}_{[x,1]}.$$

More explicitly, thanks to the previous step, if n is greater than or equal to i, we have that

$$w_n^i \coloneqq \sum_{t=1}^{\beta^i} \left|\Delta_n^{i;t}\right| \mathbb{1}_{\left[x_n^{i;t},1\right]}.$$

We also define

$$w_\infty^i \coloneqq \sum_{t=1}^{\beta^i} \left|\Delta_\infty^{i;t}\right| \mathbb{1}_{\left[x_\infty^{i;t},1\right]}.$$

If we slightly modify the procedure described in the previous step, we show that for all positive integer i the sequence $\{w_n^i\}_{n\in\mathbb{N}}$ converges toward w_∞^i with respect to L^2 norm. Since ψ is even, we notice that for all n in $\mathbb{N}$ it holds that

$$\mathcal{MS}_\psi(w_n) = \mathcal{MS}_\psi(v_n) \leq \mathcal{M}.$$

Let ε be a positive real number. We can apply lemma 2.1.17 with ε and $\mathcal{M}$. Let n_0 be an integer with the property declared in lemma 2.1.17. Let us consider k, j positive integer such that $k > j \geq n_0$. So, the following inequalities hold true:

$$\begin{aligned}
\left| \sum_{i=1}^{k} \left(\sum_{t=1}^{\beta^i} |\Delta_{\infty}^{i;t}| \right) - \sum_{i=1}^{j} \left(\sum_{t=1}^{\beta^i} |\Delta_{\infty}^{i;t}| \right) \right| &= \sum_{i=j+1}^{k} \left(\sum_{t=1}^{\beta^i} |\Delta_{\infty}^{i;t}| \right) \\
&= \lim_{n \to +\infty} \sum_{i=j+1}^{k} \left(\sum_{t=1}^{\beta^i} |\Delta_{n}^{i;t}| \right) \\
&= \lim_{n \to +\infty} \sum_{i \geq n_0} \left(\sum_{x \in \mathcal{S}(v_n)^i} |\Delta v_n(x)| \right) \\
&= \lim_{n \to +\infty} \left\| w_n - \left(|v_n(0)| + \sum_{i=1}^{n_0} w_n^i \right) \right\|_{\infty} \\
&\leq \varepsilon.
\end{aligned}$$

So we can conclude that v_∞ is well defined and it is in $\mathcal{PJ}$, as shown in further details in the proof of lemma 2.1.16. As a matter of facts, we are not assuming that v_∞ is the canonical representative of a class of functions in $\mathcal{PJ}$.

Step 4: We have to show that $\{v_n\}_{n\in\mathbb{N}}$ converges toward v_∞ with respect to L^2 norm (in deed, this holds for a specific subsequence). Let ε be a positive real number. We can apply lemma 2.1.17 with $\mathcal{M}$ and $\frac{\varepsilon}{4}$. Let n_0 be an integer with the property declared in lemma 2.1.17. By definition of v_∞, we can make the following assumptions:

- $\left\| \left(d + \sum_{j=1}^{n_0} v_\infty^j \right) - v_\infty \right\|_{L^2} \leq \frac{\varepsilon}{4}$;
- if n is an integer greater than or equal to n_0, then
$$\left\| \left(\sum_{j=1}^{n_0} v_n^j \right) - \left(\sum_{j=1}^{n_0} v_\infty^j \right) \right\|_{L^2} \leq \frac{\varepsilon}{4};$$
- if n is an integer greater than or equal to n_0, then
$$|v_n(0) - d| \leq \frac{\varepsilon}{4}.$$

Hence, if n is an integer such that $n \geq n_0$, the following inequalities hold true:

$$\begin{aligned}
\|v_n - v_\infty\|_{L^2} &\leq \|v_n(0) - d\|_{L^2} + \left\| \left(\sum_{i\geq 1} v_n^i \right) - \left(\sum_{i \geq 1} v_\infty^i \right) \right\|_{L^2} \\
&\leq \|v_n(0) - d\|_{L^2} + \left\| \left(\sum_{i\geq 1} v_n^i \right) - \left(\sum_{i=1}^{n_0} v_n^i \right) \right\|_{L^2} \\
&\quad + \left\| \left(\sum_{i=1}^{n_0} v_n^i \right) - \left(\sum_{i=1}^{n_0} v_\infty^i \right) \right\|_{L^2} + \left\| \left(\sum_{i=1}^{n_0} v_\infty^i \right) - \left(\sum_{i\geq 1} v_\infty^i \right) \right\|_{L^2} \\
&\leq 4 \cdot \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Step 5: In conclusion, we have to show that

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) \geq \mathcal{MS}_\psi(v_\infty).$$

As a matter of fact, this holds for a specific subsequence subsequence. Let n be a natural number. If i, j are positive integers and $i \neq j$, we notice that $\mathcal{S}\left(v_n\right)^i$ and $\mathcal{S}\left(v_n\right)^j$ are disjoint. Hence, it holds that

$$\mathcal{MS}_\psi(v_n) = \sum_{x\in\mathcal{S}(v_n)} \psi(\Delta v_n\left(x\right)) = \sum_{i\geq 1} \sum_{x\in\mathcal{S}(v_n)^i} \psi(\Delta v_n\left(x\right)) = \sum_{i\geq 1} \mathcal{MS}_\psi(v_n^i).$$

We have that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) &= \liminf_{n\to+\infty} \sum_{i\geq 1} \mathcal{MS}_\psi(v_n^i) && (2.5)\\
&\geq \sum_{i\geq 1} \left(\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n^i)\right)\\
&= \sum_{i\geq 1} \left(\liminf_{n\to+\infty} \sum_{x\in\mathcal{S}(v_n)^i} \psi(\Delta v_n^i\left(x\right))\right) && (2.6)\\
&= \sum_{i\geq 1} \left(\liminf_{n\to+\infty} \sum_{t=1}^{\beta^i} \psi(\Delta_n^{i;t})\right) && (2.7)\\
&\geq \sum_{i\geq 1} \left(\sum_{t=1}^{\beta^i} \psi(\Delta_\infty^{i;t})\right) && (2.8)\\
&\geq \mathcal{MS}_\psi(v_\infty).
\end{aligned}$$

In (2.5) we used the Fatou's lemma; in (2.6) we used the fact that if i, n are positive integers such that $n \geq i$, then

$$\mathcal{S}\left(v_n^i\right) = \left\{x_n^{i;1}; \ldots; x_n^{i;\beta^i}\right\};$$

in (2.7) we used the lower semicontinuity of ψ and the definition of $\Delta_\infty^{i;t}$; in (2.8) we used the minimality of the canonical representative (see 2.1.15). □

Proposition 2.1.19. *Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{PJ}$. Let $\mathcal{M}$ be a natural number such that for all n in $\mathbb{N}$ it holds that*

- $\operatorname{card} \mathcal{S}\left(v_n\right) \leq \mathcal{M}$;
- $\|v_n\|_{L^2} \leq \mathcal{M}$.

There exists a subsequence, not relabelled, with the following properties:

- *there exists a natural number β such that $\operatorname{card} \mathcal{S}\left(v_n\right) = \beta$ for all n in $\mathbb{N}$. Hence, we can denote*

  $$\mathcal{S}\left(v_n\right) := \left\{x_n^1; \ldots; x_n^\beta\right\},$$

  with the assumption that

  $$0 := x_n^0 < x_n^1 < \cdots < x_n^\beta < x_n^{\beta+1} := 1.$$

We also denote

$$v_n \coloneqq v_n(0) + \sum_{t=1}^{\beta} \Delta_n^t \mathbb{1}_{[x_n^t,1]}.$$

- *For all integer i in $\{0;\dots;\beta+1\}$ there exists x_∞^i in $[0,1]$ such that*

$$\lim_{n\to+\infty} x_n^i = x_\infty^i.$$

- *Let r,t be integers in $\{0;\dots;\beta+1\}$. We declare that r,t are equivalent if and only if $x_\infty^t = x_\infty^r$. This induces a partition on $\{0;\dots;\beta+1\}$ into disjoint sets. In other words, there exist a natural number α and a collection of pairwise disjoint classes of equivalence*

$$\{\mathcal{J}^0;\dots;\mathcal{J}^{\alpha+1}\}$$

that covers $\{0;\dots;\beta+1\}$. For all integer s in $\{0;\dots;\alpha+1\}$, we can well define $y_\infty^s \coloneqq x_\infty^r$, where r is an index in $\mathcal{J}^s$. For all s in $\{1;\dots;\alpha\}$ there exists a real number Θ_∞^s such that

$$\lim_{n\to+\infty} \sum_{t\in\mathcal{J}^s} \Delta_n^t = \Theta_\infty^s;$$

there exists a real number Θ_∞^0 such that

$$\lim_{n\to+\infty} \sum_{t\in\mathcal{J}^0} v_n(0) + \Delta_n^t = \Theta_\infty^0.$$

- *If we define*

$$v_\infty = \sum_{i=0}^{\alpha} \Theta_\infty^i \mathbb{1}_{[y_\infty^i,1]},$$

$\{v_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward v_∞. In particular, we have that $\operatorname{card}\mathcal{S}(v_\infty) \le \alpha$.

- *If we define $\mathcal{MS}_\psi$ as in 2.1.14, it holds that*

$$\mathcal{MS}_\psi(v_\infty) \le \liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n).$$

Proof. **Step 1**: It's immediate to see that there exists a natural number β such that $\operatorname{card}\mathcal{S}(v_n) = \beta$ for all n in $\mathbb{N}$. Hence, we can denote

$$\mathcal{S}(v_n) \coloneqq \left\{x_n^1;\dots;x_n^\beta\right\},$$

with the assumption that

$$0 \coloneqq x_n^0 < x_n^1 < \cdots < x_n^\beta < x_n^{\beta+1} \coloneqq 1.$$

We also denote

$$v_n \coloneqq v_n(0) + \sum_{t=1}^{\beta} \Delta_n^t \mathbb{1}_{[x_n^t,1]}.$$

Since $[0,1]^\beta$ is a compact set in $\mathbb{R}^\beta$, there exist a subsequence, not relabelled, and real numbers $\left\{x_\infty^1;\dots;x_\infty^\beta\right\}$ such that for all t in $\{1;\dots;\beta\}$ it holds that

$$x_\infty^t \coloneqq \lim_{n\to+\infty} x_n^t.$$

In particular, we have that

$$0 := x_\infty^0 \leq x_\infty^1 \leq \cdots \leq x_\infty^\beta \leq x_\infty^{\beta+1} := 1.$$

Let i, j be in $\{0; \dots; \beta+1\}$; we say that i, j are equivalent if and only if $x_\infty^i = x_\infty^j$. This induces a partition on $\{0; \dots; \beta+1\}$ into disjoint sets. In other words, there exist a natural number α and a collection of pairwise disjoint classes of equivalence

$$\left\{\mathcal{J}^0; \dots; \mathcal{J}^{\alpha+1}\right\}$$

that covers $\{0; \dots; \beta+1\}$. For all integer s in $\{0; \dots; \alpha+1\}$, we can well define $y_\infty^s := x_\infty^r$, where r is an index in $\mathcal{J}^s$. For all n in $\mathbb{N}$, we also denote

$$\Delta_n^0 := v_n(0),$$

$$\Delta_n^{\beta+1} := v_n(1).$$

Having said that, for all n in $\mathbb{N}$ we have that

$$v_n = \sum_{t=0}^{\beta+1} \Delta_n^t \mathbb{1}_{[x_n^t,1]} = \sum_{s=0}^{\alpha+1} \left(\sum_{t \in \mathcal{J}^s} \Delta_n^t \mathbb{1}_{[x_n^t,1]} \right).$$

Step 2: Let s be an integer in $\{0; \dots; \alpha+1\}$; for all n in $\mathbb{N}$ we define

$$\Theta_n^s := \sum_{t \in \mathcal{J}^s} \Delta_n^t.$$

We claim that for all s in $\{0; \dots; \alpha\}$ the sequence $\{\Theta_n^s\}_{n\in\mathbb{N}}$ is bounded. By contradiction, let us assume that we can well define

$$s_0 := \min\left\{ s \in \{0; \dots; \alpha\} \mid \exists \left\{\Theta_{n_k}^s\right\}_{k\in\mathbb{N}} : \lim_{k\to+\infty} \left|\Theta_{n_k}^s\right| = +\infty \right\}.$$

Let us assume that s_0 is a positive integer. Hence, there exists a positive real number $\mathcal{M}_1$ such that $|\Theta_n^s| \leq \mathcal{M}_1$ for all s in $\{0; \dots; s_0-1\}$ for all n in $\mathbb{N}$. By definition, for all t in $\mathcal{J}^{s_0}$ it holds that

$$\lim_{n\to+\infty} x_n^t = y_\infty^{s_0}.$$

Let t be in $\mathcal{J}^{s_0+1}$: we remark that it is fundamental to assume that $s_0 < \alpha+1$. By definition, for all t in $\mathcal{J}^{s_0+1}$ we have that

$$\lim_{n\to+\infty} x_n^t = y_\infty^{s_0+1}$$

and $y_\infty^{s_0} < y_\infty^{s_0+1}$. We denote

$$\varepsilon_0 := \max\left\{ \frac{y_\infty^{s_0+1} - y_\infty^{s_0}}{4}; \frac{y_\infty^{s_0} - y_\infty^{s_0-1}}{4} \right\}.$$

There exists a natural number n_0 such that for all n greater than or equal to n_0 it holds that

- if s is an integer in $\{0; \dots; s_0-1\}$ and t is an integer in $\mathcal{J}^s$, then

$$x_n^t < y_\infty^{s_0} - \varepsilon_0;$$

- if s is an integer in $\{s_0+1;\ldots;\alpha+1\}$ and t is an integer in $\mathcal{J}^s$, then

$$x_n^t > y_\infty^{s_0} + \varepsilon_0;$$

- if t is an integer in $\mathcal{J}^{s_0}$, then

$$y_\infty^{s_0} - \frac{\varepsilon_0}{2} < x_n^t < y_\infty^{s_0} + \frac{\varepsilon_0}{2}.$$

Having said that, if n is any integer greater than or equal to n_0, the following inequalities hold true:

$$\begin{aligned}
\mathcal{M}^2 \geq \|v_n\|_{L^2}^2 &= \int_0^1 \left(\sum_{s=0}^{\alpha+1}\left(\sum_{t\in\mathcal{J}^s} \Delta_n^t \mathbb{1}_{[x_n^t,1]}(x)\right)\right)^2 dx \\
&\geq \int_{y_\infty^{s_0}+\frac{\varepsilon_0}{2}}^{y_\infty^{s_0}+\varepsilon_0} \left[\sum_{s=0}^{s_0}\left(\sum_{t\in\mathcal{J}^s}\Delta_n^t\right)\right]^2 dx \\
&= \frac{\varepsilon_0}{2}\left[\sum_{s=0}^{s_0-1}\left(\sum_{t\in\mathcal{J}^s}\Delta_n^t\right) + \sum_{t\in\mathcal{J}^{s_0}}\Delta_n^t\right]^2 \\
&= \frac{\varepsilon_0}{2}\left[\left(\sum_{s=0}^{s_0-1}\Theta_n^s\right) + \Theta_n^{s_0}\right]^2.
\end{aligned}$$

By definition of s_0 and $\mathcal{M}_1$, for all n in $\mathbb{N}$ for all s in $\{0;\ldots;s_0-1\}$ it holds that

$$\left|\sum_{s=0}^{s_0-1}\Theta_n^s\right| \leq s_0 \cdot \mathcal{M}_1.$$

So, the absurd follows taking the superior limit.

If s_0 is equal to 0, the procedure that we have just described in many details can be easily adapted. To be precise, we remark that the sequence $\{\Theta_n^{\alpha+1}\}_{n\in\mathbb{N}}$ can be unbounded.

Step 3: Because of the compactness of closed balls in $\mathbb{R}^{\alpha+1}$, there exist a subsequence, not relabelled, and real numbers $\{\Theta_\infty^0;\ldots;\Theta_\infty^\alpha\}$ such that for all s in $\{0;\ldots;\alpha\}$ it holds that

$$\lim_{n\to+\infty}\Theta_n^s = \Theta_\infty^s.$$

We define

$$v_\infty := \sum_{s=0}^{\alpha}\Theta_\infty^s \mathbb{1}_{[y_\infty^s,1]}.$$

Obviously, v_∞ is in $\mathcal{PJ}$. We claim that $\{v_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward v_∞. For all n in $\mathbb{N}$ we have that

$$v_n = \sum_{s=0}^{\alpha}\left(\sum_{t\in\mathcal{J}^s}\Delta_n^t \mathbb{1}_{[x_n^t,1]}\right).$$

Hence, it is enough to show that

- $\left\{\sum_{t\in\mathcal{J}^s}\Delta_n^t\mathbb{1}_{[x_n^t,1]}\right\}_{n\in\mathbb{N}}$ converges pointwise for all x in $(0,1)\setminus\{y_\infty^s\}$ toward $\Theta_\infty^s\mathbb{1}_{[y_\infty^s,1]}$ for all s in $\{0;\ldots;\alpha\}$;

- $\left\{\sum_{t\in\mathcal{J}^{\alpha+1}}\Delta_n^t\mathbb{1}_{[x_n^t,1]}\right\}_{n\in\mathbb{N}}$ converges pointwise toward 0.

As for the first statement, let s be an integer in $\{0;\ldots;\alpha\}$; let and x be in $[0,y_\infty^s)$. There exists n_0 in $\mathbb{N}$ such that $x_n^t > x$ for all integer $n\geq n_0$ for all t in $\mathcal{J}^s$. So, if n is an integer greater than or equal to n_0, then

$$\sum_{t\in\mathcal{J}^s}\Delta_n^t\mathbb{1}_{[x_n^t,1]}(x) = 0 = \Theta_\infty^s\mathbb{1}_{[y_\infty^s,1]}(x).$$

If x is in $(y_\infty^s,1]$, there exists n_0 in $\mathbb{N}$ such that $x > x_n^t$ for all integer $n\geq n_0$ for all t in $\mathcal{J}^s$. So, if n is an integer greater than or equal to n_0, then

$$\lim_{n\to+\infty}\sum_{t\in\mathcal{J}^s}\Delta_n^t\mathbb{1}_{[x_n^t,1]}(x) = \lim_{n\to+\infty}\sum_{t\in\mathcal{J}^s}\Delta_n^s = \lim_{n\to+\infty}\Theta_n^s = \Theta_\infty^s = \Theta_\infty^s\mathbb{1}_{[y_\infty^s,1]}(x).$$

As for the second statement, it can be similarly proved.

Step 4: In conclusion, the following inequalities hold true:

$$\begin{aligned}
\liminf_{k\to+\infty}\mathcal{MS}_\psi(v_{n_k}) &= \liminf_{k\to+\infty}\sum_{t=1}^{\beta}\psi\left(\Delta_n^t\right)\\
&\geq \liminf_{k\to+\infty}\sum_{s=1}^{\alpha}\left(\sum_{t\in\mathcal{J}^s}\psi\left(\Delta_{n_k}^t\right)\right) &(2.9)\\
&\geq \liminf_{k\to+\infty}\sum_{s=1}^{\alpha}\psi\left(\sum_{t\in\mathcal{J}^s}\Delta_{n_k}^t\right)\\
&= \liminf_{k\to+\infty}\sum_{s=1}^{\alpha}\psi\left(\Theta_{n_k}^s\right) &(2.10)\\
&= \sum_{s=1}^{\alpha}\psi\left(\Theta_\infty^s\right) &(2.11)\\
&= \mathcal{MS}_\psi(v_\infty).
\end{aligned}$$

In (2.9) we used the fact that ψ is subadditive; in (2.10) we used the fact that ψ is lower semicontinuous; in (2.11) we used the characterization of the essential discontinuities of v_∞ and the fact that $\psi(0) = 0$. □

Theorem 2.1.20 (Compactness and lower semicontinuity of the generalized Mumford-Shah functional in $\mathcal{PJ}$)**.**
Let ψ be a weight function as in 2.1.11; let us define $\mathcal{MS}_\psi$ as in 2.1.14. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{PJ}$. We assume that there exists a real number $\mathcal{M}$ such that for all n in $\mathbb{N}$ it holds that

- *$\mathcal{MS}_\psi(v_n)\leq\mathcal{M}$;*

- $\|v_n\|_{L^2} \leq \mathcal{M}$.

Then, there exist a subsequence, not relabelled, and a function v_∞ *in* $\mathcal{PJ}$ *such that* $\{v_n\}_{n\in\mathbb{N}}$ *converges pointwise for almost every* x *in* $(0,1)$ *toward* v_∞ *and*

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) \geq \mathcal{MS}_\psi(v_\infty).$$

In particular, $\mathcal{MS}_\psi$ *is a lower semicontinuous functional.*

Proof. **Step 1**: Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{PJ}$ as in the hypothesis. As declared in 2.1.8, for all n in $\mathbb{N}$ we have that

$$v_n = \left(\sum_{i\geq 1} v_n^i\right) + v_n(0).$$

For all n in $\mathbb{N}$ we define

$$\nu_n := \sum_{i\geq 2} v_n^i.$$

We immediately notice that for all n in $\mathbb{N}$ the following properties hold true:

- if x is any point in $\mathcal{S}\left(\nu_n\right)$, then $|\Delta\nu_n\left(x\right)| = |\Delta v_n\left(x\right)| \leq 1$;
- $\nu_n(0) = 0$;
- $\mathcal{MS}_\psi(\nu_n) \leq \mathcal{MS}_\psi(v_n) \leq \mathcal{M} + 1$.

Thanks to proposition 2.1.18, up to subsequences, not relabelled, there exists ν_∞ in $\mathcal{PJ}$ such that $\{\nu_n\}_{n\in\mathbb{N}}$ converges toward ν_∞ with respect to L^2 norm and

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(\nu_n) \geq \mathcal{MS}_\psi(\nu_\infty).$$

Up to further subsequences, not relabelled, we can assume that the convergence is pointwise for almost every x in $(0,1)$.

Thanks to the triangular inequality, there exists a positive real number $\mathcal{M}_1$ such that $\|v_n^1 + v_n(0)\|_{L^2} \leq \mathcal{M}_1$ for all n in $\mathbb{N}$. Moreover, for all n in $\mathbb{N}$ we have that

- $\mathcal{MS}_\psi(v_n^1) \leq \mathcal{M} + 1$;
- $|\Delta v_n^1\left(x\right)| \geq 1$ for all x in $\mathcal{S}\left(v_n^1\right)$.

If we define $\mathcal{I}_\psi(1)$ as in 2.1.12 and we recall that is a positive real number (see 2.1.13), for all n in $\mathbb{N}$ the following inequalities hold true:

$$\mathcal{M} + 1 \geq \mathcal{MS}_\psi(v_n^1) = \sum_{x\in\mathcal{S}(v_n^1)} \psi(\Delta v_n^1\left(x\right)) \geq \operatorname{card}\mathcal{S}\left(v_n^1\right)\cdot\mathcal{I}_\psi(1);$$

in other words, we have that

$$\operatorname{card}\mathcal{S}\left(v_n^1\right) \leq \frac{\mathcal{M}+1}{\mathcal{I}_\psi(1)}.$$

Thanks to proposition 2.1.19, there exist another subsequence, not relabelled, and a function ν^1_∞ in $\mathcal{PJ}$ such that $\{v^1_n + v_n(0)\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward ν^1_∞ and

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v^1_n + v_n(0)) \geq \mathcal{MS}_\psi(\nu^1_\infty).$$

To resume, if we define $v_\infty := \nu^1_\infty + \nu_\infty$, we have that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) &= \liminf_{n\to+\infty} \mathcal{MS}_\psi\left(v^1_n + v_n(0) + \nu_n\right) && (2.12)\\
&= \liminf_{n\to+\infty} \left[\mathcal{MS}_\psi\left(v^1_n + v_n(0)\right) + \mathcal{MS}_\psi(\nu_n)\right]\\
&\geq \liminf_{n\to+\infty} \mathcal{MS}_\psi(\nu_n) + \liminf_{k\to+\infty} \mathcal{MS}_\psi\left(v^1_n\right) && (2.13)\\
&\geq \mathcal{MS}_\psi(\nu_\infty) + \mathcal{MS}_\psi(\nu^1_\infty) && (2.14)\\
&\geq \mathcal{MS}_\psi(v_\infty).
\end{aligned}$$

In (2.12) we used 2.1.8 and the definition of $\mathcal{MS}_\psi$; (2.13) have already been discussed; in (2.14) we used the fact that $\mathcal{MS}_\psi$ is subadditive (see 2.1.16).

Step 2: As for the lower semicontinuity, let v_∞ be a function in L^2; let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in L^2 that converges toward v_∞ with respect to L^2 norm. We have to show that

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) \geq \mathcal{MS}_\psi(v_\infty).$$

If the left hand side is $+\infty$, the conclusion is trivial. Hence, up to subsequences, not relabelled, we can assume that there exists a real number $\mathcal{M}$ such that $\mathcal{MS}_\psi(v_n) \leq \mathcal{M}+1$ for all n in $\mathbb{N}$, the inferior limit is actually a limit, i. e.

$$\lim_{n\to+\infty} \mathcal{MS}_\psi(v_n) = \mathcal{M},$$

and $\{v_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward v_∞. As shown in the previous step, we can conclude that

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) = \mathcal{M} \geq \mathcal{MS}_\psi(v_\infty).$$

□

2.2 Generalized Mumford-Shah functional in $\mathcal{SBV}$

We introduce the space of the Special Functions of Bounded Variations ($\mathcal{SBV}$); we define the generalized Mumford-Shah functional in $\mathcal{SBV}$; we state and prove a lower semicontinuity theorem and some compactness result.

2.2.1 The space of functions $\mathcal{SBV}$

Definition 2.2.1 ($\mathcal{SBV}$).
Let $\mathscr{U}$ be a class of functions in L^2 coinciding almost everywhere. Let us assume that $\Delta\mathscr{U}(x)$ is well defined for all x in $(0,1)$ (see 2.1.4). Let us define $\mathcal{S}(\mathscr{U})$ as in 2.1.5. Let us assume that

- $\mathcal{S}(\mathscr{U})$ is at most countable;

- $\sum_{x\in\mathcal{S}(\mathscr{U})} |\Delta\mathscr{U}\,(x)| < +\infty$;
- $\mathscr{U}\,(0)^+$ is well defined and it is a real number.

Hence, we can well define the function $v : [0,1] \to \mathbb{R}$ such that

$$v \coloneqq \mathscr{U}\,(0)^+ + \sum_{y\in\mathcal{S}(\mathscr{U})} \Delta\mathscr{U}\,(y)\, \mathbb{1}_{[y,1]}.$$

We denote as $\mathscr{V}$ the class of the functions that coincide with v for almost every x in $[0,1]$. We refer to v as the jump part of $\mathscr{U}$. We define $\mathscr{W} \coloneqq \mathscr{U} - \mathscr{V}$. We say that $\mathscr{U}$ is in $\mathcal{SBV}$ if and only if $\mathscr{W}$ is in $W^{1;1}$. We denote as $w : [0,1] \to \mathbb{R}$ the continuous representative of $\mathscr{W}$; we refer to w as the absolutely continuous part of $\mathscr{U}$. We can also define $u : [0,1] \to \mathbb{R}$ such that $u \coloneqq w + v$. We refer to u as the canonical representative of $\mathscr{U}$. We denote $\dot{u} \coloneqq \dot{w}$ and we say that it is the weak derivative of $\mathscr{U}$.

Remark 2.2.2. In the setting of definition 2.2.1, we notice that u belongs to $\mathscr{U}$. Obviously, any class in $\mathcal{SBV}$ is completely determined by its canonical representative. Let $\mathscr{U}_1, \mathscr{U}_2$ be classes in $\mathcal{SBV}$; let us denote as $u_1 \coloneqq w_1 + v_1$ and $u_2 \coloneqq w_2 + v_2$ respectively the canonical representatives of $\mathscr{U}_1$ and $\mathscr{U}_2$ as declared in definition 2.2.1. If $u_1(x) = u_2(x)$ for all x in $[0,1]$, then $\mathscr{U}_1 = \mathscr{U}_2$ in $\mathcal{SBV}$. If we assume that $\mathscr{U}_1 = \mathscr{U}_2$, by definition 2.2.1 it holds that $v_1(x) = v_2(x)$ for all x in $[0,1]$; so, w_1 and w_2 are the continuous representatives of the same class of functions in $W^{1;1}$; therefore, $w_1(x) = w_2(x)$ for all x in $[0,1]$. Having said that, we can identify $\mathscr{U}$ with its canonical representative. If we recall definition 2.1.8, it holds that

$$u = u(0) + w + \sum_{i\geq 1} v^i.$$

We also introduce the following notation, that will be very useful later.

Definition 2.2.3 ($\mathcal{S}\tilde{\mathcal{B}}\mathcal{V}$)**.**
Let u be in $\mathcal{SBV}$ as declared in 2.2.2. We say that u is in $\mathcal{S}\tilde{\mathcal{B}}\mathcal{V}$ if $\operatorname{card}\mathcal{S}\,(u) < +\infty$. From now on, unless otherwise specified, we represent u as

$$u \coloneqq w + u(0) + \sum_{i=1}^{k} \Delta^i \mathbb{1}_{[x^i,1]},$$

with the following assumption:

- w is the absolutely continuous part of u, i. e. w is in $W^{1;1}$, and $w(0) = 0$;
- $\mathcal{S}\,(u) \coloneqq \left\{x^1; \ldots; x^k\right\}$ and $0 \coloneqq x^0 < x^1 < \cdots < x^k < x^{k+1} \coloneqq 1$;
- for all integer i in $\{1; \ldots; k\}$, we denote $\Delta^i \coloneqq \Delta u\,(x^i)$;
- v is the jump part of u, i. e. v is in $\mathcal{PJ}$, and $v(0) = 0$; more precisely, it holds that
$$v \coloneqq \sum_{i=1}^{k} \Delta^i \mathbb{1}_{[x^i,1]}.$$

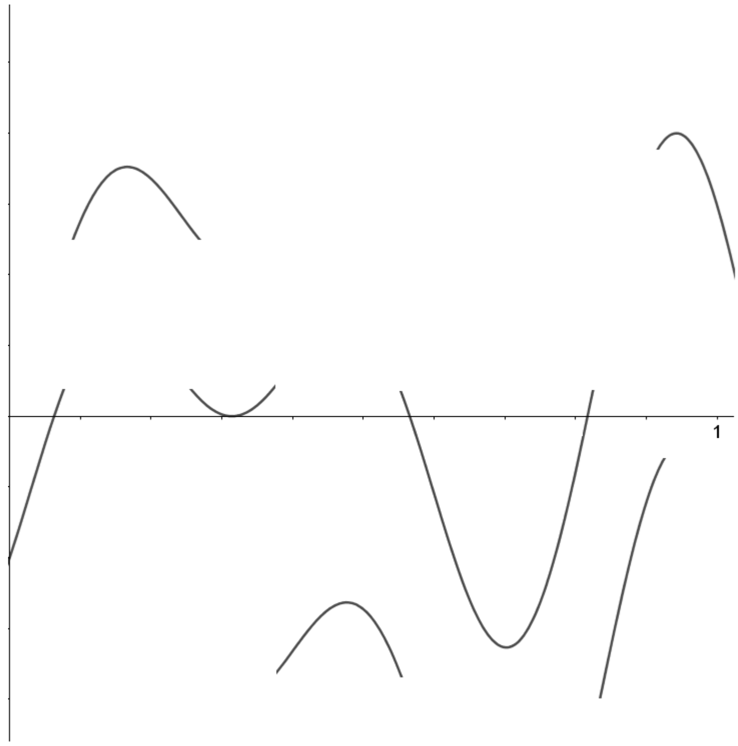

Figure 2.2: Example of a function u in $\tilde{\mathcal{SBV}}$

2.2.2 Weak formulation in $\mathcal{SBV}$

Definition 2.2.4 (Generalized Mumford-Shah functional in $\mathcal{SBV}$).
Let φ be a Young function as in 1.1.1; let ψ be a weight function as in 2.1.11. Let us define $\mathcal{D}_\varphi$ as in 1.3.1 and $\mathcal{MS}_\psi$ as in 2.1.14. We define the generalized Mumford-Shah functional in $\mathcal{SBV}$ $\mathcal{MS}_{\varphi;\psi} : L^2 \to [0, +\infty]$ as follows:

- if u is in $\mathcal{SBV}$ and $u = w + v$ is the canonical decomposition as in 2.2.2, then

$$\mathcal{MS}_{\varphi;\psi}(u) := \mathcal{D}_\varphi(w) + \mathcal{MS}_\psi(v) = \int_0^1 \varphi(\dot{u}(x))\, dx + \sum_{x \in \mathcal{S}(u)} \psi(\Delta u\,(x));$$

- if u is in $L^2 \setminus \mathcal{SBV}$, then

$$\mathcal{MS}_{\varphi;\psi}(u) := +\infty.$$

2.2.3 Compactness and lower semicontinuity in $\mathcal{SBV}$

Theorem 2.2.5 (Compactness and lower semicontinuity of the generalized Mumford-Shah functional in $\mathcal{SBV}$).
Let φ be a Young function as in 1.1.1; let ψ be a weight function as in 2.1.11; let us define $\mathcal{MS}_{\varphi;\psi}$ as in 2.2.4. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in L^2. We assume that there exists a real number $\mathcal{M}$ such that for all n in $\mathbb{N}$ it holds that:

- $\mathcal{MS}_{\varphi;\psi}(u_n) \leq \mathcal{M}$;

- $\|u_n\|_{L^2} \leq \mathcal{M}$.

There exist a subsequence, not relabelled, and a function u_∞ with the following properties:

- *u_∞ is in $\mathcal{SBV}$;*
- *if we consider the usual decomposition $u_n := w_n + v_n$, we have that $\{w_n\}_{n\in\mathbb{N}}$ converges uniformly in $[0,1]$ toward w_∞ and $\{v_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward v_∞. In other words, the sequence of the jump parts and the sequence of the absolutely continuous parts converge separately toward the jump part of u_∞ and the absolutely continuous part of u_∞, respectively.*
- $\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \geq \mathcal{MS}_{\varphi;\psi}(u_\infty)$.

In particular, $\mathcal{MS}_{\varphi;\psi}$ is a lower semicontinuous functional.

Proof. **Step 1**: Under our hypothesis, $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{SBV}$. Let $u_n = w_n + v_n$ be the canonical decomposition as declared in 2.2.1. By definition of $\mathcal{MS}_{\varphi;\psi}$, we immediately notice that $\mathcal{D}_\varphi(w_n) \leq \mathcal{M}$ and $\mathcal{MS}_\psi(v_n) \leq \mathcal{M}$ for all n in $\mathbb{N}$. Moreover, $w_n(0)$ is equal to 0 for all n in $\mathbb{N}$. Thanks to theorem 1.3.9, there exists a subsequence, not relabelled, and a function w_∞ in $W^{1;1}$ such that $\{w_n\}_{n\in\mathbb{N}}$ converges toward w_∞ uniformly in $[0,1]$, $\{\dot{w}_n\}_{n\in\mathbb{N}}$ converges L^1-weakly toward $\dot{w}_\infty$ and

$$\liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n) \geq \mathcal{D}_\varphi(w_\infty).$$

In particular, $\{w_n\}_{n\in\mathbb{N}}$ converges toward w_∞ with respect to L^2 norm and $w_\infty(0) = 0$.

Thanks to the triangular inequality, there exists $\mathcal{M}_1$ in $\mathbb{R}$ such that $\|v_n\|_{L^2} \leq \mathcal{M}_1$ for all n in $\mathbb{N}$. Thanks to theorem 2.1.20, there exist a function v_∞ in $\mathcal{PJ}$ and a subsequence, not relabelled, such that $\{v_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward v_∞ and

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) \geq \mathcal{MS}_\psi(v_\infty).$$

If we define $u_\infty := w_\infty + v_\infty$, we have that

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) &= \liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n) + \mathcal{MS}_\psi(v_n)\\ &\geq \liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n) + \liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n)\\ &\geq \mathcal{D}_\varphi(w_\infty) + \mathcal{MS}_\psi(v_\infty)\\ &= \mathcal{MS}_{\varphi;\psi}(u_\infty).\end{aligned}$$

Step 2: As for the lower semicontinuity, let u_∞ be a function in L^2; let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in L^2 that converges toward u_∞ with respect to L^2 norm. We have to show that

$$\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \geq \mathcal{MS}_{\varphi;\psi}(u_\infty).$$

If the left hand side is $+\infty$, the conclusion is trivial. Hence, up to subsequences, not relabelled, it is not restrictive to assume that there exists a real number $\mathcal{M}$ such that $\mathcal{MS}_{\varphi;\psi}(u_n) \leq \mathcal{M} + 1$ for all n in $\mathbb{N}$, the inferior limit is actually a limit, i. e.

$$\lim_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) = \mathcal{M},$$

$\{w_n\}_{n\in\mathbb{N}}$ converges uniformly in $[0,1]$ toward w_∞ and $\{v_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward v_∞. As shown in the previous step, we can conclude that

$$\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) = \mathcal{M} \geq \mathcal{MS}_{\varphi;\psi}(u_\infty).$$

□

2.3 Generalized Mumford-Shah energy

Finally, we can define the functional $\mathscr{E}_{\varphi;\psi}$ as in 0.1 and show that it admits minimum.

Definition 2.3.1 (Generalized Mumford-Shah energy).
Let φ be a Young function as in 1.1.1; let ψ be a weight function as in 2.1.11; let us define $\mathcal{MS}_{\varphi;\psi}$ as in 2.2.4. Let $h : [0,1] \to \mathbb{R}$ be a function in L^2. We define the functional $\mathcal{E}_{\varphi;\psi} : L^2 \to [0,+\infty]$ such that

$$\mathscr{E}_{\varphi;\psi}(u) := \begin{cases} \displaystyle\int_0^1 (u-h)^2 \, dx + \mathcal{MS}_{\varphi;\psi}(u) & \text{if } u \in \mathcal{SBV}; \\ +\infty & \text{if } u \in L^2 \setminus \mathcal{SBV}. \end{cases}$$

Theorem 2.3.2 (Existence of the minimum via direct method).
Let $h : [0,1] \to \mathbb{R}$ be a function in L^2; let φ be a Young function as in 1.1.1; let ψ be a weight function as in 2.1.11. We define the functional $\mathscr{E}_{\varphi;\psi}$ as in 2.3.1. Then, the generalized Mumford-Shah energy admits minimum in $\mathcal{SBV}$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in L^2 such that

$$\lim_{n\to+\infty} \left| \mathscr{E}_{\varphi;\psi}(u_n) - \inf_{L^2} \mathscr{E}_{\varphi;\psi} \right| = 0.$$

We define $\gamma := \|h\|_{L^2}$; since $\mathscr{E}_{\varphi;\psi}(0)$ is equal to γ^2, it is not restrictive to assume that $\mathscr{E}_{\varphi;\psi}(u_n) \leq \gamma^2$ for all n in $\mathbb{N}$. In particular, $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{SBV}$ such that for all n in $\mathbb{N}$ it holds that:

- $\mathcal{MS}_{\varphi;\psi}(u_n) \leq \gamma^2$;
- $\|u_n\|_{L^2} \leq \|u_n - h\|_{L^2} + \|h\|_{L^2} \leq \sqrt{\mathscr{E}_{\varphi;\psi}(u_n)} + \|h\|_{L^2} \leq 2\gamma$.

Thanks to theorem 2.2.5, there exist a subsequence, not relabelled, and a function u_∞ is $\mathcal{SBV}$ such that $\{u_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward u_∞ and

$$\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \geq \mathcal{MS}_{\varphi;\psi}(u_\infty). \tag{2.15}$$

Thanks to the Fatou's lemma, we have that

$$\liminf_{n\to+\infty} \int_0^1 (h-u_n)^2 \, dx \geq \int_0^1 (h-u_\infty)^2 \, dx. \tag{2.16}$$

If we join (2.15) and (2.16), we obtain that

$$\liminf_{n\to+\infty} \mathscr{E}_{\varphi;\psi}(u_n) \geq \mathscr{E}_{\varphi;\psi}(u_\infty).$$

In conclusion, we have that

$$\begin{aligned}
\inf_{L^2} \mathscr{E}_{\varphi;\psi} &\leq \mathscr{E}_{\varphi;\psi}(u_\infty) \\
&\leq \liminf_{n\to+\infty} \mathscr{E}_{\varphi;\psi}(u_n) \\
&= \lim_{n\to+\infty} \left(\mathscr{E}_{\varphi;\psi}(u_n) - \inf_{L^2} \mathscr{E}_{\varphi;\psi} \right) + \inf_{L^2} \mathscr{E}_{\varphi;\psi} \\
&= \inf_{L^2} \mathscr{E}_{\varphi;\psi}.
\end{aligned}$$

So, u_∞ is a function that minimizes the Mumford-Shah energy. □

Chapter 3

A discrete approximation

We introduce the notion of Γ-convergence. We define a family of problems that approximate $\mathscr{E}_{\varphi;\psi}$ is the sense of the Γ-convergence; this allows to obtain the minimum and the minimizers of $\mathscr{E}_{\varphi;\psi}$ as limit of the sequences of the minima and minimizers of the approximating problems. The simplification turn out to be very relevant, because the approximating problems are set in finite-dimensional spaces.

3.1 Γ-convergence

Definition 3.1.1 (Γ-convergence)**.**
Let $(\mathbb{X}; d)$ be a metric space; let $\{F_n\}_{n\in\mathbb{N}}$ and F be functionals from $\mathbb{X}$ to $[0, +\infty]$. We say that F_n Γ-converges toward F as n approaches $+\infty$ with respect to the distance d if the following inequalities hold:

- (liminf inequality) if u is in $\mathbb{X}$ and $\{u_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathbb{X}$ that converges toward u, then we have that

$$F(u) \leq \liminf_{n\to+\infty} F_n(u_n); \tag{3.1}$$

- (limsup inequality) if u is in $\mathbb{X}$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ that converges toward u such that

$$F(u) \geq \limsup_{n\to+\infty} F_n(u_n). \tag{3.2}$$

  $\{u_n\}_{n\in\mathbb{N}}$ is called recovery sequence of u.

Proposition 3.1.2 (Stability under continuous perturbations)**.**
Let $(\mathbb{X}; d)$ be a metric space; let $\{F_n\}_{n\in\mathbb{N}}$ and F be functionals from $\mathbb{X}$ to $[0, +\infty]$. Let us assume that $\{F_n\}_{n\in\mathbb{N}}$ Γ-converges toward F with respect to the distance d. Let $G : \mathbb{X} \to [0, +\infty)$ be a continuous functional. Then, $\{F_n + G\}_{n\in\mathbb{N}}$ Γ-converges toward $F + G$ with respect to the distance d.

Proof. We have to show that (3.1) and (3.2) hold for $\{F_n + G\}_{n\in\mathbb{N}}$ and $F + G$.

- As for (3.1), let u be in $\mathbb{X}$; let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$ that converges toward u. Then, we have that

$$\liminf_{n\to+\infty} F_n(u_n) + G(u_n) \geq \liminf_{n\to+\infty} F_n(u_n) + \liminf_{n\to+\infty} G(u_n) \geq F(u) + G(u).$$

- As for (3.2), let u be in $\mathbb{X}$; let $\{u_n\}_{n\in\mathbb{N}}$ be a recovery sequence of u for F. We claim that it works for $F+G$. We have that

$$\limsup_{n\to+\infty} F_n(u_n) + G(u_n) \leq \limsup_{n\to+\infty} F_n(u_n) + \limsup_{n\to+\infty} G(u_n) \leq F(u) + G(u).$$

□

Definition 3.1.3 (Dense in energy)**.**
Let $(\mathbb{X}; d)$ be a metric space; let $F : \mathbb{X} \to [0, +\infty]$ be a functional. Let $\mathbb{D}$ a subset in $\mathbb{X}$ with the following property: for all u in $\mathbb{X}$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that

$$\lim_{n\to+\infty} d(u; u_n) = 0,$$

$$\lim_{n\to+\infty} F(u_n) = F(u).$$

We say that $\mathbb{D}$ is dense in energy for F.

Lemma 3.1.4. *Let $(\mathbb{X}; d)$ be a metric space; let $\{F_n\}_{n\in\mathbb{N}}$ and F be functionals between $\mathbb{X}$ and $[0, +\infty]$. Let $\mathbb{D}$ be a subset dense in energy for F_∞ as in 3.1.3. Let us suppose that for all b in $\mathbb{D}$ there exists a recovery sequence sequence $\{b_n\}_{n\in\mathbb{N}}$ for F in $\mathbb{D}$ (see 3.1.1), i. e.*

$$\lim_{n\to+\infty} d(b_n; b) = 0,$$

$$\limsup_{n\to+\infty} F_n(b_n) \leq F(b).$$

Then, for all x in $\mathbb{X}$ there exists a recovery sequence for F in $\mathbb{D}$.

Proof. Let x be in $\mathbb{X}$. By hypothesis, there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that

$$\lim_{n\to+\infty} d(b_n; x) = 0,$$

$$\lim_{n\to+\infty} F(b_n) = F(x).$$

For all n in $\mathbb{N}$ there exists a sequence $\left\{b_n^k\right\}_{k\in\mathbb{N}}$ in $\mathbb{D}$ such that

$$\lim_{k\to+\infty} d\left(b_n^k; b_n\right) = 0,$$

$$\limsup_{k\to+\infty} F_k\left(b_n^k\right) \leq F(b_n).$$

Let n be a natural number. There exists a positive integer k_n such that for all integer i greater than or equal to k_n it holds that

$$d\left(b_n^i; b_n\right) \leq \frac{1}{n},$$

$$F_i\left(b_n^i\right) \leq F(b_n) + \frac{1}{n}.$$

Without loss of generality, we can assume that the sequence $\{k_n\}_{n\in\mathbb{N}}$ is strictly monotonically increasing. For all integer i in $\{0; \ldots; k_1 - 1\}$ we define $x_i \coloneqq b_1^i$; let n be an integer greater than 1 for all integer i in $\{k_n; \ldots; k_{n+1} - 1\}$ we define $x_i \coloneqq b_n^i$. We claim that $\{x_i\}_{i\in\mathbb{N}}$ is a recovery sequence for F_∞ in $\mathbb{D}$. Let i be an integer greater than k_1;

there corresponds a natural number n such that i is in $\{k_n; \ldots; k_{n+1} - 1\}$; so, we have that

$$d(x_i; x) \leq d(x_i; b_n) + d(b_n; x) \leq \frac{1}{n} + d(x; b_n),$$

$$F_i(x_i) = F_i\left(b_n^i\right) \leq F(b_n) + \frac{1}{n}.$$

Let ε be a positive real number. By definition, there exists a positive integer n_0 such that for all integer n greater than n_0 it holds that

$$d(b_n; x) + \frac{1}{n} \leq \varepsilon,$$

$$F(b_n) + \frac{1}{n} \leq F(x) + \varepsilon.$$

Therefore, for all i greater than k_{n_0} we have that

$$d(x_i; x) \leq \varepsilon,$$

$$F_i(x_i) \leq F(x) + \varepsilon.$$

So, the thesis follows immediately. □

3.2 A family of approximating problems

The aim of this section is to introduce a discrete approximation of $\mathscr{E}_{\varphi;\psi}$ in the sense of the Γ-convergence.

Remark 3.2.1. Let φ be a Young function as in 1.1.1; let ψ be a weight function as in 2.1.11; let h be a function in L^2. We define $\mathcal{MS}_{\varphi;\psi}$ as in 2.2.4 and $\mathscr{E}_{\varphi;\psi}$ as in 2.3.1. So, we can consider the functional $\mathscr{E}_{\varphi;\psi} - \mathcal{MS}_{\varphi;\psi} : L^2 \to [0, +\infty)$ such that

$$\mathscr{E}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u) \coloneqq \int_0^1 (h - u)^2 \, dx.$$

Obviously, it is continuous with respect to L^2 norm. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a family of functionals between L^2 and $[0, +\infty]$ that Γ-converges toward $\mathcal{MS}_{\varphi;\psi}$ with respect to L^2 norm. Thanks to proposition 3.1.2, the sequence $\{\mathcal{F}_n + \mathscr{E}_{\varphi;\psi} - \mathcal{MS}_{\varphi;\psi}\}_{n\in\mathbb{N}}$ Γ-converges toward $\mathscr{E}_{\varphi;\psi}$ with respect to L^2 norm. So, it is enough to approximate $\mathcal{MS}_{\varphi;\psi}$ in the sense of the Γ-convergence.

Definition 3.2.2 (Sublinear weight function)**.**
Let ψ be a weight function as in 2.1.11. Let us assume that there exist positive real numbers A, B such that $\psi(x) \leq Ax + B$ for all x in $[0, +\infty)$. We say that ψ is a sublinear weight function.

Definition 3.2.3. Let ψ be a sublinear weight function as in 3.2.2. For all natural number n, we define the function $\psi_n : \mathbb{R} \to [0, +\infty)$ such that

$$\psi_n(x) \coloneqq 2^n \psi\left(\frac{x}{2^n}\right).$$

Let φ be a Young function as in 1.1.1. We also define

$$\Xi_n \coloneqq \{a \in (0, +\infty) \mid \forall x \geq a \ : \varphi(x) \geq \psi_n(x)\},$$

$$\xi_n \coloneqq \inf \Xi_n.$$

Remark 3.2.4. Since φ is a superlinear function and ψ is a sublinear function, for all natural number n the set Ξ_n is not empty; so, the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is well defined.

Lemma 3.2.5. *Let φ be a Young function as in 1.1.1; let ψ be a sublinear weight function as in 3.2.2. Let us define the sequences $\{\Xi_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ as in 3.2.3. The following conclusions hold true:*

- *ξ_n belongs to Ξ_n for all n in $\mathbb{N}$, i. e. $\Xi_n = [\xi_n, +\infty)$;*
- $\lim\limits_{n\to+\infty} \xi_n = +\infty$*;*
- $\lim\limits_{n\to+\infty} \dfrac{\xi_n}{2^n} = 0.$

Proof. **Step 1**: Let n be any natural number. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in Ξ_n that converges toward ξ_n. Since φ is continuous and ψ_n is lower semicontinuous, we have that

$$\varphi(\xi_n) = \lim_{k\to+\infty} \varphi(x_k) \geq \liminf_{k\to+\infty} \psi_n(x_k) \geq \psi_n(\xi_n).$$

Having said that, it immediately follows that $\Xi_n = [\xi_n, +\infty)$.

Step 2: Let x be a positive real number. By hypothesis on ψ, we have that

$$\lim_{n\to+\infty} 2^n \psi\left(\frac{x}{2^n}\right) = \lim_{n\to+\infty} x \frac{\psi\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}} = +\infty.$$

By contradiction, let us assume that there exist a positive real number $\mathcal{M}$ and a subsequence $\{\xi_{n_k}\}_{k\in\mathbb{N}}$ such that $\xi_{n_k} \leq \mathcal{M}$ for all k in $\mathbb{N}$. Then, for all x in $[\mathcal{M}+1, +\infty)$ for all k in $\mathbb{N}$ it holds that

$$\varphi(x) \geq 2^{n_k} \psi\left(\frac{x}{2^{n_k}}\right).$$

The absurd follows taking the limit as k approaches $+\infty$.

Step 3: We claim that $\left\{\frac{\xi_n}{2^n}\right\}_{n\in\mathbb{N}}$ is a bounded sequence. By definition of infimum, for all natural number n there exists x_n in $(\xi_n - 1, \xi_n) \cap (0, +\infty)$ such that $\psi_n(x_n) \geq \varphi(x_n)$, i. e.

$$\frac{\psi\left(\frac{x_n}{2_n}\right)}{\frac{x_n}{2^n}} \geq \frac{\varphi(x_n)}{x_n}.$$

So, we can conclude that $\{x_n\}_{n\in\mathbb{N}}$ converges toward $+\infty$. Since φ is a superlinear function, we have that

$$\lim_{n\to+\infty} \frac{\psi\left(\frac{x_n}{2_n}\right)}{\frac{x_n}{2^n}} = +\infty.$$

By contradiction, if there exists a subsequence $\left\{\frac{\xi_{n_k}}{2^{n_k}}\right\}_{k\in\mathbb{N}}$ that converges toward $+\infty$, then

$$\limsup_{x\to+\infty} \frac{\psi(x)}{x} = +\infty.$$

This is absurd because ψ is a sublinear growth function.

Step 4: We claim that

$$\lim_{n\to+\infty} \frac{\xi_n}{2^n} = 0.$$

By contradiction, let us assume that there exists a positive real number ε_0 and a subsequence, not relabelled, such that $\frac{\xi_n}{2^n} > \varepsilon_0$ for all n in $\mathbb{N}$. By definition of infimum, for all n in $\mathbb{N}$ there exists x_n in $(\varepsilon_0 2^n, \xi_n)$ such that $\varphi(x_n) < \psi_n(x_n)$, i. e.

$$\frac{\varphi(x_n)}{2^n} < \psi\left(\frac{x_n}{2^n}\right).$$

Since φ is monotonically increasing in $[0, +\infty)$, for all n in $\mathbb{N}$ it holds that

$$\varepsilon_0 \frac{\varphi(\varepsilon_0 2^n)}{\varepsilon_0 2^n} < \psi\left(\frac{x_n}{2^n}\right).$$

We have shown that there exists a positive real number $\mathcal{M}$ such that for all n in $\mathbb{N}$ it holds that

$$\frac{x_n}{2^n} \in \left[\varepsilon_0, \frac{\xi_n}{2^n}\right] \subseteq [\varepsilon_0, \mathcal{M}].$$

Since ψ is sublinear, we know that

$$\sup_{[\varepsilon_0, \mathcal{M}]} \{\psi(x)\} < +\infty.$$

So, for all n in $\mathbb{N}$ we have that

$$\varepsilon_0 \frac{\varphi(\varepsilon_0 2^n)}{\varepsilon_0 2^n} < \psi\left(\frac{x_n}{2^n}\right) \leq \sup_{[\varepsilon_0, \mathcal{M}]} \{\psi(x)\}.$$

Since φ is superlinear, the absurd follows taking the limit as n approaches $+\infty$. □

Definition 3.2.6 (Truncated potential)**.**
Let φ be a Young function as in 1.1.1; let ψ be a sublinear weight function as in 3.2.2. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be as in 3.2.3. For all n in $\mathbb{N}$ we define

$$f_n \coloneqq \min\{\varphi; \psi_n\}.$$

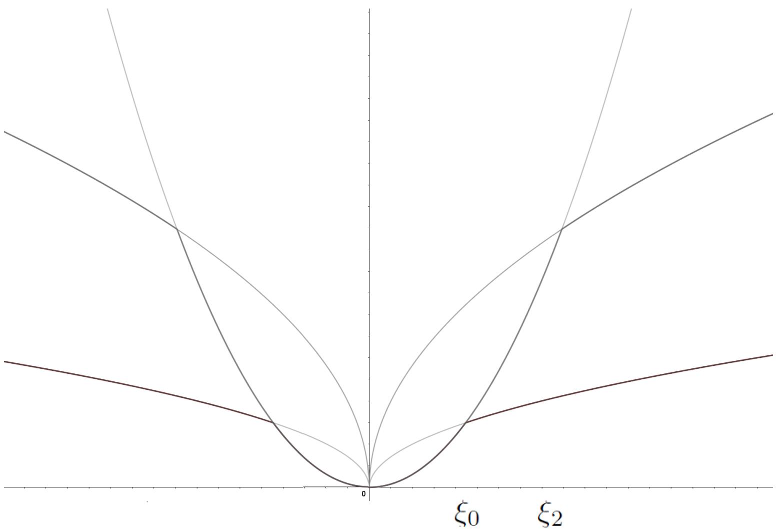

Figure 3.1: Example of truncated potentials f_0 and f_2, where $\varphi(x) = \frac{3}{10}x^2$, $\psi(x) = \sqrt{|x|}$

Remark 3.2.7. By definitions 3.2.6 and 3.2.3, if $|x| \geq \xi_n$, then $\varphi(x) \geq \psi_n(x)$ and $f_n(x) = \psi_n(x)$.

Definition 3.2.8 ($\mathcal{PC}_n$)**.**
Let n be a natural number. We define

$$\mathcal{PC}_n := \left\{ v \in \mathcal{PJ} \;\middle|\; \mathcal{S}(v) \subseteq \left\{ \frac{1}{2^n}; \dots; \frac{2^n - 1}{2^n} \right\} \right\}.$$

Definition 3.2.9 (Piecewise affine interpolation)**.**
Let n be in $\mathbb{N}$; let v be a function in $\mathcal{PC}_n$. We define ρ_v as the piecewise affine function that joins the points

$$\left(\frac{i}{2^n}; v\left(\frac{i}{2^n} \right) \right) \to \left(\frac{i+1}{2^n}; v\left(\frac{i+1}{2^n} \right) \right)$$

for all i in $\{0; \dots; 2^n - 1\}$.

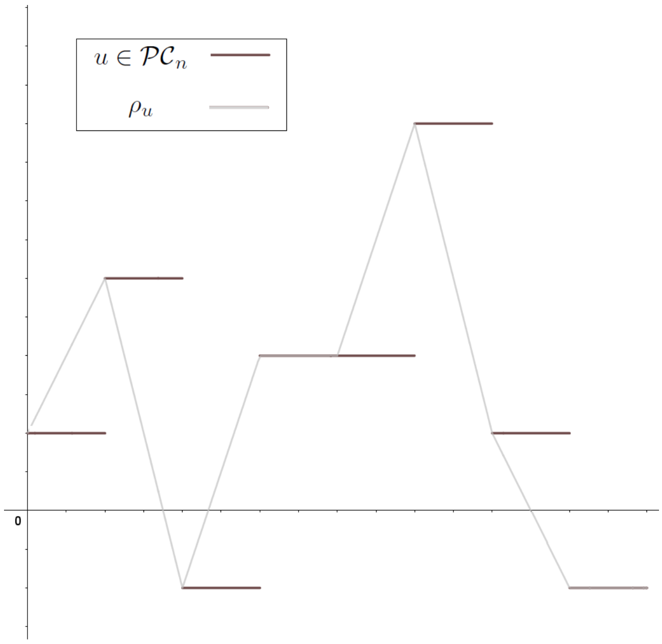

Figure 3.2: Example of a function u in $\mathcal{PC}_n$ and its piecewise affine interpolation ρ_u

Remark 3.2.10. In the setting of definition 3.2.9, to v in $\mathcal{PC}_n$ there corresponds a finite set $\{\Delta^0; \dots; \Delta^{2^n - 1}\}$ in $\mathbb{R}$ such that

$$v := \sum_{i=0}^{2^n - 1} \Delta^i \mathbb{1}_{\left[\frac{i}{2^n}, 1\right]}.$$

So, for all integer i in $\{0; \dots; 2^n - 2\}$ we have that

$$\dot{\rho_v}\Big|_{\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)} = 2^n \Delta^{i+1}$$

and

$$\dot{\rho_v}\Big|_{\left(\frac{2^n - 1}{2^n}, 1\right)} = 0.$$

Definition 3.2.11 (Approximating functionals)**.**
Let φ be a Young function as in 1.1.1; let ψ be a sublinear weight function as in 3.2.2. For all n in $\mathbb{N}$ we define f_n as in 3.2.6. We define $\mathcal{F}_n : L^2 \to [0, +\infty]$ such that

$$\mathcal{F}_n(u) \coloneqq \begin{cases} \displaystyle\sum_{i=1}^{2^n-1} \frac{1}{2^n} f_n(2^n \Delta^i) & \text{if } u = \displaystyle\sum_{i=0}^{2^n-1} \Delta^i \mathbb{1}_{[\frac{i}{2^n},1]} \in \mathcal{PC}_n \\ +\infty & \text{if } u \in L^2 \setminus \mathcal{PC}_n \end{cases}.$$

3.3 Γ-convergence of the approximating problems

Let φ be a Young function as in 1.1.1; let ψ be a sublinear weight function as in 3.2.2. We define the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ as in 3.2.11 and $\mathcal{MS}_{\varphi;\psi}$ as in 2.2.4. We claim that $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ Γ-converges toward $\mathcal{MS}_{\varphi;\psi}$ with respect to L^2 norm. By definition 3.1.1, we have to show limsup inequality and liminf inequality. So, in this section we consider φ, ψ, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\mathcal{MS}_{\varphi;\psi}$ as we have just declared.

3.3.1 Limsup inequality

Lemma 3.3.1. *$\tilde{\mathcal{SBV}}$ is dense in energy in L^2 for $\mathcal{MS}_{\varphi;\psi}$ as in 3.1.3.*

Proof. Let u be in L^2. If $\mathcal{MS}_{\varphi;\psi}(u)$ is equal to $+\infty$, we can consider any sequence $\{u_n\}_{n\in\mathbb{N}}$ in $C^\infty((0,1))$ that converges toward u with respect to L^2 norm. Thanks to the lower semicontinuity theorem of the generalized Mumford-Shah functional (see 2.2.5), we have that

$$\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \geq \mathcal{MS}_{\varphi;\psi}(u) = +\infty.$$

So, we can assume that $\mathcal{MS}_{\varphi;\psi}(u)$ is a real number. We have that u belongs to $\mathcal{SBV}$; we consider the canonical representation as in 2.2.1, namely

$$u = w + v = u(0) + w + \sum_{i\geq 1} v^i,$$

where w is the absolutely continuous part and v is the jump part that can be decomposed as described 2.1.8. For all positive integer n, we define

$$u_n \coloneqq u(0) + w + \sum_{i=1}^{n} v^i.$$

By definition 2.2.1, it immediately follows that the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ converges toward u with respect to L^2 norm. Let us define $\mathcal{I}_\psi$ as in 2.1.12; as shown in 2.1.13, if a is a positive real number, then $\mathcal{I}_\psi(a)$ is greater than 0. So, for all n in $\mathbb{N}^*$ the following inequalities hold true:

$$\begin{aligned} \mathcal{MS}_{\varphi;\psi}(u) &\geq \int_0^1 \varphi(\dot{u})\ dx + \sum_{i=1}^{n} \left(\sum_{x\in\mathcal{S}(u)^i} \psi(\Delta u\,(x)) \right) \\ &\geq \sum_{i=1}^{n} \left(\sum_{x\in\mathcal{S}(u)^i} \mathcal{I}_\psi \left(\frac{1}{i}\right) \right) \\ &\geq \left(\sum_{i=1}^{n} \operatorname{card} \mathcal{S}\,(u)^i \right) \cdot \mathcal{I}_\psi \left(\frac{1}{n}\right). \end{aligned}$$

In particular, we have that

$$\operatorname{card}\mathcal{S}\left(u_n\right) = \sum_{i=1}^{n} \operatorname{card}\mathcal{S}\left(u\right)^i \leq \frac{\mathcal{MS}_{\varphi;\psi}(u)}{\mathcal{I}_\psi\left(\frac{1}{n}\right)} < \infty.$$

So, $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}\tilde{\mathcal{B}}\mathcal{V}$. To conclude, we just remark that

$$\mathcal{MS}_{\varphi;\psi}(u) = \int_0^1 \varphi(\dot{u})\ dx + \lim_{n\to+\infty} \sum_{i=1}^{n} \left(\sum_{x\in\mathcal{S}(u)^i} \psi(\Delta u\left(x\right)) \right) = \lim_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n).$$

□

Theorem 3.3.2 (Limsup inequality)**.**
Let u be in L^2. There exists a recovery sequence $\{u_n\}_{n\in\mathbb{N}}$ for $\mathcal{MS}_{\varphi;\psi}$, i. e.

$$\lim_{n\to+\infty} \|u_n - u\|_{L^2} = 0,$$

$$\limsup_{n\to+\infty} \mathcal{F}_n(u_n) \leq \mathcal{MS}_{\varphi;\psi}(u).$$

Proof. **Step 1**: Let u be in L^2. If we join lemmas 3.1.4 and 3.3.1, we can assume that u belongs to $\mathcal{S}\tilde{\mathcal{B}}\mathcal{V}$. Let us consider the canonical representative of u as declared in 2.2.3, i. e.

$$u := u(0) + w + \sum_{i=1}^{k} \Delta^i \mathbb{1}_{[x^i,1]}.$$

There exists a positive integer n_0 with the following property: for all integer n greater than or equal to n_0 for all integer i in $\{1;\ldots;k\}$ there exists an integer $j(n;i)$ in $\{1;\ldots;2^n-1\}$ such that, if we set

$$I_{j(n;i)} := \left[\frac{j(n;i)}{2^n}, \frac{j(n;i)+1}{2^n}\right],$$

it holds that

- if $i_1 \neq i_2$, then $I_{j(n;i_1)} \cap I_{j(n;i_2)} = \emptyset$;
- x^i is in $I_{j(n;i)}$.

Moreover, the sequence $\left\{\frac{j(n;i)+1}{2^n}\right\}_{n\geq n_0}$ is monotonically decreasing and it converges toward x^i; the sequence $\left\{\frac{j(n;i)}{2^n}\right\}$ is monotonically increasing and it converges toward x^i.

Let n be an integer greater than or equal to n_0; for all integer t in $\{0;\ldots;2^n-1\}$, we define

$$\chi_n^t := \begin{cases} \Delta^t + u\left(\frac{t-1}{2^n}\right) & \text{if } \exists i \in \{1;\ldots;k\}:\ t = j(n;i); \\ u\left(\frac{t}{2^n}\right) & \text{if } \forall i \in \{1;\ldots;k\}:\ t \neq j(n;i). \end{cases}$$

We also define

$$u_n := \sum_{t=0}^{2^n-1} \chi_n^t \mathbb{1}_{\left[\frac{t}{2^n}, \frac{t+1}{2^n}\right)}.$$

Step 2: Let x_0 be any point in $(0,1)\setminus S(u)$; we claim that $\{u_n(x_0)\}_{n\geq n_0}$ converges toward $u(x_0)$. There exists a positive real number η such that $(x_0-\eta, x_0+\eta)$ is completely contained in $(0,1)$ and it is disjoint by $\mathcal{S}(u)$. Let ε be a positive real number. Since w is uniformly continuous, there exists a positive real number δ that corresponds to ε in the definition of uniform continuity. There exists an integer n_1 greater than n_0 such that

- for all integer n greater than or equal to n_1 for all integer i in $\{1;\dots;k\}$ it holds that $I_{j(n;i)}\cap(x_0-\eta,x_0+\eta)=\emptyset$;

- $\dfrac{1}{2^n}\leq\min\{\eta;\delta\}$.

For all integer $n\geq n_1$ there exists i_n in $\{0;\dots;2^n-1\}\setminus\{j(n;1);\dots j(n;N)\}$ such that x_0 is in $\left[\frac{i_n}{2^n},\frac{i_n+1}{2^n}\right)$. For all integer $n\geq n_1$, it holds that

$$\left|x_0-\frac{i_n}{2^n}\right|\leq\frac{1}{2^n}\leq\min\{\eta;\delta\};$$

so, we can state that

$$|u_n(x_0)-u(x_0)|=\left|u\left(\frac{i_n}{2^n}\right)-u(x_0)\right|=\left|w\left(\frac{i_n}{2^n}\right)-w(x_0)\right|\leq\varepsilon.$$

So, the sequence $\{u_n\}_{n\geq n_0}$ converges pointwise toward u for almost every x in $(0,1)$. Moreover, for all integer n greater than or equal to n_0 it holds

$$\|u_n\|_\infty\leq\max_{t\in\{1;\dots;k\}}\left|\chi_n^t\right|\leq\|u\|_\infty+\max_{t\in\{1;\dots;k\}}\left|\Delta^t\right|<+\infty.$$

Thanks to the dominated convergence theorem, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm.

Step 3: For all integer n greater than or equal to n_0, we define

$$\Theta_n\coloneqq\{0;\dots;2^n-1\}\setminus\{j(n;1);\dots j(n;N)\}.$$

So, for all integer n greater than or equal to n_0 the following inequalities hold true:

$$\begin{aligned}
\mathcal{MS}_{\varphi;\psi}(u_0) &= \int_0^1 \varphi(\dot{w})\ dx + \sum_{t=1}^{k} \psi(\Delta^t) \\
&= \int_0^1 \varphi(\dot{w})\ dx + \sum_{t=1}^{k} \psi\left(\frac{2^n\Delta^t}{2^n}\right) && (3.3) \\
&\geq \int_0^1 \varphi(\dot{w})\ dx + \sum_{t=1}^{k} \frac{1}{2^n} f_n\left(2^n\Delta^t\right) \\
&\geq \sum_{t\in\Theta_n}\left(\int_{\frac{t}{2^n}}^{\frac{t+1}{2^n}} \varphi(\dot{w})\ dx\right) + \sum_{t=1}^{k} \frac{1}{2^n} f_n\left(2^n\Delta^t\right) && (3.4) \\
&\geq \sum_{t\in\Theta_n} \frac{1}{2^n}\varphi\left(2^n\left(w\left(\frac{t+1}{2^n}\right) - w\left(\frac{t}{2^n}\right)\right)\right) + \sum_{t=1}^{k} \frac{1}{2^n} f_n\left(2^n\Delta^t\right) && (3.5) \\
&= \sum_{t\in\Theta_n} \frac{1}{2^n}\varphi\left(2^n\left(u\left(\frac{t+1}{2^n}\right) - u\left(\frac{t}{2^n}\right)\right)\right) + \sum_{t=1}^{k} \frac{1}{2^n} f_n\left(2^n\Delta^t\right) && (3.6) \\
&\geq \sum_{t=0}^{2^n-1} \frac{1}{2^n} f_n\left(2^n\left(u_n\left(\frac{t+1}{2^n}\right) - u_n\left(\frac{t}{2^n}\right)\right)\right) && (3.7) \\
&= \mathcal{F}_n(u_n).
\end{aligned}$$

In (3.3) we used the definition of truncated potential (see 3.2.6); in (3.4) we used the fact that the straight line that joins the points $(a;b)$ and $(c;d)$ minimizes the functional $\mathcal{D}_\varphi$ with Dirichlet boundary conditions (see 1.3.2); in (3.5) we used the definition of Θ_n; in (3.6) we used the definition of u_n and the definition of the truncated potential; in (3.7) we used the definition of $\mathcal{F}_n$ (see (3.2.11)). So, the thesis follows taking the superior limit. □

3.3.2 Liminf inequality

Lemma 3.3.3 (Replacing technique)**.**
Let u be in L^2. Let $\{u_n\}_{n\in\mathbb{N}}$ a sequence such that u_n belongs to $\mathcal{PC}_n$ for all n in $\mathbb{N}$. There exists a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ in $\mathcal{SBV}$ with the following properties:

- *$\mathcal{F}_n(u_n) = \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)$ for all n in $\mathbb{N}$;*
- *$\lim_{n\to+\infty} \|\tilde{u}_n - u_n\|_{L^2} = 0$.*

We say that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is the replaced sequence.

Proof. Since u_n belongs to $\mathcal{PC}_n$ for all n in $\mathbb{N}$, we denote

$$u_n \coloneqq \sum_{i=0}^{2^n-1} \Delta_n^i \mathbb{1}_{\left[\frac{i}{2^n},1\right]}.$$

For all n in $\mathbb{N}$ we define

$$\Lambda_n \coloneqq \left\{ i \in \{1;\ldots;2^n-1\} \;\middle|\; \varphi(\Delta_n^i) \geq 2^n\psi\left(\frac{\Delta_n^i}{2^n}\right)\right\}.$$

We also define

$$v_n \coloneqq \Delta_n^0 + \sum_{i\in\Lambda_n} \Delta_n^i \mathbb{1}_{\left[\frac{i}{2^n},1\right]},$$

$$w_n \coloneqq \sum_{i\notin\Lambda_n} \Delta_n^i \mathbb{1}_{\left[\frac{i}{2^n},1\right]}.$$

We say that v_n is the "jump part" of u_n and w_n is the "absolutely continuous part" of u_n. For all n in $\mathbb{N}$ we define ρ_{w_n} as the piecewise affine function that joins the points

$$\left(\frac{i}{2^n}; w_n\left(\frac{i}{2^n}\right)\right) \to \left(\frac{i+1}{2^n}; w_n\left(\frac{i+1}{2^n}\right)\right)$$

for all i in $\{0; \dots; 2^n-1\}$, as declared in 3.2.9. We also define $\tilde{u}_n \coloneqq v_n + \rho_{w_n}$. By definition of $\mathcal{F}_n$ (see 3.2.11) and $\tilde{u}_n$, for all n in $\mathbb{N}$ it holds that

$$\begin{aligned}
\mathcal{F}_n(u_n) &= \sum_{i\notin\Lambda_n} \frac{1}{2^n} f_n\left(2^n\Delta_n^i\right) + \sum_{i\in\Lambda_n} \frac{1}{2^n} f_n\left(2^n\Delta_n^i\right) \\
&= \sum_{i\notin\Lambda_n} \frac{1}{2^n} \varphi\left(2^n\Delta_n^i\right) + \sum_{i\in\Lambda_n} \psi\left(\Delta_n^i\right) \\
&= \sum_{i\notin\Lambda_n} \left(\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \varphi\left(\dot{\rho_{w_n}}\right) dx\right) + \sum_{i\in\Lambda_n} \psi\left(\Delta_n^i\right) \\
&= \int_0^1 \varphi\left(\dot{\rho_{w_n}}\right) dx + \sum_{i\in\Lambda_n} \psi\left(\Delta_n^i\right) \\
&= \mathcal{MS}_{\varphi;\psi}\left(\rho_{w_n} + v_n\right) = \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n).
\end{aligned}$$

We claim that

$$\lim_{n\to+\infty} \|\tilde{u}_n - u_n\|_{L^2} = 0.$$

We have that $\tilde{u}_n - u_n = \rho_{w_n} - w_n$ for all n in $\mathbb{N}$. Let us define $\{\xi_n\}_{n\in\mathbb{N}}$ as in 3.2.3. We notice that if i does not belong to Λ_n, then $|\Delta_i^n| \leq \frac{\xi_n}{2^n}$. Hence, for all n in $\mathbb{N}$ the following inequalities hold true:

$$\begin{aligned}
\|\rho_{w_n} - w_n\|_{L^2}^2 &= \sum_{i\notin\Lambda_n} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} (\rho_{w_n} - w_n)^2 dx \\
&\leq \sum_{i\notin\Lambda_n} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \left(\Delta_n^i\right)^2 dx \\
&= \sum_{i\notin\Lambda_n} \frac{1}{2^n} \left(\Delta_n^i\right)^2 \\
&\leq \sum_{i\notin\Lambda_n} \frac{1}{2^n} \left(\frac{\xi_n}{2^n}\right)^2 \leq \left(\frac{\xi_n}{2^n}\right)^2.
\end{aligned}$$

Thanks to lemma 3.2.5, we can conclude that

$$\lim_{n\to+\infty} \|\rho_{w_n} - w_n\|_{L^2} = 0.$$

□

Theorem 3.3.4 (Liminf inequality).
Let u be in L^2. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in L^2 that converges toward u with respect to L^2 norm. Then, the following inequality holds true:

$$\mathcal{MS}_{\varphi;\psi}(u) \leq \liminf_{n\to+\infty} \mathcal{F}_n(u_n).$$

Proof. If the right hand side is equal to $+\infty$, the conclusion is trivial. Therefore, up to subsequences, not relabelled, we can assume that the inferior limit is actually a limit and it is real, i. e. there exists a real number $\mathcal{M}$ such that

$$\lim_{n\to+\infty} \mathcal{F}_n(u_n) = \mathcal{M}$$

and that $\mathcal{F}_n(u_n) \leq \mathcal{M} + 1$ for all n in $\mathbb{N}$. In particular, u_n belongs to $\mathcal{PC}_n$ for all n in $\mathbb{N}$. Let $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ be the replaced sequence given by lemma 3.3.3. It's easy to see that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm. So, the conclusion is an immediate consequence of the lower semicontinuity theorem of the generalized Mumford-Shah functional (see 2.2.5). In fact, the following inequalities hold true:

$$\liminf_{n\to+\infty} \mathcal{F}_n(u_n) = \liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n) \geq \mathcal{MS}_{\varphi;\psi}(u).$$

□

3.4 Approximation of minima and minimizers

Finally, we show how to approximate the minimum and the minimizers of the functional $\mathscr{E}_{\varphi;\psi}$.

In this section, we assume that

- φ is a Young function as in 1.1.1;
- ψ is a sublinear weight function as in 3.2.2;
- $\mathcal{MS}_{\varphi;\psi}$ is defined as in 2.2.4;
- $h : [0,1] \to \mathbb{R}$ is a function in L^2;
- $\mathcal{E}_{\varphi;\psi}$ is defined as in 2.3.1;
- the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is defined as in 3.2.11.

Definition 3.4.1 (Approximating functionals of the generalized Mumford-Shah energy).

For all n in $\mathbb{N}$ we define $\mathcal{G}_n : L^2 \to [0,+\infty]$ such that

$$\mathcal{G}_n(u) := \begin{cases} \displaystyle\int_0^1 (u-h)^2 \, dx + \mathcal{F}_n(u) & \text{if } u \in \mathcal{PC}_n; \\ +\infty & \text{if } u \in L^2 \setminus \mathcal{PC}_n. \end{cases}$$

Remark 3.4.2. As explained in 3.2.1, the sequence $\{\mathcal{G}_n\}_{n\in N}$ Γ-converges toward $\mathcal{E}_{\varphi;\psi}$ with respect to L^2 norm.

Definition 3.4.3 (Quasi-minima sequence)**.**
Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be defined as in 3.4.1. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in L^2 such that

$$\lim_{n\to+\infty}\left|\mathcal{G}_n(u_n)-\inf_{L^2}\mathcal{G}_n\right|=0.$$

We say that $\{u_n\}_{n\in\mathbb{N}}$ is a quasi-minima sequence.

Theorem 3.4.4. *Let us define $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ as in 3.4.1. Let $\{u_n\}_{n\in\mathbb{N}}$ be a quasi-minima sequence as in 3.4.3. There exists a subsequence, not relabelled, and a function u in $\mathcal{SBV}$ such that $\{u_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward u. Moreover, it holds that*

$$\liminf_{n\to+\infty}\mathcal{G}_n(u_n)\geq\mathscr{E}_{\varphi;\psi}(u).$$

and u is a function that minimizes $\mathscr{E}_{\varphi;\psi}$.

Proof. **Step 1**: Let us define $\gamma\coloneqq\|h\|_{L^2}$. Since $\mathcal{G}_n(0)$ is equal to γ^2 for all n in $\mathbb{N}$, we can assume that $\mathcal{G}_n(u_n)\leq\gamma^2$ for all n in $\mathbb{N}$. In particular, u_n belongs to $\mathcal{PC}_n$ for all n in $\mathbb{N}$. So, we can consider the replaced sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ in $\mathcal{SBV}$ as in 3.3.3. For all n in $\mathbb{N}$ we have that

$$\mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)=\mathcal{F}_n(u_n)\leq\mathcal{G}_n(u_n)\leq\gamma^2.$$

As shown in lemma 3.3.3, the sequence $\{\|u_n-\tilde{u}_n\|_{L^2}\}_{n\in\mathbb{N}}$ is infinitesimal. Thanks to the triangular inequality, for all n in $\mathbb{N}$ we have that

$$\begin{aligned}\|u_n\|_{L^2}&\leq\|u_n-\tilde{u}_n\|_{L^2}+\|h-u_n\|_{L^2}+\|h\|_{L^2}\\&\leq\|u_n-\tilde{u}_n\|_{L^2}+\sqrt{\mathcal{G}_n(u_n)}+\gamma\\&\leq\|u_n-\tilde{u}_n\|_{L^2}+2\gamma.\end{aligned}$$

In particular, we can conclude that there exists a real number $\mathcal{M}$ such that $\|\tilde{u}_n\|_{L^2}\leq\mathcal{M}$ for all n in $\mathbb{N}$. So, we can use theorem 2.2.5 and we obtain that there exist a subsequence, not relabelled, and a function u in $\mathcal{SBV}$ such that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ converges pointwise for almost every x in $(0,1)$ toward u. Since $\{u_n-\tilde{u}_n\}_{n\in\mathbb{N}}$ converges toward zero function with respect to L^2 norm, up to further subsequences, not relabelled, we can assume that the convergence is pointwise for almost every x in $(0,1)$. By difference, we can conclude that $\{u_n\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere toward u. If we join theorem 2.2.5 and lemma 3.3.3, we obtain that

$$\mathcal{MS}_{\varphi;\psi}(u)\leq\liminf_{n\to+\infty}\mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)=\liminf_{n\to+\infty}\mathcal{F}_n(u_n).$$

Thanks to the Fatou's lemma, we can state that

$$\int_0^1(u-h)^2\,dx\leq\liminf_{n\to+\infty}\int_0^1(h-u_n)^2\,dx.$$

So, we have that

$$\begin{aligned}\mathscr{E}_{\varphi;\psi}(u)&=\int_0^1(h-u)^2\,dx+\mathcal{MS}_{\varphi;\psi}(u)\\&\leq\liminf_{n\to+\infty}\int_0^1(h-u_n)^2\,dx+\liminf_{n\to+\infty}\mathcal{F}_n(u_n)\\&\leq\liminf_{n\to+\infty}\mathcal{G}_n(u_n).\end{aligned}$$

Step 2: Thanks to theorem 2.3.2, there exists a function y that minimizes $\mathscr{E}_{\varphi;\psi}$. Thanks to 3.4.2, there exists a recovery sequence (see 3.1) for y. So, the following inequalities hold true:

$$\begin{aligned}\min_{L^2} \mathscr{E}_{\varphi;\psi} = \mathscr{E}_{\varphi;\psi}(y) &\geq \limsup_{n\to+\infty} \mathcal{G}_n(u_n) \\ &\geq \liminf_{n\to+\infty} \left(\inf_{L^2} \mathcal{G}_n \right) \\ &= \liminf_{n\to+\infty} \left(\inf_{L^2} \mathcal{G}_n - \mathcal{G}_n(u_n) + \mathcal{G}_n(u_n) \right) \\ &= \liminf_{n\to+\infty} \mathcal{G}_n(u_n) \\ &\geq \mathscr{E}_{\varphi;\psi}(u) \\ &\geq \min_{L^2} \mathscr{E}_{\varphi;\psi}.\end{aligned}$$

So, we conclude that u is a function that minimizes $\mathscr{E}_{\varphi;\psi}$. □

Chapter 4

Descending metric slope

We introduce the notion of descending metric slope; it describes how regular is the functional in a neighborhood of a fixed point. We want to compute the descending metric slope of the generalized Mumford-Shah functional.

4.1 Definition and main properties

Definition 4.1.1 (Descending metric slope)**.**
Let $(\mathbb{X}; d)$ be a metric space; let $F : \mathbb{X} \to [0, +\infty]$ be any function; let x_0 be any point in $\mathbb{X}$ such that $F(x_0)$ is a real number. The descending metric slope of F in x_0 is defined as

$$\left|\nabla F\right|(x_0) := \limsup_{r \to 0^+} \frac{F(x_0) - \inf\{f(x) \mid d(x; x_0) \leq r\}}{r}.$$

If $F(x_0) = +\infty$ we define

$$\left|\nabla F\right|(x_0) := +\infty.$$

Remark 4.1.2. In the setting of definition 4.1.1, the descending metric slope of F in x_0 measures how much it is possible to decrease the value of the functional with respect to the distance from x_0. We notice that $\left|\nabla F\right|$ is a nonnegative function on $\mathbb{X}$.

Lemma 4.1.3. *Let $(\mathbb{X}; d)$ be a metric space; let $F : \mathbb{X} \to [0, +\infty]$ be a function; let x_0 be any point in $\mathbb{X}$. Let us define the descending metric slope as in 4.1.1. If we assume that $F(x_0)$ is a real number and x_0 is not an isolated point, then the following identity holds true:*

$$\left|\nabla F\right|(x_0) = \limsup_{x \to x_0} \frac{\max\{F(x_0) - F(x); 0\}}{d(x_0; x)}.$$

Proof. Let us define

$$\mathcal{M} := \limsup_{x \to x_0} \frac{\max\{F(x_0) - F(x); 0\}}{d(x_0; x)}.$$

Since x_0 is a cluster point for $\mathbb{X}$, we notice that $\mathcal{M}$ is well defined.

Step 1: We show that $\left|\nabla F\right|(x_0) \geq \mathcal{M}$. If $\mathcal{M}$ is equal to 0, the conclusion is trivial; therefore, we can assume $\mathcal{M}$ is in $(0, +\infty]$. By definition of superior limit, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathbb{X} \setminus \{x_0\}$ that converges toward x_0 such that

$$\lim_{n \to +\infty} \frac{\max\{F(x_0) - F(x_n); 0\}}{d(x_n; x_0)} = \mathcal{M}.$$

Since $\mathcal{M}$ is greater than 0, the numerator must be eventually positive, hence

$$\lim_{n\to+\infty} \frac{F(x_0) - F(x_n)}{d(x_n; x_0)} = \mathcal{M}.$$

For all natural number n we set

$$r_n := d(x_n; x_0),$$
$$i_n := \inf \{f(x) \mid d(x; x_0) \leq r_n\}.$$

So, we have that $i_n \leq F(x_n)$ for all n in $\mathbb{N}$. Therefore, the following inequalities hold true:

$$\begin{aligned} \left|\nabla F\right|(x_0) &= \limsup_{r\to 0^+} \frac{F(x_0) - \inf \{f(x) \mid d(x; x_0) \leq r\}}{r} \\ &\geq \limsup_{n\to+\infty} \frac{F(x_0) - i_n}{r_n} \\ &\geq \limsup_{n\to+\infty} \frac{F(x_0) - F(x_n)}{r_n} = \mathcal{M}. \end{aligned}$$

Step 2: We show that $\left|\nabla F\right|(x_0) \leq \mathcal{M}$. If $\left|\nabla F\right|(x_0)$ is equal to 0, the conclusion is trivial; therefore, we can assume that $\left|\nabla F\right|(x_0)$ is greater than 0. Let $\{r_n\}_{n\in\mathbb{N}}$ be an infinitesimal positive sequence such that, if we define i_n as in previous step, the following identity holds true:

$$\left|\nabla F\right|(x_0) = \lim_{n\to 0^+} \frac{F(x_0) - i_n}{r_n}.$$

Up to subsequences, not relabelled, we can assume that $i_n < F(x_0)$ for all n in $\mathbb{N}$. By definition of infimum, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}\backslash\{x_0\}$ such that $d(x_0; x_n) \leq r_n$ and $F(y_n) \leq i_n + r_n^2$ for all n in $\mathbb{N}$. So, the following inequalities hold true:

$$\begin{aligned} \mathcal{M} &\geq \limsup_{n\to+\infty} \frac{\max \{F(x_0) - F(y_n); 0\}}{r_n} \\ &\geq \limsup_{n\to+\infty} \frac{F(x_0) - F(y_n)}{r_n} \\ &\geq \limsup_{n\to+\infty} \frac{F(x_0) - i_n}{r_n} - r_n \\ &= \left|\nabla F\right|(x_0). \end{aligned}$$

□

Remark 4.1.4. Let φ be a Young function as in 1.1.1; let ψ be a weight function as in 2.1.11. We define the generalized Mumford-Shah functional $\mathcal{MS}_{\varphi;\psi}$ as in 2.2.4. Let u_0 be in $\mathcal{SBV}$. We notice that $\mathcal{MS}_{\varphi;\psi}(u_0 + c) = \mathcal{MS}_{\varphi;\psi}(u_0)$ for all c in $\mathbb{R}$. Hence, if $\{c_n\}_{n\in\mathbb{N}}$ is any infinitesimal sequence, the following identity holds true:

$$\frac{\mathcal{MS}_{\varphi;\psi}(u_0) - \mathcal{MS}_{\varphi;\psi}(u_0 + c_n)}{|c_n|} = 0.$$

This is enough to state that

$$\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u_0) = \limsup_{u\to u_0} \frac{\mathcal{MS}_{\varphi;\psi}(u_0) - \mathcal{MS}_{\varphi;\psi}(u)}{\|u_0 - u\|_{L^2}}.$$

4.2 Slope of $\mathcal{MS}_{\varphi;\psi}$

From now on we assume that φ is a Young function as in 1.1.1, ψ is a weight function as in 2.1.11, $\mathcal{MS}_{\varphi;\psi}$ is the generalized Mumford-Shah functional defined in 2.2.4 and $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|$ is the descending metric slope defined in 4.1.1.

Our aim is to compute the descending metric slope of the generalized Mumford-Shah functional. We find as more necessary conditions as possible for the slope to be finite and we write a lower bound for the slope. Surprisingly enough, these conditions prove sufficient and we find an upper bound for the slope that involves the regularity of φ and ψ.

4.2.1 Lower bound for the slope

Theorem 4.2.1 (Finiteness of the essential discontinuities)**.**
Let u be a function in $\mathcal{SBV}$ such that $\mathcal{MS}_{\varphi;\psi}(u)$ and $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)$ are real numbers. Then, the set of the essential discontinuities of u is finite; in other words, u belongs to $\tilde{\mathcal{SBV}}$ (see 2.2.3).

Proof. We consider the canonical representative of u, i. e. $u = w + v + u(0)$, where w is the absolutely continuous part and v is the jump part (see 2.2.1). As defined in 2.1.8, we can decompose v as follows:

$$v = \sum_{i\geq 1} v^i = \sum_{i\geq 1}\left(\sum_{x\in\mathcal{S}(u)^i}\psi\left(\Delta u\left(x\right)\right)\right).$$

By definition 2.1.8, if i is a positive integer and x belongs to $\mathcal{S}\left(u\right)^i$, then $\left|\Delta u\left(x\right)\right|$ is in $\left(\frac{1}{i}, \frac{1}{i-1}\right]$. For all positive integer n, we define

$$u_n \coloneqq u(0) + w + \sum_{i=1}^{n} v^i.$$

Let θ be a positive real number; let $\Gamma_\psi(\theta)$ be as in 2.1.10. By definition 2.1.11, we have that

$$\liminf_{\theta\to 0^+}\Gamma_\psi(\theta) = \liminf_{\theta\to 0^+}\frac{\psi(\theta)}{\theta} = +\infty.$$

Let ε be a positive real number; there exists a positive integer N_0 such that if x is in $\left(0, \frac{1}{N_0}\right)$, then $\Gamma_\psi(x) \geq \frac{1}{\varepsilon}$. So, if i is a natural number greater than or equal to N_0 and x is a point in $\mathcal{S}\left(u\right)^i$, then $\frac{1}{\varepsilon} \leq \Gamma_\psi(\left|\Delta u\left(x\right)\right|)$; in other words, we have that

$$\left|\Delta u\left(x\right)\right| \leq \varepsilon\psi(\left|\Delta u\left(x\right)\right|).$$

For all integer n greater than or equal to N_0 the following inequalities hold true:

$$\begin{aligned}\|u-u_n\|_{L^2} &= \left\| v - \sum_{i=1}^{n} v^i \right\|_{L^2} \\ &\leq \left\| v - \sum_{i=1}^{n} v^i \right\|_{\infty} \\ &\leq \sum_{i\geq n+1} \left\|v^i\right\|_{\infty} \\ &\leq \sum_{i\geq n+1} \left(\sum_{x\in\mathcal{S}(u)^i} |\Delta u\,(x)| \right) \\ &\leq \sum_{i\geq n+1} \left(\sum_{x\in\mathcal{S}(u)^i} \varepsilon\,|\psi(\Delta u\,(x))| \right) \\ &= \varepsilon\left[\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n)\right].\end{aligned}$$

Obviously, $\{u_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm (see 2.1.8). We also remark that $\mathcal{MS}_{\varphi;\psi}(u)$ is a real number; if we assume that $\mathcal{S}\,(u)$ is not finite, then $u_n \neq u$ for all positive integer n. We have just shown that if n is an integer greater than or equal to N_0, then

$$\frac{1}{\varepsilon} \leq \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n)}{\|u-u_n\|_{L^2}}.$$

So, we can state that

$$\liminf_{n\to+\infty} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n)}{\|u-u_n\|_{L^2}} = +\infty.$$

As shown in lemma 4.1.3, we can conclude that $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) = +\infty$. □

Theorem 4.2.2 (Regularity and Neumann boundary conditions).
Let u be in $\mathcal{S}\tilde{\mathcal{B}}\mathcal{V}$ represented as in 2.2.3, namely

$$u = w + u(0) + \sum_{i=1}^{k} \Delta^i \mathbb{1}_{[x^i;1]}.$$

For all integer i in $\{0;\ldots;k\}$ we denote $\Omega^i := [x^i, x^{i+1}]$; we also denote $\Omega := [0,1]$. We suppose that $\mathcal{MS}_{\varphi;\psi}(u)$ and $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)$ are both real numbers. We assume that:

- *ψ is in $C^1((0,+\infty))$;*
- *φ is in $C^1(\mathbb{R})$;*
- *for all ρ in $C^\infty(\Omega)$ there exist a positive real number τ and a function η in $L^1(\Omega)$ such that for all t in $(-\tau,\tau)$ for almost every x in Ω the following inequality holds true:*

$$|\varphi'(\dot{u}(x) + t\dot{\rho}(x))| \leq \eta(x).$$

Then, the following conclusions hold true:

- *(regularity)* $\varphi'(\dot u)$ *is in* $W^{1;2}(\Omega)$ *and it holds that*
$$\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) \geq \left\|[\varphi'(\dot u)]'\right\|_{L^2(\Omega)}; \tag{4.1}$$

- *(Neumann boundary conditions)*
  1. $\varphi'\left(\dot u\left(x^i\right)\right) = \psi'\left(\Delta^i\right)$ *for all integer* i *in* $\{1; \ldots; k\}$*;*
  2. $\varphi'\left(\dot u(0)\right) = 0$*;*
  3. $\varphi'\left(\dot u(1)\right) = 0$.

Proof. **Step 1**: Let ρ be a function in $C_c^\infty(\Omega)$. Since $\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u)$ is a real number, we can state that

$$\limsup_{t\to 0} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u+t\rho)}{\|t\rho\|_{L^2(\Omega)}} \leq \left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u).$$

In other words, the following inequalities hold true:

$$\begin{aligned}\limsup_{t\to 0} \int_\Omega \frac{\varphi(\dot u) - \varphi(\dot u + t\dot\rho)}{|t|}\, dx &= \limsup_{t\to 0} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u+t\rho)}{|t|} \\ &\leq \left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u)\, \|\rho\|_{L^2(\Omega)}.\end{aligned}$$

We consider a positive real number τ and a function η in $L^1(\Omega)$ as in the hypothesis; thanks to the theorem of derivation under integral, the following identities holds true:

$$\lim_{t\to 0^+} \int_\Omega \frac{\varphi(\dot u) - \varphi(\dot u + t\dot\rho)}{t}\, dx = -\int_\Omega \varphi'(\dot u)\dot\rho\, dx;$$
$$\lim_{t\to 0^-} \int_\Omega \frac{\varphi(\dot u) - \varphi(\dot u + t\dot\rho)}{-t}\, dx = \int_\Omega \varphi'(\dot u)\dot\rho\, dx.$$

Hence, if ρ is a function in $C_c^\infty(\Omega)$, we have that

$$\left|\int_\Omega \varphi'(\dot u)\dot\rho\, dx\right| \leq \left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u)\, \|\rho\|_{L^2(\Omega)}. \tag{4.2}$$

So, if we define $\Xi(u) : C_c^\infty(\Omega) \to \mathbb{R}$ such that

$$[\Xi(u)](\rho) := \int_\Omega \varphi'(\dot u)\dot\rho\, dx,$$

the functional $\Xi(u)$ is linear and continuous. Thanks to the Riesz's representation theorem (see [2]), there exists a function ξ in $L^2(\Omega)$ such that for all ρ in $C_c^\infty(\Omega)$ the following identities hold true:

$$\int_\Omega \varphi'(\dot u)\dot\rho\, dx = [\Xi(u)](\rho) = \int_\Omega \xi\rho\, dx.$$

In other words, we have just shown that $\varphi'(\dot u)$ is in $W^{1;2}(\Omega)$ and $-\xi$ is the weak derivative. So, for all ρ in $C_c^\infty(\Omega)$, we can integrate by parts (4.2) and we obtain that

$$\left|\int_\Omega [\varphi'(\dot u)]'\, \rho\, dx\right| \leq \left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u)\, \|\rho\|_{L^2(\Omega)}. \tag{4.3}$$

Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ that converges toward $[\varphi'(\dot{u})]'$ with respect to L^2 norm. If we take the limit as n approaches $+\infty$ in (4.3), we obtain that

$$\int_\Omega \left([\varphi'(\dot{u})]'\right)^2 dx \leq \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) \left\|[\varphi'(\dot{u})]'\right\|_{L^2(\Omega)}.$$

Then, we immediately conclude that

$$\left\|[\varphi'(\dot{u})]'\right\|_{L^2(\Omega)} \leq \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u).$$

Step 2: Since $\varphi'(\dot{u})$ is in $W^{1;2}(\Omega)$, it is continuous in $[0,1]$; so, the boundary conditions make sense. Moreover, we have that $\varphi'(\dot{u})$ is in $W^{1;2}(\Omega^i)$ for all i in $\{0;\ldots;k\}$ and the weak derivative is the restriction of the weak derivative defined in $(0,1)$. We also notice that for all integer i in $\{0;\ldots;k\}$ for all ρ in $C^\infty(\Omega^i)$ it holds that

$$\int_{\Omega^i} \varphi'(\dot{u})\dot{\rho}\ dx = \int_{\Omega^i} \xi\rho\ dx + \Big[\varphi'(\dot{u})\rho\Big]_{x^i}^{x^{i+1}}.$$

Let us assume that i is in $\{1;\ldots;k\}$. Let ρ be any function in $C^\infty(\Omega^i)$ such that $\rho\left(x^i\right) = 1$ and $\rho\left(x^{i+1}\right) = 0$; we define $\tilde{\rho} : [0,1] \to \mathbb{R}$ such that $\tilde{\rho}(x) \coloneqq \rho(x)\mathbb{1}_{\Omega^i}(x)$. Then, the following inequalities hold true:

$$\begin{aligned}
\limsup_{t\to 0} &\left[\int_{\Omega^i} \frac{\varphi(\dot{u}) - \varphi(\dot{u}+t\dot{\rho})}{|t|}\ dx + \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i + t\right)}{|t|}\right] \\
&= \limsup_{t\to 0} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u+t\tilde{\rho})}{|t|} \\
&\leq \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)\left\|\tilde{\rho}\right\|_{L^2(\Omega)} \\
&= \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)\left\|\rho\right\|_{L^2(\Omega^i)}.
\end{aligned}$$

We consider a positive real number τ and a function η in $L^1(\Omega^i)$ as in the hypothesis. Thanks to the theorem of derivation under integral and the regularity of ψ, the following identities hold true:

$$\lim_{t\to 0^+}\left[\int_{\Omega^i} \frac{\varphi(\dot{u}) - \varphi(\dot{u}+t\dot{\rho})}{t}\ dx + \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i + t\right)}{t}\right] = -\int_{\Omega^i} \varphi'(\dot{u})\dot{\rho}\ dx - \psi'\left(\Delta^i\right),$$

$$\lim_{t\to 0^-}\left[\int_{\Omega^i} \frac{\varphi(\dot{u}) - \varphi(\dot{u}+t\dot{\rho})}{-t}\ dx + \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i + t\right)}{-t}\right] = \int_{\Omega^i} \varphi'(\dot{u})\dot{\rho}\ dx + \psi'\left(\Delta^i\right).$$

So, for all function ρ is in $C^\infty(\Omega^i)$ such that $\rho\left(x^i\right) = 1$ and $\rho\left(x^{i+1}\right) = 0$ it holds that

$$\left|\int_{\Omega^i} \varphi'(\dot{u})\dot{\rho}\ dx + \psi'\left(\Delta^i\right)\right| \leq \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)\left\|\rho\right\|_{L^2(\Omega^i)}.$$

We integrate by parts and we obtain that

$$\left|\int_{\Omega^i} \xi\rho\ dx - \varphi'\left(\dot{u}\left(x^i\right)\right) + \psi'\left(\Delta^i\right)\right| \leq \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)\left\|\rho\right\|_{L^2(\Omega^i)}.$$

If we use the triangular inequality, we find that

$$-\left|\int_{\Omega^i} \xi\rho\ dx\right| + \left|-\varphi'\left(\dot{u}\left(x^i\right)\right) + \psi'\left(\Delta^i\right)\right| \leq \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)\left\|\rho\right\|_{L^2(\Omega^i)}.$$

We can rearrange terms and use the Hölder's inequality; so, we have that

$$\begin{aligned}\left|-\varphi'\left(\dot u\left(x^i\right)\right)+\psi'\left(\Delta^i\right)\right| &\le \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)\,\|\rho\|_{L^2(\Omega^i)}+\left|\int_{\Omega^i}\xi\rho\,dx\right|\\ &\le \left(\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)+\|\xi\|_{L^2(\Omega^i)}\right)\|\rho\|_{L^2(\Omega^i)}\,.\end{aligned}$$

Having said that, we can choose a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $C^\infty(\Omega^i)$ with the following properties:

- $\rho_n\left(x^i\right)=1$ and $\rho_n\left(x^{i+1}\right)=0$ for all natural number n;
- $\{\rho_n\}_{n\in\mathbb{N}}$ converges toward zero function with respect to L^2 norm in Ω^i.

Hence, we obtain that

$$\left|-\varphi'\left(\dot u\left(x^i\right)\right)+\psi'\left(\Delta^i\right)\right|=0.$$

Step 3: To conclude, we can easily adapt the procedure described in the previous step, taking different sets of test functions:

- if we consider ρ in $C^\infty(\Omega^0)$ such that $\rho(0)=1$ and $\rho\left(x^1\right)=0$, we obtain that $|\varphi'(\dot u(0))|=0$;
- if we consider ρ in $C^\infty(\Omega^k)$ such that $\rho(1)=1$ and $\rho\left(x^k\right)=0$, we obtain that $|\varphi'(\dot u(1))|=0$.

Then, the theorem is completely proved. □

Remark 4.2.3. Under the hypothesis of the theorem 4.2.2, $\varphi'(\dot u)$ is a continuous function. If we also assume that $\varphi':\mathbb{R}\to\mathbb{R}$ is an homeomorphism, then $\dot u$ is a continuous function and u is in $C^1(\Omega)$.

Corollary 4.2.4 (Characterization of the global minimum)**.**
Under the hypothesis of theorem 4.2.2, we also assume that

- *if* Δ *is in* $(0,+\infty)$*, then* $\psi'(\Delta)\neq 0$*;*
- $\varphi(x)=0$ *if and only if* $x=0$*;*
- $\varphi'(x)=0$ *if and only if* $x=0$*.*

Then, $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)=0$ *if and only if* u *is a global minimum point for* $\mathcal{MS}_{\varphi;\psi}$*. In particular, the set of local minimum points that are not global minimum points for* $\mathcal{MS}_{\varphi;\psi}$ *is empty.*

Proof. It's easy to see that u is a global minimum for $\mathcal{MS}_{\varphi;\psi}$ if and only if there exists a constant c such that $u(x)=c$ for almost every x in Ω.

If $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)$ is equal to 0 we have that $[\varphi'(\dot u)]'$ is equal to 0 almost everywhere in Ω (see (4.1)). We integrate by parts and we use the Neumann boundary conditions in 0 and 1; so, for all ρ in $C^\infty(\Omega)$ we have that

$$0=\int_\Omega[\varphi'(\dot u)]'\rho\,dx=-\int_\Omega\varphi'(\dot u)\dot\rho\,dx.$$

Thanks to the Du Bois-Reymond's lemma, there exists a constant c such that $\varphi'(\dot{u}(x))$ is equal to c for almost every x in Ω. We know that $\varphi'(\dot{u}(1)) = \varphi'(\dot{u}(0)) = 0$ (see 4.2.2); so, c is equal to 0. In particular, we have that $\dot{u}(x) = 0$ for almost every x in Ω. We have shown that w coincides almost everywhere with a globally constant function. If $\mathcal{S}(u) \neq \emptyset$, let x be in $\mathcal{S}(u)$ and $\Delta u(x)$ be the height of the corresponding jump; then, $\psi'(\Delta u(x)) = \varphi'(u(x)) = 0$ that is against our assumption on ψ'. In particular, u is a globally constant function.

If u is a local minimum, by definition of descending metric slope it immediately follows that $\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) = 0$ (see 4.1.1). In particular u is a global minimum. □

Remark 4.2.5. We remark that the necessary conditions on u and the lower bound for the slope given by theorem 4.2.2 are consistent with those obtained in the classical case, where φ is the quadratic potential and ψ is the function that counts jumps. However, it was studied by Clara Antonucci in her master thesis (see [1]). She find out that the conditions given by theorem 4.2.2 are also sufficient for the slope to be finite and 4.1 is actually an identity. As for the generalized Mumford-Shah functional, the metric slope is strictly related to the regularity of φ and ψ.

Theorem 4.2.6. *Under the hypothesis of theorem 4.2.2, we also assume that*

- φ *is in* $C^2(\mathbb{R})$;
- $\varphi' : \mathbb{R} \to \mathbb{R}$ *is an homeomorphism.*

There exists a positive real number $\mathcal{M}$ *such that for all integer* i *in* $\{1; \dots; k\}$ *the following inequality holds true:*

$$\limsup_{\varepsilon \to 0} \frac{\psi(\Delta^i) - \psi(\Delta^i + \varepsilon) + \varepsilon\psi'(\Delta^i)}{|\varepsilon|^{\frac{4}{3}}} \leq \mathcal{M}.$$

Proof. Thanks to 4.2.3, we know that w is in $C^1(\Omega)$; in particular $\dot{w}$ is bounded. Let μ be a positive real number such that $\dot{w}(x)$ is in $[-\mu, \mu]$ for all x in Ω. Let us fix i in $\{1; \dots; k\}$; let ε be a positive real numbers such that $\varepsilon < x^i - x^{i-1}$. We set

$$\Omega^i_-(\varepsilon) := \left(x^i - \varepsilon, x^i\right].$$

Let a be in $[1, +\infty)$. We define $\rho_\varepsilon : \Omega \to \mathbb{R}$ such that

$$\rho_\varepsilon(x) := [x - (x^i - \varepsilon)]^a \mathbb{1}_{\Omega^i_-(\varepsilon)}(x).$$

We notice that if x is in $\Omega \setminus \{x^i\}$ then $\dot{\rho}_\varepsilon(x)$ is in $[-a\varepsilon^{a-1}, a\varepsilon^{a-1}]$. Without loss of generality, we can assume that if x is in $\Omega \setminus \{x^i\}$ then $\dot{w}(x) + \dot{\rho}_\varepsilon(x)$ is in $[-\mu, \mu]$. Since φ is a C^2 function, to x in Ω there corresponds ξ_x in $[\dot{w}(x) - |\dot{\rho}_\varepsilon(x)|, \dot{w}(x) + |\dot{\rho}_\varepsilon(x)|]$ such that

$$\varphi(\dot{u}(x) + \dot{\rho}_\varepsilon(x)) = \varphi(\dot{u}(x)) + \dot{\rho}_\varepsilon(x)\varphi'(\dot{u}(x)) + \frac{\dot{\rho}_\varepsilon(x)^2}{2}\varphi''(\xi_x).$$

Moreover, it is not restrictive to assume that if x is in Ω then $\varphi''(\xi_x)$ is in $[0, \mu]$. In other words, for all x in $\Omega \setminus \{x^i\}$ we have that

$$\begin{aligned}\varphi(\dot{u}(x)) - \varphi(\dot{u}(x) + \dot{\rho}_\varepsilon(x)) &= -\dot{\rho}_\varepsilon(x)\varphi'(\dot{u}(x)) - \frac{\dot{\rho}_\varepsilon(x)^2}{2}\varphi''(\xi_x) \\ &\geq -\dot{\rho}_\varepsilon(x)\varphi'(\dot{u}(x)) - \frac{\mu}{2}\dot{\rho}_\varepsilon(x)^2.\end{aligned}$$

So, we have that

$$
\begin{aligned}
\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) &\geq \limsup_{\varepsilon\to0^+}\frac{\mathcal{MS}_{\varphi;\psi}(u)-\mathcal{MS}_{\varphi;\psi}(u+\rho_\varepsilon)}{\|\rho_\varepsilon\|_{L^2(\Omega)}}\\
&\geq \limsup_{\varepsilon\to0^+}\frac{-\displaystyle\int_\Omega\varphi'(\dot u)\dot\rho_\varepsilon\,dx-\frac{\mu}{2}\int_\Omega\dot\rho_\varepsilon^2\,dx+\psi\left(\Delta^i\right)-\psi\left(\Delta^i-\varepsilon^a\right)}{\|\rho_\varepsilon\|_{L^2(\Omega)}}\\
&= \limsup_{\varepsilon\to0^+}\frac{-\displaystyle\int_{\Omega^i_-(\varepsilon)}\varphi'(\dot u)\dot\rho_\varepsilon\,dx-\frac{\mu}{2}\int_\Omega\dot\rho_\varepsilon^2\,dx+\psi\left(\Delta^i\right)-\psi\left(\Delta^i-\varepsilon^a\right)}{\|\rho_\varepsilon\|_{L^2(\Omega)}}.
\end{aligned}
$$

Since we can integrate by parts and use the Neumann boundary condition in x^i (see theorem 4.2.2), we obtain that

$$
\begin{aligned}
\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) &\geq \limsup_{\varepsilon\to0^+}\frac{\displaystyle\int_{\Omega^i_-(\varepsilon)}\left[\varphi'(\dot u)\right]'\rho_\varepsilon\,dx-\frac{\mu}{2}\|\dot\rho_\varepsilon\|^2_{L^2(\Omega)}}{\|\rho_\varepsilon\|_{L^2(\Omega)}}\\
&\qquad+\frac{-\varphi'\left(\dot u\left(x^i\right)\right)\varepsilon^a+\psi\left(\Delta^i\right)-\psi\left(\Delta^i-\varepsilon^a\right)}{\|\rho_\varepsilon\|_{L^2(\Omega)}}\\
&= \limsup_{\varepsilon\to0^+}\frac{\displaystyle\int_{\Omega}\left[\varphi'(\dot u)\right]'\rho_\varepsilon\,dx-\frac{\mu}{2}\|\dot\rho_\varepsilon\|^2_{L^2(\Omega)}}{\|\rho_\varepsilon\|_{L^2(\Omega)}}\\
&\qquad+\frac{-\varepsilon^a\psi'\left(\Delta^i\right)+\psi\left(\Delta^i\right)-\psi\left(\Delta^i-\varepsilon^a\right)}{\|\rho_\varepsilon\|_{L^2(\Omega)}}.
\end{aligned}
$$

If we use the Hölder's inequality, we have that

$$
-\left\|\left[\varphi'(\dot u)\right]'\right\|_{L^2(\Omega)}\|\rho_\varepsilon\|_{L^2(\Omega)}\leq\int_\Omega\left[\varphi'(\dot u)\right]'\rho_\varepsilon\,dx\leq\left\|\left[\varphi'(\dot u)\right]'\right\|_{L^2(\Omega)}\|\rho_\varepsilon\|_{L^2(\Omega)}.
$$

It's easy to see that there exist two positive real numbers $c_1(a)$ and $c_2(a)$ such that

$$
\begin{aligned}
\|\rho_\varepsilon\|_{L^2(\Omega)} &= \left(\int_0^\varepsilon x^{2a}\,dx\right)^{\frac12}=\frac{1}{\sqrt{2a+1}}\varepsilon^{a+\frac12}=c_1(a)\varepsilon^{a+\frac12},\\
\|\dot\rho_\varepsilon\|^2_{L^2(\Omega)} &= \int_0^\varepsilon a^2x^{2a-2}\,dx=\frac{a^2}{2a-1}\varepsilon^{2a-1}=c_2(a)\varepsilon^{2a-1}.
\end{aligned}
$$

Therefore, we can rearrange terms and we obtain that

$$
\begin{aligned}
&\limsup_{\varepsilon\to0^+}\frac{-\frac{\mu}{2}c_2(a)\varepsilon^{2a-1}-\varepsilon^a\psi'(\Delta^i)+\psi(\Delta^i)-\psi(\Delta^i-\varepsilon^a)}{\|\rho_\varepsilon\|_{L^2(\Omega)}}\\
&\qquad\leq\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)+\left\|\left[\varphi'(\dot u)\right]'\right\|_{L^2(\Omega)}.
\end{aligned}
$$

In other words, we have that

$$
\begin{aligned}
&\limsup_{\varepsilon\to0^+}\frac{\psi\left(\Delta^i\right)-\psi\left(\Delta^i-\varepsilon^a\right)-\varepsilon^a\psi'\left(\Delta^i\right)}{c_1(a)\varepsilon^{a+\frac12}}\\
&\qquad\leq\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)+\left\|\left[\varphi'(\dot u)\right]'\right\|_{L^2(\Omega)}+\frac{\mu}{2}\lim_{\varepsilon\to0}\frac{c_2(a)\varepsilon^{2a-1}}{c_1(a)\varepsilon^{a+\frac12}}.
\end{aligned}
$$

Since $\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u)$ is a real number and $\varphi'(\dot{w})$ is in $W^{1;2}(\Omega)$ (see 4.2.2), the right hand side is finite if and only if $a \geq \frac{3}{2}$. So, if a is greater than or equal to $\frac{3}{2}$, it holds that

$$\limsup_{\varepsilon \to 0^+} \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i - \varepsilon^a\right) - \varepsilon^a \psi'\left(\Delta^i\right)}{c_1(a)\varepsilon^{a+\frac{1}{2}}} \leq \left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) + \left\|[\varphi'(\dot{u})]'\right\|_{L^2(\Omega)} + \frac{\mu c_2(a)}{2c_1(a)}.$$

If we replace ε^a with ε, we have shown that there exists a positive real number $\mathcal{M}$ such that

$$\limsup_{\varepsilon \to 0^+} \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i - \varepsilon\right) - \varepsilon \psi'\left(\Delta^i\right)}{\varepsilon^{1+\frac{1}{2a}}} \leq \mathcal{M}.$$

The condition is optimal if a is equal to $\frac{3}{2}$; hence, the following inequality holds true:

$$\limsup_{\varepsilon \to 0^+} \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i - \varepsilon\right) - \varepsilon \psi'\left(\Delta^i\right)}{\varepsilon^{\frac{4}{3}}} \leq \mathcal{M}. \tag{4.4}$$

To conclude, let us consider any positive real number ε such that $\varepsilon < x^{i+1} - x^i$. We set

$$\Omega^i_+(\varepsilon) \coloneqq [x^i, x^i + \varepsilon).$$

Let a be in $[1, +\infty)$. We define $\rho_\varepsilon : \Omega \to \mathbb{R}$ such that

$$\rho_\varepsilon(x) \coloneqq -[x - (x^i - \varepsilon)]^a \mathbb{1}_{\Omega^i_-(\varepsilon)}(x).$$

If we slightly modify the procedure that we have just described, we easily obtain that

$$\limsup_{\varepsilon \to 0^-} \frac{\psi\left(\Delta^i\right) - \psi\left(\Delta^i - \varepsilon\right) - \varepsilon \psi'\left(\Delta^i\right)}{(-\varepsilon)^{\frac{4}{3}}} \leq \mathcal{M}. \tag{4.5}$$

Joining (4.4) and (4.5), the thesis follows immediately. □

Example 4.2.7. Let ψ be a weight function with the following properties:

- ψ is in $C^1((0, +\infty))$;
- if x is in $[1, +\infty)$, then $\psi(x) = 1$;
- if x is in $\left(\frac{1}{2}, 1\right)$ then $\psi(x) = 1 - (1 - x)^{\frac{4}{3}}$.

We notice that

$$\limsup_{\varepsilon \to 0} \frac{\psi(1) - \psi(1 + \varepsilon) + \psi'(1)\varepsilon}{|\varepsilon|^{\frac{4}{3}}} = 1.$$

Let us define $u \coloneqq \mathbb{1}_{\left[\frac{1}{2}, 1\right]}$. In particular, we have that $\mathcal{MS}_{\varphi;\psi}(u) = \psi(1) = 1$. Let a be in $[1, +\infty)$; let θ be a positive real number. Let us fix a positive real number ε such that ε^a is in $\left(0, \frac{1}{2\theta}\right)$; we define $\rho_\varepsilon : [0, 1] \to \mathbb{R}$ such that

$$\rho_\varepsilon(x) = \theta \left[x - \left(\frac{1}{2} - \varepsilon\right)\right]^a \mathbb{1}_{\left(\frac{1}{2} - \varepsilon, \frac{1}{2}\right]}(x).$$

We notice that

$$\|\rho_\varepsilon\|_{L^2} = \left(\int_0^\varepsilon \theta^a x^{2a} \, dx\right)^{\frac{1}{2}} = \frac{\theta}{\sqrt{2a+1}} \varepsilon^{a+\frac{1}{2}};$$

$$\|\dot{\rho}_\varepsilon\|_{L^2}^2 = \int_0^\varepsilon \theta^2 a^2 x^{2a-2}\, dx = \theta^2 \frac{a^2}{2a-1}\varepsilon^{2a-1};$$
$$\psi(1-\theta\varepsilon^a) = 1 - \theta^{\frac{4}{3}}\varepsilon^{a\frac{4}{3}}.$$

Hence, the following identities hold true:

$$\frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u+\rho_\varepsilon)}{\|\rho_\varepsilon\|_{L^2}} = \frac{-\|\dot{\rho}_\varepsilon\|_{L^2}^2 + \psi(1) - \psi(1-\theta\varepsilon^a)}{\|\rho_\varepsilon\|_{L^2}} = \frac{-\theta^2\frac{a^2}{2a-1}\varepsilon^{2a-1} + \theta^{\frac{4}{3}}\varepsilon^{a\frac{4}{3}}}{\frac{\theta}{\sqrt{2a+1}}\varepsilon^{a+\frac{1}{2}}}.$$

It's easy to see that if a is equal to $\frac{3}{2}$, then

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u+\rho_\varepsilon)}{\|\rho_\varepsilon\|_{L^2}} = -\frac{9}{4}\theta + 2\theta^{\frac{1}{3}}.$$

Therefore, we obtain that

$$\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) \geq \max_{\theta>0}\left\{-\frac{9}{4}\theta + 2\theta^{\frac{1}{3}}\right\} = \frac{8}{9}\sqrt{\frac{2}{3}}.$$

The example show that the lower bound given by 4.1 can be strict.

4.2.2 Upper bound for the slope

Definition 4.2.8 (Approximating sequence)**.**
Let u be in $\mathcal{SBV}$ such that $\mathcal{MS}_{\varphi;\psi}(u)$ is a real number. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{SBV}$ that converges toward u with respect to L^2 norm. We say that it is an approximating sequence for u if the following identity holds true:

$$\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) = \lim_{n\to+\infty} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n)}{\|u-u_n\|_{L^2}}.$$

Lemma 4.2.9. *Let u be in $\mathcal{SBV}$ such that $\mathcal{MS}_{\varphi;\psi}(u)$ is a real number. Let $\{u_n\}_{n\in\mathbb{N}}$ be an approximating sequence in $\mathcal{SBV}$ for u in the sense of definition 4.2.8. The following conclusions hold true:*

- $\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) = \limsup_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) = \mathcal{MS}_{\varphi;\psi}(u)$.
- *Let us consider the canonical decomposition $u_n = w_n + v_n$ as in 2.2.1, where w_n is the absolutely continuous part and v_n is the jump part. Similarly, we decompose $u = w + v$. Then, there exists a subsequence, not relabelled, with the following properties:*

  1. *$\{w_n\}_{n\in\mathbb{N}}$ converges uniformly in $[0,1]$ toward w and $\{v_n\}_{n\in\mathbb{N}}$ converges toward v with respect to L^2 norm and pointwise for almost every x in $(0,1)$;*
  2. $\liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n) = \limsup_{n\to+\infty} \mathcal{D}_\varphi(w_n) = \mathcal{D}_\varphi(w)$;
  3. $\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) = \limsup_{n\to+\infty} \mathcal{MS}_\psi(v_n) = \mathcal{MS}_\psi(v)$.

Proof. As for the first statement, we notice that $\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u)$ is nonnegative; hence, we can state that:

$$\liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) \geq 0.$$

In other words, we have

$$\limsup_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \leq \mathcal{MS}_{\varphi;\psi}(u).$$

As shown in theorem 2.2.5, $\mathcal{MS}_{\varphi;\psi}$ is a lower semicontinuous functional in $\mathcal{SBV}$. So, the following inequalities are proved:

$$\mathcal{MS}_{\varphi;\psi}(u) \leq \liminf_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \leq \limsup_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) \leq \mathcal{MS}_{\varphi;\psi}(u).$$

We remark that $\mathcal{MS}_{\varphi;\psi}(u)$ is a real number: then, up to further subsequences, not relabelled, there exists a real number $\mathcal{M}$ such that $\mathcal{MS}_{\varphi;\psi}(u_n) \leq \mathcal{M}$ and $\|u_n\|_{L^2} \leq \mathcal{M}$ for all n in $\mathbb{N}$. Thanks to theorem 2.2.5, up to further subsequences, $\{w_n\}_{n\in\mathbb{N}}$ converges uniformly in $[0,1]$ toward w; by difference $\{v_n\}_{n\in\mathbb{N}}$ converges toward v with respect to L^2 norm and pointwise for almost every x in $(0,1)$. Thanks to theorems 1.3.11 and 2.1.20, we also know that

$$\liminf_{n\to+\infty} \mathcal{D}_\varphi(w_n) \geq \mathcal{D}_\varphi(w),$$
$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n) \geq \mathcal{MS}_\psi(v).$$

Let us assume that there exists $\varepsilon_0 > 0$ such that

$$\limsup_{n\to+\infty} \mathcal{D}_\varphi(w_n) \geq \mathcal{D}_\varphi(w) + \varepsilon_0$$

Up to further subsequence, not relabelled, we can assume that for all n in $\mathbb{N}$ it holds that

$$\mathcal{D}_\varphi(w) \leq \mathcal{D}_\varphi(w_n) - \frac{\varepsilon_0}{2}.$$

Up to further subsequences, not relabelled, we can assume that if n is in $\mathbb{N}$, then

$$\mathcal{MS}_\psi(v) \leq \mathcal{MS}_\psi(v_n) + \frac{\varepsilon_0}{4}.$$

Hence, for all n in $\mathbb{N}$ the following inequalities hold true:

$$\begin{aligned}\mathcal{MS}_{\varphi;\psi}(u) &= \mathcal{D}_\varphi(w) + \mathcal{MS}_\psi(v)\\ &\leq \mathcal{D}_\varphi(w_n) - \frac{\varepsilon_0}{2} + \mathcal{MS}_\psi(v_n) + \frac{\varepsilon_0}{4}\\ &= \mathcal{MS}_{\varphi;\psi}(u_n) - \frac{\varepsilon_0}{4},\end{aligned}$$

that is against the first statement. We have shown that

$$\limsup_{n\to+\infty} \mathcal{D}_\varphi(w_n) \leq \mathcal{D}_\varphi(w).$$

Then, we can immediately conclude that

$$\limsup_{n\to+\infty} \mathcal{MS}_\psi(v_n) \leq \mathcal{MS}_\psi(v).$$

□

Definition 4.2.10 (Strictly weight function)**.**
Let ψ be a weight function as in 2.1.11 with the following properties:

- for all a, b in $\mathbb{R} \setminus \{0\}$ it holds that

$$\psi(a+b) < \psi(a) + \psi(b);$$

- if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence that converges toward $+\infty$, $\{b_n\}_{n\in\mathbb{N}}$ is a sequence that converges toward $-\infty$ and $\{a_n + b_n\}_{n\in\mathbb{N}}$ converges toward a real number, then

$$\liminf_{n\to+\infty} \psi(a_n + b_n) < \liminf_{n\to+\infty} \psi(a_n) + \liminf_{n\to+\infty} \psi(b_n).$$

We say that ψ is a strictly weight function.

Lemma 4.2.11 (Regularity of the approximating sequence)**.**
Let u be in $\tilde{\mathcal{SBV}}$ represented as in 2.2.3, namely

$$u = w + u(0) + \sum_{i=1}^{k} \Delta^i \mathbb{1}_{[x^i,1]}.$$

For all integer i in $\{0; \dots; k\}$ we denote $\Omega^i := [x^i, x^{i+1}]$ and $\Omega := [0,1]$. Let us assume that ψ is a strictly weight function as in 4.2.10. Let us assume that $\mathcal{MS}_{\varphi;\psi}(u)$ is a real number. Let $\mathcal{M}$ be any real number such that $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) > \mathcal{M}$. There exists a sequence $\{\bar{u}_n\}_{n\in\mathbb{N}}$ with the following properties:

- *$\{\bar{u}_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm;*
- *if n is in $\mathbb{N}$, then $\mathcal{S}(\bar{u}_n) = \mathcal{S}(u)$;*
- *if we represent the sequence $\{u_n\}_{n\in\mathbb{N}}$ as in 2.2.3, namely*

$$\bar{u}_n := \bar{u}_n(0) + \bar{w}_n + \bar{v}_n = \bar{u}_n(0) + \bar{w}_n + \sum_{i=1}^{k} \Delta_n^i \mathbb{1}_{[x^i,1]},$$

  then, for all integer i in $\{1; \dots; k\}$ it holds that $\{\Delta_n^i\}_{n\in\mathbb{N}}$ converges toward Δ^i and $\{\bar{u}_n(0)\}_{n\in\mathbb{N}}$ converges toward $u(0)$.
- $\displaystyle\limsup_{n\to+\infty} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\bar{u}_n)}{\|u - \bar{u}_n\|_{L^2(\Omega)}} \geq \mathcal{M}.$

Proof. **Step 1**: Let $\{u_n\}_{n\in\mathbb{N}}$ be any approximating sequence for u (see 4.2.8) represented as in 2.2.1, namely

$$u_n := u_n(0) + w_n + v_n.$$

Let $\mathcal{M}$ be any real number such that $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) > \mathcal{M}$. We set

$$\Delta_0 := \min\left\{\left|\Delta^i\right| \;\middle|\; i \in \{1; \dots; k\}\right\},$$

$$\Theta_{\mathcal{M}} := \sup\left\{\delta \in (0,+\infty) \;\middle|\; \forall x \in (0,\delta)\ \frac{\psi(x)}{x} \geq \mathcal{M}\right\}.$$

Let us fix $\varepsilon_0 > 0$ such that

$$\varepsilon_0 \leq \min\left\{\frac{\Delta_0}{4}; \Theta_{\mathcal{M}}\right\}. \tag{4.6}$$

Thanks to lemma 2.1.17, there exists a positive real number i_0 such that for all n in $\mathbb{N}$ the following inequalities hold true:

$$\left\|\sum_{i>i_0} v_n^i\right\|_\infty \leq \sum_{i>i_0}\left(\sum_{x\in\mathcal{S}(v_n^i)} |\Delta v_n(x)|\right) \leq \varepsilon_0. \tag{4.7}$$

For all n in $\mathbb{N}$ we consider the following decomposition:

$$a_n := u_n(0) + \sum_{i=1}^{i_0} v_n^i,$$

$$b_n := \sum_{i>i_0} v_n^i.$$

We notice that

$$\mathcal{S}(a_n) = \bigcup_{i=1}^{i_0} \mathcal{S}\left(v_n^i\right),$$

$$\mathcal{S}(b_n) = \bigcup_{i>i_0} \mathcal{S}\left(v_n^i\right).$$

For all n in $\mathbb{N}$ we have that

- if x is in $\mathcal{S}(b_n)$, then $|\Delta b_n(x)| \leq \frac{1}{i_0}$;
- $b_n(0) = 0$;
- $\mathcal{MS}_\psi(b_n) \leq \mathcal{MS}_{\varphi;\psi}(u_n)$.

Thanks to lemma 4.2.9, we have that

$$\lim_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) = \mathcal{MS}_{\varphi;\psi}(u) < +\infty;$$

in particular, the sequence $\{\mathcal{MS}_\psi(b_n)\}_{n\in\mathbb{N}}$ is bounded. Therefore, the proposition 2.1.18 guarantees the existence of a subsequence, not relabelled, with the following properties:

- for all integer $p > i_0$ there exists β^p in $\mathbb{N}$ such that if $n \geq p$, then $\operatorname{card}\mathcal{S}(v_n^p) = \beta^p$;
- for all integer $p > i_0$, for all $n \geq p$ we represent v_n^p as in 2.2.3, namely

  $$v_n^p = \sum_{t=1}^{\beta^p} \Delta_n^{p;t} \mathbb{1}_{[x_n^{p;t},1]}.$$

  Then, for all integer t in $\{1;\ldots;\beta^p\}$ there exists $x_\infty^{p;t}$ in $[0,1]$ and $\Delta_\infty^{p;t}$ whose absolute value is in $\left[\frac{1}{p}, \frac{1}{p-1}\right]$, such that

  $$\lim_{n\to+\infty} x_n^{p;t} = x_\infty^{p;t},$$

  $$\lim_{n\to+\infty} \Delta_n^{p;t} = \Delta_\infty^{p;t}.$$

- For all integer $p > i_0$, if we define

$$v_\infty^p := \sum_{t=1}^{\beta^p} \Delta_\infty^{p;t} \mathbb{1}_{\left[x_\infty^{p;t},1\right]}$$

then $\{v_n^p\}_{n\in\mathbb{N}}$ converges toward v_∞^p with respect to L^2 norm and pointwise for almost every x in $[0,1]$; moreover, if we define

$$\mathcal{B}^p := \left\{t \in \{1;\ldots;\beta^p\} \mid x_\infty^{p;t} = 0\right\},$$

then

$$v_\infty^p(0) = \sum_{t\in\mathcal{B}^p} \Delta_\infty^{p;t}.$$

- $\left\{\sum_{t=i_0+1}^{p} v_\infty^t\right\}_{p>i_0}$ is a Cauchy sequence with respect to L^2 norm; if we define

$$b := \sum_{p>i_0} v_\infty^p,$$

then, $\{b_n\}_{n\in\mathbb{N}}$ converges toward b with respect to L^2 norm.

In particular, b is in $\mathcal{PJ}$. Moreover, for all integer $p > i_0$ we have that:

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(v_n^p) \geq \mathcal{MS}_\psi(v_\infty^p);$$

we also know that

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(b_n) \geq \mathcal{MS}_\psi(b).$$

Obviously, we can state that $\|b\|_\infty \leq \varepsilon_0$.

The sequence $\{a_n\}_{n\in\mathbb{N}}$ is such that for all natural number n the following properties hold true:

- if x is in $\mathcal{S}(a_n)$, then $|\Delta a_n(x)| \geq \dfrac{1}{i_0}$;
- $\mathcal{MS}_\psi(a_n) \leq \mathcal{MS}_{\varphi;\psi}(u_n)$;
- $\|a_n\|_{L^2(\Omega)} \leq \|v_n + u_n(0)\|_{L^2(\Omega)} + \|b_n\|_{L^2(\Omega)} \leq \|v_n + u_n(0)\|_{L^2(\Omega)} + \varepsilon_0$.

In particular, we can assume that the sequences $\{\mathcal{MS}_\psi(a_n)\}_{n\in\mathbb{N}}$ and $\left\{\|a_n\|_{L^2(\Omega)}\right\}_{n\in\mathbb{N}}$ are bounded, because

$$\lim_{n\to+\infty} \mathcal{MS}_{\varphi;\psi}(u_n) = \mathcal{MS}_{\varphi;\psi}(u) < +\infty,$$

$$\lim_{n\to+\infty} \|v_n + u_n(0)\|_{L^2(\Omega)} = \|v + u(0)\|_{L^2(\Omega)} < +\infty$$

as shown in lemma 4.2.9. If we define $a := v + u(0) - b$, it's easy to see that $\{a_n\}_{n\in\mathbb{N}}$ converges toward a with respect to L^2 norm. Up to further subsequences, not relabelled, we can assume that the convergence is pointwise for almost every x in $(0,1)$. If we recall definition 2.1.12 and the fact that $\mathcal{I}_\psi\left(\frac{1}{i_0}\right) > 0$, it's immediate to see that the following inequality holds for all n in $\mathbb{N}$:

$$\mathcal{I}_\psi\left(\frac{1}{i_0}\right) \operatorname{card} S(a_n) \leq \mathcal{MS}_\psi(a_n) \leq \mathcal{MS}_{\varphi;\psi}(u_n);$$

hence, there exist another subsequence, not relabelled, and a natural number j such that $\operatorname{card}\mathcal{S}(a_n) = j$ for all n in $\mathbb{N}$. Thanks to proposition 2.1.19, we can state that a is in $\mathcal{PJ}$, $\operatorname{card}\mathcal{S}(a) \leq j$ and

$$\liminf_{n\to+\infty} \mathcal{MS}_\psi(a_n) \geq \mathcal{MS}_\psi(a).$$

Step 2: By definition, $v + u(0) = a + b$; we know that the sets $\mathcal{S}(v)$ and $\mathcal{S}(a)$ are finite; therefore, $\mathcal{S}(b)$ is finite too. We set $A := \operatorname{card}\mathcal{S}(a)$ and $B := \operatorname{card}\mathcal{S}(b)$; let a and b be represented as in 2.2.3, namely

$$a := a(0) + \sum_{i=1}^{A} \alpha^i \mathbb{1}_{[y^i,1]},$$

$$b := b(0) + \sum_{i=1}^{B} \beta^i \mathbb{1}_{[z^i,1]}.$$

We know that $\mathcal{S}(v)$ is contained in $\mathcal{S}(a) \cup \mathcal{S}(b)$ and

$$\begin{aligned}
\mathcal{MS}_\psi(v + u(0)) &= \lim_{n\to+\infty} \mathcal{MS}_\psi(v_n + u_n(0)) \\
&= \lim_{n\to+\infty} \mathcal{MS}_\psi(a_n) + \mathcal{MS}_\psi(b_n) \\
&\geq \liminf_{n\to+\infty} \mathcal{MS}_\psi(a_n) + \liminf_{n\to+\infty} \mathcal{MS}_\psi(b_n) \\
&\geq \mathcal{MS}_\psi(a) + \mathcal{MS}_\psi(b) \\
&\geq \mathcal{MS}_\psi(v + u(0)).
\end{aligned}$$

We claim that $\mathcal{S}(b) \cap \mathcal{S}(a) = \emptyset$. Since ψ is a strictly weight function (see 4.2.10), if there exists x in $\mathcal{S}(b) \cap \mathcal{S}(a)$, then

$$\mathcal{MS}_\psi(a) + \mathcal{MS}_\psi(b) > \mathcal{MS}_\psi(v + u(0)),$$

that is absurd, obviously. We claim that $\mathcal{S}(b) = \emptyset$. We know that

$$\|b\|_\infty \leq \varepsilon_0 \leq \frac{\Delta_0}{4}.$$

If there exists x_0 in $\mathcal{S}(b)$, then

$$|\Delta b(x)| \leq 2\varepsilon_0 \leq \frac{\Delta_0}{2}.$$

In particular, we find that x_0 must be in $\mathcal{S}(a)$, that is absurd. We also claim that $\mathcal{S}(v) = \mathcal{S}(a)$. If there exists x_0 in $\mathcal{S}(a) \setminus \mathcal{S}(v)$, then x_0 must be in $\mathcal{S}(b)$, that is absurd.

We have just shown that $b(x) = b(0)$ for all x in Ω. We claim that $b(0) = 0$. First of all, we have that

$$\limsup_{n\to+\infty} \mathcal{MS}_\psi(b_n) = 0.$$

By contradiction, if there exists a positive real number ε such that

$$\limsup_{n\to+\infty} \mathcal{MS}_\psi(b_n) > \varepsilon,$$

then, up to further subsequences, not relabelled, $\mathcal{MS}_\psi(b_n) \geq \varepsilon$ for all n in $\mathbb{N}$. Thanks to theorem 2.2.5, up to further subsequences, not relabelled, $\mathcal{MS}_\psi(a) - \frac{\varepsilon}{2} \leq \mathcal{MS}_\psi(a_n)$ for all n in $\mathbb{N}$. Hence, if n is in $\mathbb{N}$, then

$$\mathcal{MS}_\psi(a) + \frac{\varepsilon}{2} \leq \mathcal{MS}_\psi(a_n) + \mathcal{MS}_\psi(b_n) = \mathcal{MS}_\psi(a_n + b_n).$$

Since we have that

$$\mathcal{MS}_\psi(v + u(0)) + \frac{\varepsilon}{2} = \mathcal{MS}_\psi(a) + \frac{\varepsilon}{2} \leq \lim_{n\to+\infty} \mathcal{MS}_\psi(a_n + b_n) = \mathcal{MS}_\psi(v + u(0)),$$

that is absurd. If we recall that

$$b_n = \sum_{i>i_0} v_n^i,$$

we have shown that

$$0 = \lim_{n\to+\infty} \mathcal{MS}_\psi\left(\sum_{i>i_0} v_n^i\right) = \lim_{n\to+\infty} \sum_{i>i_0} \mathcal{MS}_\psi(v_n^i).$$

It immediately follows that for all integer $i > i_0$, the following identity holds true:

$$\lim_{n\to+\infty} \mathcal{MS}_\psi(v_n^i) = 0.$$

If there exists $i_1 > i_0$, $\varepsilon > 0$ and a specific subsequence, not relabelled, such that for all n in $\mathbb{N}$ it holds that $\mathcal{MS}_\psi(v_n^{i_1}) > \varepsilon$, then

$$0 = \lim_{n\to+\infty} \sum_{i>i_0} \mathcal{MS}_\psi(v_n^i) \geq \limsup_{n\to+\infty} \mathcal{MS}_\psi(v_n^{i_1}) \geq \varepsilon,$$

that is absurd. Therefore, for all integer $i > i_0$, the following identity holds true:

$$0 = \lim_{n\to+\infty} \mathcal{MS}_\psi(v_n^i) = \mathcal{MS}_\psi(v_\infty^i).$$

Hence, for all x in Ω for all integer $i > i_0$ it holds that $v_\infty^i(x) = v_\infty^i(0)$. We claim that $v_\infty^i(0) = 0$ for all integer $i > i_0$. Let us fix $i_1 > i_0$; we know that

$$v_\infty^{i_1}(0) = \sum_{t\in\mathcal{B}^{i_1}} \Delta_\infty^{i_1;t}.$$

Since $|\Delta_\infty^{t;i_1}|$ is in $\left[\frac{1}{i_1}, \frac{1}{i_1-1}\right]$, if $\mathcal{B}^{i_1}$ is not empty we find that

$$0 = \lim_{n\to+\infty} \mathcal{MS}_\psi(v_n^{i_1}) \geq \lim_{n\to+\infty} \sum_{t\in\mathcal{B}^{i_1}} \psi(\Delta_n^{i_1;t}) = \sum_{t\in\mathcal{B}^{i_1}} \psi(\Delta_\infty^{i_1;t}) > 0,$$

that is absurd. By definition of b, we can easily conclude $b(x) = 0$ for all x in Ω.

Step 3: We define $\tilde{u}_n := w_n + a_n$. We claim that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ has the following properties:

- $\operatorname{card} \mathcal{S}(a_n) = j$ for all n in $\mathbb{N}$;
- $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm;
- $\limsup_{n\to+\infty} \dfrac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)}{\|u-\tilde{u}_n\|_{L^2(\Omega)}} \geq \mathcal{M}$.

Since $\left\{\|b_n\|_{L^2(\Omega)}\right\}_{n\in\mathbb{N}}$ is an infinitesimal sequence, we notice that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm. More precisely, the following inequalities hold true:

$$\|u-\tilde{u}_n\|_{L^2(\Omega)} \leq \|u-u_n\|_{L^2(\Omega)} + \|b_n\|_{L^2(\Omega)} \leq \|u-u_n\|_{L^2(\Omega)} + \sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|.$$

If we join (4.6), (4.7) and the definition of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, the following inequality are proved:

$$\begin{aligned}
\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n) &= \mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) + \sum_{x\in\mathcal{S}(b_n)} \psi(\Delta b_n(x)) \\
&= \mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) + \sum_{x\in\mathcal{S}(b_n)} \psi(|\Delta b_n(x)|) \\
&\geq \mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) + \psi\left(\sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|\right) \\
&\geq \mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) + \mathcal{M}\sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|.
\end{aligned}$$

Let n_0 be a natural number such that for all integer $n \geq n_0$ it holds that

$$\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) \geq \mathcal{M}\,\|u-u_n\|_{L^2(\Omega)}.$$

Hence, if n is any integer greater than or equal to n_0, the following inequalities hold true:

$$\begin{aligned}
\frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)}{\|u-\tilde{u}_n\|_{L^2(\Omega)}} &\geq \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n) + \mathcal{M}\sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|}{\|u-u_n\|_{L^2(\Omega)} + \sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|} \\
&\geq \frac{\mathcal{M}\,\|u-u_n\|_{L^2(\Omega)} + \mathcal{M}\sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|}{\|u-u_n\|_{L^2(\Omega)} + \sum_{x\in\mathcal{S}(b_n)} |\Delta b_n(x)|} = \mathcal{M}.
\end{aligned}$$

In other words, we have just shown that:

$$\limsup_{n\to+\infty} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)}{\|u-\tilde{u}_n\|_{L^2(\Omega)}} \geq \mathcal{M}.$$

Step 4: Having said that, for all n in $\mathbb{N}$ we represent a_n as in 2.2.3, namely

$$a_n = w_n + u_n(0) + \sum_{i=1}^{j} \Delta_n^i \mathbb{1}_{[x_n^i,1]}$$

where $0 \coloneqq x_n^0 < x_n^1 < \cdots < x_n^j < x_n^{j+1} \coloneqq 1$. We notice that $|\Delta_n^i| \geq \frac{1}{i_0}$ for all n in $\mathbb{N}$ for all i in $\{1; \ldots; j\}$. Up to further subsequence, not relabelled, we can assume that for all i in $\{0; \ldots; j+1\}$ there exists x_∞^i in $[0,1]$ such that

$$\lim_{n \to +\infty} x_n^i = x_\infty^i.$$

Let r, t be integers in $\{0; \ldots; j+1\}$; we declare that r, t are equivalent if and only if $x_\infty^t = x_\infty^r$. This induces a partition on $\{0; \ldots; j+1\}$ into disjoint sets. In other words, there exist a natural number h and a collection of pairwise disjoint sets

$$\{\mathcal{A}^0; \ldots; \mathcal{A}^{h+1}\}$$

that cover $\{0; \ldots; j+1\}$ (see 2.1.19). So, for all i in $\{0; \ldots; h+1\}$ we can well define $y^i \coloneqq x_\infty^r$, where r is any index in $\mathcal{A}^i$. We recall that r belongs to $\mathcal{A}^0$ if and only if $x_\infty^r = 0$; similarly, r belongs to $\mathcal{A}^{h+1}$ if and only if $x_\infty^r = 1$. For all i in $\{0; \ldots; h+1\}$ for all n in $\mathbb{N}$ we define

$$\Theta_n^i \coloneqq \sum_{t \in \mathcal{A}^i} \Delta_n^t.$$

As shown in 2.1.19, up to further subsequences, not relabelled, for all i in $\{0; \ldots; h\}$ there exists a real number Θ^i such that

$$\lim_{n \to +\infty} \Theta_n^i = \Theta^i.$$

We set

$$\mathcal{B} \coloneqq \left\{ i \in \{1; \ldots; h\} \;\middle|\; \forall t \in \{0; \ldots; k+1\} \; y^i \neq x^t \right\},$$
$$\mathcal{C} \coloneqq \{1; \ldots; h\} \setminus \mathcal{B}.$$

As shown in proposition 2.1.19, for all i in $\{1; \ldots; h\}$ we have that $\Theta^i \neq 0$ if and only i belongs to $\mathcal{C}$ and it holds that $\Theta^i = \Delta v\,(y^i)$.

We claim that $\mathcal{A}^0 = \mathcal{A}^{h+1} = \mathcal{B} = \emptyset$ and if i is in $\{1; \ldots; h\}$ then $\operatorname{card} \mathcal{A}^i = 1$. We have shown that

$$\mathcal{MS}_\psi(v + u(0)) = \lim_{n \to +\infty} \mathcal{MS}_\psi(a_n) \geq \mathcal{MS}_\psi(a).$$

If we rearrange terms, we find that

$$\begin{aligned}
0 = {} & \lim_{n \to +\infty} \sum_{i \in \mathcal{C}} \left(\sum_{t \in \mathcal{A}^i} \psi(\Delta_n^t) - \psi(\Delta v\left(y^i\right)) \right) \\
& + \sum_{t \in \mathcal{A}^0} \psi(\Delta_n^t) + \sum_{t \in \mathcal{A}^{h+1}} \psi(\Delta_n^t) + \sum_{i \in \mathcal{B}} \left(\sum_{t \in \mathcal{A}^i} \psi(\Delta_n^i) \right) \\
\geq {} & \sum_{i \in \mathcal{C}} \liminf_{n \to +\infty} \left(\sum_{t \in \mathcal{A}^i} \psi(\Delta_n^t) - \psi(\Delta v\left(y^i\right)) \right) \\
& + \sum_{t \in \mathcal{A}^0} \liminf_{n \to +\infty} \psi(\Delta_n^t) + \sum_{t \in \mathcal{A}^{h+1}} \liminf_{n \to +\infty} \psi(\Delta_n^t) + \sum_{i \in \mathcal{B}} \liminf_{n \to +\infty} \left(\sum_{t \in \mathcal{A}^i} \psi(\Delta_n^t) \right) \\
\geq {} & \sum_{i \in \mathcal{C}} \liminf_{n \to +\infty} \left(\sum_{t \in \mathcal{A}^i} \psi(\Delta_n^t) - \psi(\Delta v\left(y^i\right)) \right) \\
& + \left(\operatorname{card} \mathcal{A}^0 + \operatorname{card} \mathcal{A}^{h+1} + \sum_{i \in \mathcal{B}} \operatorname{card} \mathcal{A}^i \right) \mathcal{I}_\psi \left(\frac{1}{i_0} \right),
\end{aligned} \tag{4.8}$$

where $\mathcal{I}_\psi\left(\frac{1}{i_0}\right)$ is defined as in 2.1.12 and it is positive (see 2.1.13). We also notice that for all i in $\{1;\ldots;h\}$ it holds that

$$\begin{aligned}\liminf_{n\to+\infty}\sum_{t\in\mathcal{A}^i}\psi(\Delta_n^t)-\psi(\Delta v\left(y^i\right))&\geq\liminf_{n\to+\infty}\psi\left(\sum_{t\in\mathcal{A}^i}\Delta_n^t\right)-\psi(\Delta v\left(y^i\right))\\&\geq\psi\left(\liminf_{n\to+\infty}\Theta_n^i\right)-\psi(\Delta v\left(y^i\right))=0.\end{aligned}$$

In other words, each addendum of the (4.8) is nonnegative. To be valid, each addendum of the sum must be zero. In particular, $\mathcal{A}^0=\mathcal{A}^{h+1}=\mathcal{B}=\emptyset$.

We show that if i is in $\mathcal{C}$, then $\operatorname{card}\mathcal{S}\left(\mathcal{A}^i\right)=1$. Obviously, $\mathcal{S}\left(\mathcal{A}^i\right)\neq\emptyset$. By contradiction, we assume that there exists an integer i in $\mathcal{C}$ such that $\operatorname{card}\mathcal{A}^i>1$; in particular, there exist two disjoint, non empty sets $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P}\cup\mathcal{Q}=\mathcal{A}^i$ and, up to further subsequences, not relabelled, there exist p in $[0,+\infty]$ and q in $[-\infty,0]$ such that

$$\lim_{n\to+\infty}\sum_{t\in\mathcal{P}}\Delta_n^t=p,$$

$$\lim_{n\to+\infty}\sum_{t\in\mathcal{Q}}\Delta_n^t=q.$$

So, we have that

$$\begin{aligned}0&=\liminf_{n\to+\infty}\sum_{t\in\mathcal{A}^i}\psi(\Delta_n^t)-\psi(\Delta v\left(y^i\right))\\&\geq\liminf_{n\to+\infty}\sum_{t\in\mathcal{P}}\psi(\Delta_n^t)+\liminf_{n\to+\infty}\sum_{t\in\mathcal{Q}}\psi(\Delta_n^t)-\psi(\Delta v\left(y^i\right))\\&\geq\liminf_{n\to+\infty}\psi\left(\sum_{t\in\mathcal{P}}\Delta_n^t\right)+\liminf_{n\to+\infty}\psi\left(\sum_{t\in\mathcal{Q}}\Delta_n^t\right)-\psi(\Delta v\left(y^i\right))\\&\geq\liminf_{n\to+\infty}\psi\left(\sum_{t\in\mathcal{P}}\Delta_n^t\right)+\liminf_{n\to+\infty}\psi\left(\sum_{t\in\mathcal{Q}}\Delta_n^t\right)-\liminf_{n\to+\infty}\psi\left(\sum_{t\in\mathcal{P}}\Delta_n^t+\sum_{t\in\mathcal{Q}}\Delta_n^t\right)\\&>0,\end{aligned}$$

because ψ is a strictly weight function (see 4.2.10); this is absurd.

Step 5: To conclude, we show that we can assume that $x_n^i=x^i$ for all n in $\mathbb{N}$ for all i in $\{1;\ldots;k\}$. If there exists a specific subsequence, not relabelled, such that for all i in $\{1;\ldots;k\}$ for all n in $\mathbb{N}$ it holds that $x_n^i=x^i$, the conclusion is trivial. Hence, we can assume that there exists i_0 in $\{1;\ldots;k\}$ such that $x_n^{i_0}\neq x^{i_0}$ for all n in $\mathbb{N}$. We define $y_n^i:=x_n^i$ if $i\neq i_0$ and $y_n^{i_0}:=x^{i_0}$. Moreover, we define

$$\bar{u}_n:=w_n+u_n(0)+\sum_{i=1}^k\Delta_n^i\mathbb{1}_{[y_n^i,1]}.$$

We claim that $\{\bar{u}_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm and

$$\limsup_{n\to+\infty}\frac{\mathcal{MS}_{\varphi;\psi}(u)-\mathcal{MS}_{\varphi;\psi}(\bar{u}_n)}{\|u-\bar{u}_n\|_{L^2(\Omega)}}\geq\mathcal{M}.$$

We notice that $\mathcal{MS}_{\varphi;\psi}(\bar{u}_n) = \mathcal{MS}_{\varphi;\psi}(u_n)$ for all n in $\mathbb{N}$. If we show that for all n in $\mathbb{N}$ it holds that

$$\|u - \bar{u}_n\|_{L^2(\Omega)} \leq \|u - \tilde{u}_n\|_{L^2(\Omega)},$$

then, we immediately obtain that for all n in $\mathbb{N}$ it holds that

$$\frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\tilde{u}_n)}{\|u - \tilde{u}_n\|_{L^2(\Omega)}} \leq \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(\bar{u}_n)}{\|u - \bar{u}_n\|_{L^2(\Omega)}}.$$

This is enough to conclude that $\{\bar{u}_n\}_{n\in\mathbb{N}}$ satisfies our requests.

We know that $\{w_n \mid n \in \mathbb{N}\} \cup \{w\}$ is a family of equicontinuous functions (see theorem 1.3.4) and $\{\tilde{u}_n(x)\}_{n\in\mathbb{N}}$ converges toward $u(x)$ for all x in E, where E is a subset in $[0,1]$ such that $\mathscr{L}(E) = 1$. Without loss of generality, we can assume that $x^{i_0} > x_n^{i_0}$ for all n in $\mathbb{N}$. Let us denote

$$\varepsilon_1 := \frac{|\Delta^{i_0}|}{10}.$$

Let $\delta_1 > 0$ be corresponding to ε_1 in the definition of uniform continuity. We can assume that

$$\delta_1 \leq \min\left\{ \frac{x^{i+1} - x^i}{10} \;\middle|\; i \in \{0;\ldots;k\} \right\}.$$

Let x_{δ_1} be in $\left(x^{i_0}, x^{i_0} + \frac{\delta_1}{2}\right) \cap E$; let y_{δ_1} be in $\left(x^{i_0} - \delta_1, x^{i_0} - \frac{\delta_1}{2}\right) \cap E$. Let n_0 be a natural number such that for all $n \geq n_0$ it holds that

- $|x_n^i - x^i| \leq \frac{\delta_1}{2}$ for all $i \in \{1;\ldots;k\}$;
- $|u(x_{\delta_1}) - \tilde{u}_n(x_{\delta_1})| \leq \varepsilon_1$;
- $|u(y_{\delta_1}) - \tilde{u}_n(y_{\delta_1})| \leq \varepsilon_1$.

Thanks to our assumption, we can state that if $n \geq n_0$ then $\tilde{u}_n$ is a continuous function in $\left(x_n^{i_0}, x^{i_0} + \frac{\delta_1}{2}\right)$; moreover, for all x in $(x_n^{i_0}, x^{i_0})$ the following inequalities hold true:

$$\begin{aligned}
|\tilde{u}_n(x) - u(x)| \geq{}& \left|u(x^{i_0})^+ - u(x^{i_0})^-\right| \\
&- |\tilde{u}_n(x) - \tilde{u}_n(x_{\delta_1})| \\
&- |\tilde{u}_n(x_{\delta_1}) - u(x_{\delta_1})| \\
&- \left|u(x_{\delta_1}) - u(x^{i_0})^+\right| \\
&- \left|u(x) - u(x^{i_0})^-\right| \\
\geq{}& \left|\Delta^{i_0}\right| - 4\varepsilon_0 = \frac{3}{5}\left|\Delta^{i_0}\right| \geq \frac{|\Delta^{i_0}|}{2}
\end{aligned}$$

Similarly, we have $\bar{u}_n$ is a continuous function in $(x^{i_0} - \delta_1, x^{i_0})$. Hence, for all x in $(x_n^{i_0}, x^{i_0}]$ we have that

$$\begin{aligned}
|\bar{u}_n(x) - u(x)| \leq{}& |\bar{u}_n(x) - \bar{u}_n(y_{\delta_1})| \\
&+ |\bar{u}_n(y_{\delta_1}) - u(y_{\delta_1})| \\
&+ |u(y_{\delta_1}) - u(x)| \\
\leq{}& 3\varepsilon_0 = \frac{3}{10}\left|\Delta^{i_0}\right| \leq \frac{|\Delta^{i_0}|}{2}
\end{aligned}$$

Having said that, we conclude with the following inequalities:

$$\begin{aligned}\|u-\bar{u}_n\|^2_{L^2(\Omega)} &= \int_0^{x_n^{i_0}} (u-\bar{u}_n)^2\, dx + \int_{x_n^{i_0}}^{x^{i_0}} (u-\bar{u}_n)^2\, dx + \int_{x^{i_0}}^1 (u-\bar{u}_n)^2\, dx \\ &= \int_0^{x_n^{i_0}} (u-\tilde{u}_n)^2\, dx + \int_{x_n^{i_0}}^{x^{i_0}} (u-\bar{u}_n)^2\, dx + \int_{x^{i_0}}^1 (u-\tilde{u}_n)^2\, dx \\ &\le \int_0^{x_n^{i_0}} (u-\tilde{u}_n)^2\, dx + \int_{x^{i_0}}^1 (u-\tilde{u}_n)^2\, dx + (x^{i_0}-x_n^{i_0})\left(\frac{|\Delta^{i_0}|}{2}\right)^2 \\ &\le \int_0^{x_n^{i_0}} (u-\tilde{u}_n)^2\, dx + \int_{x^{i_0}}^1 (u-\tilde{u}_n)^2\, dx + \int_{x_n^{i_0}}^{x^{i_0}} (u-\tilde{u}_n)^2\, dx \\ &= \|u-\tilde{u}_n\|^2_{L^2(\Omega)}.\end{aligned}$$

□

Theorem 4.2.12 (Upper bound for the descending metric slope)**.**
Let ψ be a strictly weight function as in 4.2.10. Let us assume that φ is in $C^2(\mathbb{R})$ and that there exists $\gamma > 0$ such that $\varphi''(x) \ge 2\gamma$ for all x in $\mathbb{R}$. Let u be in $\tilde{\mathcal{SBV}}$ represented as in 2.2.3, namely

$$u := w + u(0) + \sum_{i=1}^{k} \Delta^i \mathbb{1}_{[x^i,1]}.$$

If i is any integer in $\{0;\dots;k\}$, we denote $\Omega^i = [x^i, x^{i+1}]$; we also denote $\Omega := [0,1]$. Let us assume that

- *$\varphi'(\dot{u})$ is in $W^{1;2}(\Omega)$;*
- *$\varphi'(\dot{u}(0)) = \varphi'(\dot{u}(1)) = 0$;*
- *if i is any integer in $\{1;\dots;k\}$, then $\varphi'(\dot{u}(x^i)) = \psi'(\Delta^i)$;*
- *if i is any integer in $\{1;\dots;k\}$, there exist β^i in $[0,+\infty)$ such that*

$$\limsup_{\delta\to 0} \frac{\psi(\Delta^i) - \psi(\Delta^i+\delta) + \delta\psi'(\Delta^i)}{|\delta|^{\frac{4}{3}}} \le \beta^i.$$

Then, the following inequality holds true:

$$\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) \le \left\|[\varphi'(\dot{u})]'\right\|_{L^2(\Omega)} + \frac{16}{3\sqrt{3\gamma}} \sum_{i=1}^{k} (\beta^i)^{\frac{3}{2}}.$$

Proof. **Step 1**: Let $\mathcal{M}$ be any real number such that $\left|\nabla \mathcal{MS}_{\varphi;\psi}\right|(u) > \mathcal{M}$. Thanks to lemma 4.2.11, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{SBV}$ with the following properties:

- $\{u_n\}_{n\in\mathbb{N}}$ converges toward u with respect to L^2 norm;
- if n is any natural number, then $\mathcal{S}(u_n) = \mathcal{S}(u)$;
- $\displaystyle\limsup_{n\to+\infty} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n)}{\|u-u_n\|_{L^2(\Omega)}} = \lim_{n\to+\infty} \frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u_n)}{\|u-u_n\|_{L^2(\Omega)}} \ge \mathcal{M}.$

- For all n in $\mathbb{N}$ we represent u_n as in 2.2.3, namely

$$u_n \coloneqq w_n + u_n(0) + \sum_{i=1}^{k} \Delta_n^i \mathbb{1}_{[x^i,1]};$$

for all i in $\{1;\ldots;k\}$ we have that

$$\lim_{n\to+\infty} \Delta_n^i = \Delta^i.$$

If n is any natural number and i is any integer in $\{1;\ldots;k\}$, we define $\rho_n \coloneqq u_n - u$ and $\chi_n^i \coloneqq \Delta_n^i - \Delta^i$. We set

$$A \coloneqq \max\left\{ \frac{2}{x^i - x^{i-1}} \;\middle|\; i \in \{1;\ldots;k+1\} \right\}, \quad B \coloneqq (4A)^{\frac{2}{3}}, \quad C \coloneqq \frac{1}{A}.$$

If i is any integer in $\{1;\ldots;k\}$, we set

$$\Lambda_+^i \coloneqq (x^i, x^i + C), \quad \Lambda_-^i \coloneqq (x^i - C, x^i), \quad \Lambda^i \coloneqq \Lambda_+^i \cup \Lambda_-^i.$$

We remark that if $i \neq j$ then $\Lambda^i \cap \Lambda^j = \emptyset$.

Since φ is a function in $C^2(\mathbb{R})$ and $\dot{u}$ and $\dot{u}_n$ are in $L^2(\Omega)$ for all n in $\mathbb{N}$, we notice that for all natural number n for almost every x in Ω there exists ξ_x in $\mathbb{R}$ with the following properties:

- $\varphi(\dot{u}(x) + \dot{\rho}_n(x)) = \varphi(\dot{u}(x)) + \dot{\rho}_n(x)\varphi'(\dot{u}(x)) + \dfrac{\dot{\rho}_n(x)^2}{2}\varphi''(\xi_x);$

- $|\xi_x - \dot{u}(x)| \leq |\dot{\rho}_n(x)|$.

If we rearrange terms and use the definition of γ, we obtain that the following inequality holds for all n in $\mathbb{N}$ for almost every x in Ω:

$$\begin{aligned}\varphi(\dot{u}(x)) - \varphi(\dot{u}(x) + \dot{\rho}_n(x)) &= -\dot{\rho}_n(x)\varphi'(\dot{u}(x)) - \frac{\dot{\rho}_n(x)^2}{2}\varphi''(\xi_x) \\ &\leq -\dot{\rho}_n(x)\varphi'(\dot{u}(x)) - \gamma\dot{\rho}_n(x)^2.\end{aligned}$$

Hence, if n is any natural number, the following inequalities hold true:

$$\frac{\mathcal{MS}_{\varphi;\psi}(u) - \mathcal{MS}_{\varphi;\psi}(u+\rho_n)}{\|\rho_n\|_{L^2(\Omega)}} \tag{4.9}$$

$$= \frac{\sum_{i=0}^{k}\left[\int_{\Omega^i} (\varphi(\dot u) - \varphi(\dot u + \dot\rho_n))\, dx\right] + \sum_{i=1}^{k}\left[\psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)\right]}{\|\rho_n\|_{L^2(\Omega)}} \tag{4.10}$$

$$\leq \frac{\sum_{i=0}^{k}\left[-\int_{\Omega^i} \dot\rho_n \varphi'(\dot u)\; dx - \gamma\,\|\dot\rho_n\|^2_{L^2(\Omega^i)}\right] + \sum_{i=1}^{k}\left[\chi_n^i\psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)\right]}{\|\rho_n\|_{L^2(\Omega)}}$$

$$= \frac{\sum_{i=0}^{k}\left[\int_{\Omega^i} \rho_n\,[\varphi'(\dot u)]'\, dx - \gamma\,\|\dot\rho_n\|^2_{L^2(\Omega^i)}\right] + \sum_{i=1}^{k}\left[\chi_n^i\psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)\right]}{\|\rho_n\|_{L^2(\Omega)}}$$

$$= \frac{\int_{\Omega} \rho_n\,[\varphi'(\dot u)]'\, dx - \sum_{i=0}^{k}\gamma\,\|\dot\rho_n\|^2_{L^2(\Omega^i)} + \sum_{i=1}^{k}\left[\chi_n^i\psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)\right)}{\|\rho_n\|_{L^2(\Omega)}} \tag{4.11}$$

$$\leq \left\|[\varphi'(\dot u)]'\right\|_{L^2(\Omega)} + \frac{-\sum_{i=0}^{k}\gamma\,\|\dot\rho_n\|^2_{L^2(\Omega^i)} + \sum_{i=1}^{k}\left[\chi_n^i\psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)\right]}{\|\rho_n\|_{L^2(\Omega)}}$$

$$\leq \left\|[\varphi'(\dot u)]'\right\|_{L^2(\Omega)} + \sum_{i=1}^{k}\left[\frac{-\gamma\,\|\dot v_n\|^2_{L^2(\Lambda^i)} + \chi_n^i\psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)}{\|\rho_n\|_{L^2(\Omega)}}\right]$$

In (4.9) we used the definition of ρ_n and χ_n^i; in (4.10) we integrate by parts and we used the Neumann boundary conditions; in (4.11) we used the Hölder's inequality.

Let i be any integer in $\{1;\ldots;k\}$; we claim that

$$\limsup_{n\to+\infty} \frac{-\gamma\,\|\dot\rho\|^2_{L^2(\Lambda^i)} + \chi_n^i\psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi_n^i)}{\|\rho_n\|_{L^2(\Omega)}} \leq \frac{16}{3\sqrt{3\gamma}}(\beta^i)^{\frac{3}{2}}. \tag{4.12}$$

We notice that, if we show (4.12), then the thesis follows immediately.

Step 2: Let us fix i in $\{1;\ldots;k\}$; let us assume that

$$\limsup_{n\to+\infty} \frac{\psi(\Delta^i) - \psi(\Delta^i + \chi_n^i) + \chi_n^i\psi'(\Delta^i)}{|\chi_n^i|^{\frac{4}{3}}} < \beta^i.$$

So, there exists n_i in $\mathbb{N}$ such that if n is any integer greater than or equal to n_i, then

$$\frac{\psi(\Delta^i) - \psi(\Delta^i + \chi_n^i) + \chi_n^i\psi'(\Delta^i)}{|\chi_n^i|^{\frac{4}{3}}} \leq \beta^i.$$

In other words, we obtain that

$$\psi(\Delta^i) - \psi(\Delta^i + \chi_n^i) + \chi_n^i\psi'(\Delta^i) \leq \; \beta^i \left|\chi_n^i\right|^{\frac{4}{3}}. \tag{4.13}$$

Hence, if n in any integer such that $n \geq n_i$, the following inequality holds true:

$$\frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \chi^i_n \psi'(\Delta^i) + \psi(\Delta^i) - \psi(\Delta^i + \chi^i_n)}{\left\|\rho_n\right\|_{L^2(\Omega)}} \leq \frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Omega)}}. \quad (4.14)$$

Let us denote

$$\mathcal{K}^i := \limsup_{n\to+\infty} \frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Omega)}}; \quad (4.15)$$

if we show that

$$\mathcal{K}^i \leq \frac{16}{3\sqrt{3\gamma}} (\beta^i)^{\frac{3}{2}},$$

then (4.12) follows immediately. If $\mathcal{K}^i \leq 0$, the conclusion is trivial; hence, we can assume that $\mathcal{K}^i > 0$. Up to further subsequences, not relabelled, we can suppose that if n is any integer greater than or equal to n_i, then

$$\frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Omega)}} > 0.$$

Having said that, we can state that for all integer $n \geq n_i$, it holds that

$$\frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Omega)}} \leq \frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Lambda^i)}}.$$

Therefore, it is enough to show that

$$\limsup_{n\to+\infty} \frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Lambda^i)}} \leq \frac{16}{3\sqrt{3\gamma}} (\beta^i)^{\frac{3}{2}}.$$

Let n be an integer greater than or equal to n_i. By definition 2.1.4, we have that $\chi^i_n = \rho_n(x^i)^+ - \rho_n(x^i)^-$; thanks to the triangular inequality, we obtain that

$$\left|\chi^i_n\right| \leq \left|\rho_n(x^i)^+\right| + \left|\rho_n(x^i)^-\right|. \quad (4.16)$$

Thanks to the mean value theorem, there exists x^i_n in Λ^i_- such that

$$\left|\rho(x^i_n)\right|^2 \leq \frac{\left\|\rho_n\right\|^2_{L^2(\Lambda^i_-)}}{C} \leq A \left\|\rho_n\right\|^2_{L^2(\Lambda^i_-)}. \quad (4.17)$$

Since ρ_n is in $W^{1;2}(\Lambda^i_-)$, the following inequalities hold true:

$$\begin{aligned} \left|\rho_n(x^i)^+\right|^2 &= \left|\rho_n(x^i_n)^2 + \lim_{x\to x^{i+}} \int_{x^i_n}^{x} 2\dot{\rho}_n(t)\rho_n(t)\, dt\right| && (4.18) \\ &\leq A \left\|\rho_n\right\|^2_{L^2(\Lambda^i)} + 2 \left\|\rho_n\right\|_{L^2(\Lambda^i)} \left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)} \\ &\leq A \left\|\rho_n\right\|^2_{L^2(\Lambda^i)} + 2 \left\|\rho_n\right\|_{L^2(\Lambda^i)} \left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)}. && (4.19) \end{aligned}$$

In (4.18) we used (4.17) and the Hölder's inequality. We also remark that (4.19) is a very specific case of the Gagliardo-Nirenberg' inequalities (see [5]). In particular, we can state that

$$\left|\rho_n(x^i)^-\right| \leq \left(A \left\|\rho_n\right\|^2_{L^2(\Lambda^i)} + 2 \left\|\rho_n\right\|_{L^2(\Lambda^i)} \left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)}\right)^{\frac{1}{2}}. \quad (4.20)$$

Similarly, we prove that

$$\left|\rho_n(x^i)^+\right| \leq \left(A\left\|\rho_n\right\|^2_{L^2(\Lambda^i)} + 2\left\|\rho_n\right\|_{L^2(\Lambda^i)}\left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)}\right)^{\frac{1}{2}}. \tag{4.21}$$

If we join (4.16), (4.20) and (4.21), we find that if n is any integer greater than or equal to n_i, then the following inequality holds true:

$$\left|\chi^i_n\right|^2 \leq 4A\left\|\rho_n\right\|^2_{L^2(\Lambda^i)} + 8\left\|\rho_n\right\|_{L^2(\Lambda^i)}\left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)}. \tag{4.22}$$

We remark that $f(x) \coloneqq x^{\frac{2}{3}}$ is a subadditive function; so, if we elevate to the power of $\frac{2}{3}$ both sides in (4.22), we obtain that

$$\begin{aligned}\left|\chi^i_n\right|^{\frac{4}{3}} &\leq \left(4A\left\|\rho_n\right\|^2_{L^2(\Lambda^i)} + 8\left\|\rho_n\right\|_{L^2(\Lambda^i)}\left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)}\right)^{\frac{2}{3}} \\ &\leq B\left\|\rho_n\right\|^{\frac{4}{3}}_{L^2(\Lambda^i)} + 4\left\|\rho_n\right\|^{\frac{2}{3}}_{L^2(\Lambda^i)}\left\|\dot{\rho}_n\right\|^{\frac{2}{3}}_{L^2(\Lambda^i)}.\end{aligned} \tag{4.23}$$

Let θ be any positive real number. If we use the Young's inequality, we find that

$$\begin{aligned}\left|\chi^i_n\right|^{\frac{4}{3}} &\leq B\left\|\rho_n\right\|^{\frac{4}{3}}_{L^2(\Lambda^i)} + 4\left(\frac{\left\|\rho_n\right\|_{L^2(\Lambda^i)}}{\theta}\right)^{\frac{2}{3}}\left(\theta\left\|\dot{\rho}_n\right\|_{L^2(\Lambda^i)}\right)^{\frac{2}{3}} \\ &\leq B\left\|\rho_n\right\|^{\frac{4}{3}}_{L^2(\Lambda^i)} + 4\left(\frac{2}{3\theta}\left\|\rho_n\right\|_{L^2(\Lambda^i)} + \frac{\theta^2}{3}\left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)}\right).\end{aligned} \tag{4.24}$$

Having said that, if n is any integer greater than or equal to n_i, the following inequalities hold true:

$$\begin{aligned}&\frac{-\gamma\left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i\left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Lambda^i)}} \\ &\leq \frac{-\gamma\left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i\left[B\left\|\rho_n\right\|^{\frac{4}{3}}_{L^2(\Lambda^i)} + 4\left(\frac{2}{3\theta}\left\|\rho_n\right\|_{L^2(\Lambda^i)} + \frac{\theta^2}{3}\left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)}\right)\right]}{\left\|\rho_n\right\|_{L^2(\Lambda^i)}} \\ &= \frac{\left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)}\left[-\gamma + \dfrac{4\theta^2\beta^i}{3}\right]}{\left\|\rho_n\right\|_{L^2(\Lambda^i)}} + B\beta^i\left\|\rho_n\right\|^{\frac{1}{3}}_{L^2(\Lambda^i)} + \frac{8\beta^i}{3\theta}.\end{aligned} \tag{4.25}$$

As (4.25) holds for all positive real number θ, we can choose θ such that

$$-\gamma + \frac{4\theta^2\beta^i}{3} \leq 0,$$

that is

$$\theta \leq \sqrt{\frac{3\gamma}{4\beta^i}}.$$

It's easy to see that (4.25) is optimal if we choose

$$\theta = \sqrt{\frac{3\gamma}{4\beta^i}}.$$

Hence, we have shown that there exists n_i in $\mathbb{N}$ such that for all integer n greater than or equal to n_i the following inequality holds true:

$$\frac{-\gamma \left\|\dot{\rho}_n\right\|^2_{L^2(\Lambda^i)} + \beta^i \left|\chi^i_n\right|^{\frac{4}{3}}}{\left\|\rho_n\right\|_{L^2(\Lambda^i)}} \leq B\beta^i \left\|\rho_n\right\|^{\frac{1}{3}}_{L^2(\Lambda^i)} + \frac{16}{3\sqrt{3\gamma}} \left(\beta^i\right)^{\frac{3}{2}}.$$

If we recall the definition of $\mathcal{K}^i$ (see (4.15)), we have that

$$\mathcal{K}^i \leq \limsup_{n\to+\infty} B\beta^i \left\|\rho_n\right\|^{\frac{1}{3}}_{L^2(\Lambda^i)} + \frac{16}{3\sqrt{3\gamma}} \left(\beta^i\right)^{\frac{3}{2}} = \frac{16}{3\sqrt{3\gamma}} \left(\beta^i\right)^{\frac{3}{2}}.$$

Step 3: In conclusion, we have just shown that if $\mathcal{M}$ is any real number such that $\mathcal{M} < \left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u)$ and $\{\beta^1; \dots; \beta^k\}$ are real numbers in $[0, +\infty)$ such that for all i in $\{1; \dots; k\}$ it holds that

$$\limsup_{\delta\to0} \frac{\psi(\Delta^i) - \psi(\Delta^i + \delta) + \delta\psi'(\Delta^i)}{|\delta|^{\frac{4}{3}}} < \beta^i,$$

then the following inequality holds true:

$$\mathcal{M} \leq \left\|[\varphi'(\dot{u}_0)]'\right\|_{L^2(\Omega)} + \frac{16}{3\sqrt{3\gamma}} \sum_{i=1}^{k} (\beta^i)^{\frac{3}{2}}.$$

This is enough to state that

$$\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) \leq \left\|[\varphi'(\dot{u}_0)]'\right\|_{L^2(\Omega)} + \frac{16}{3\sqrt{3\gamma}} \sum_{i=1}^{k} (\beta^i)^{\frac{3}{2}}.$$

It is immediate to see that it is not restrictive to assume that for all i in $\{1; \dots; k\}$ we have that

$$\limsup_{\delta\to0} \frac{\psi(\Delta^i) - \psi(\Delta^i + \delta) + \delta\psi'(\Delta^i)}{|\delta|^{\frac{4}{3}}} \leq \beta^i.$$

Then, the theorem is completely proved. □

Corollary 4.2.13. *In the hypothesis of theorem 4.2.12, if we also assume that $\beta^i = 0$ for all i in $\{1; \dots k\}$, then*

$$\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) = \left\|[\varphi'(\dot{u})']\right\|_{L^2(\Omega)}.$$

Proof. We notice that $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) \geq \left\|[\varphi'(\dot{u})]'\right\|_{L^2(\Omega)}$ because of theorem 4.2.2; by theorem 4.2.12, it immediately follows that $\left|\nabla\mathcal{MS}_{\varphi;\psi}\right|(u) \leq \left\|[\varphi'(\dot{u})]'\right\|_{L^2(\Omega)}$. □

Bibliography

[1] ANTONUCCI, C. A discrete variational approximation of the mumford-shah functional in dimension one, 2017.

[2] BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer New York, 2010.

[3] DAL MASO, G., T. R. *On a notion of unilateral slope for the Mumford-Shah functional.* Nonlinear Differential Equations Appl. 13, 2007.

[4] DE GIORGI, E., F. T. *Su un tipo di convergenza variazionale.* Rend. Sem. Mat. Brescia 3, 1979.

[5] GAGLIARDO, E. *Ulteriori proprietà di alcune classi di funzioni in più variabili.* Ricerche Mat., 8:24-51, 1959.

[6] GOBBINO, M. *Gradient Flow for the One-Dimensional Mumford-Shah Functional.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) Vol. XXVII, 1998.

[7] LÉONARD, C. *Orlicz Space.*

[8] MUMFORD, D., S. J. *Optimal approximations by piecewisesmooth functions and associated variational problems.* Comm. Pure Appl. Math. 42, 1989.