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Chapter 1

Notation

Let us fix some useful notation.

Definition 1.0.1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be any function, let h be any vector in $\mathbb{R}^d$. We define $\tau_h f : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\tau_h f(x) \coloneqq f(x-h).$$

Definition 1.0.2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be any function, let δ be any positive real number. We define $\sigma_\delta f : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\sigma_\delta(x) f \coloneqq \frac{1}{\delta^d} f\left(\frac{x}{\delta}\right).$$

Chapter 2

Measure and integration

2.1 Introduction to measure theory

2.1.1 Definition and main properties

Definition 2.1.1 (algebra of sets).
Let $\mathbb{E}$ be a set. Let $\mathscr{A}$ any collection in $\mathbb{P}(\mathbb{E})$ with the following properties:

- $\emptyset$ is in $\mathscr{E}$;
- A is in $\mathscr{E}$ if and only if A^c is in $\mathscr{E}$;
- if A_1, A_2 are sets in $\mathscr{A}$, then $A_1 \cup \mathcal{A}_2$ is in $\mathscr{A}$.

We say that $\mathscr{A}$ is finite-additive algebra of sets over $\mathbb{E}$.

Definition 2.1.2 (finite-additive measure).
Let $\mathbb{E}$ be a set with a finite-additive algebra $\mathscr{A}$. Let $m : \mathscr{A} \to [0; +\infty]$ be any function with the following properties:

- $m(\emptyset) = 0$;
- if A_1, A_2 are disjoint sets in $\mathscr{A}$, then $m(A_1 \cup A_2) = m(A_1) + m(A_2)$.

We say that m is a finite-additive measure.

Definition 2.1.3 (σ-algebra).
Let $\mathbb{E}$ be a set. Let $\mathscr{E}$ any collection in $\mathbb{P}(\mathbb{E})$ with the following properties:

- $\emptyset$ is in $\mathscr{E}$;
- A is in $\mathscr{E}$ if and only if A^c is in $\mathscr{E}$;
- let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets in $\mathscr{E}$. If we define

$$A \coloneqq \bigcup_{n\in\mathbb{N}} A_n,$$

  then A is in $\mathscr{E}$.

We say that $\mathscr{E}$ is a σ-algebra (or a countable-additive algebra) over $\mathbb{E}$ and $(\mathbb{E}; \mathscr{E}; \mu)$ is called measurable space.

Remark 2.1.4. By definition 2.1.3 it immediately follows that if $\mathscr{E}$ is a σ-algebra in $\mathbb{E}$, then $\mathbb{E}$ is in $\mathscr{E}$ and it is closed under countable intersection.

Definition 2.1.5 (generated σ-algebra)**.**
Let $\mathbb{E}$ be a set; let $\mathscr{G}$ be any collection of sets in $\mathbb{P}(\mathbb{E})$. We denote as $\sigma(\mathscr{G})$ the intersection of the σ-algebras in $\mathbb{E}$ that contain $\mathscr{G}$.

Remark 2.1.6. In the setting of definition 2.1.5, it's immediate to see that $\sigma(\mathscr{G})$ is a σ-algebra and it is the smallest one that contains $\mathscr{G}$.

Definition 2.1.7 (σ-additive measure)**.**
Let $\mathbb{E}$ be a set with a σ-algebra $\mathscr{E}$. Let $\mu : \mathscr{E} \to [0; +\infty]$ be any function with the following properties:

- $\mu(\emptyset) = 0$;
- if $\{A_n\}$ is a sequence in $\mathscr{E}$ of pairwise disjoint set, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

We say that μ is a σ-additive measure.

Definition 2.1.8. Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. We say that a property $\mathcal{P}$ holds true for almost every x in $\mathbb{E}$ if the set for which the property is not valid is completely contained in a measurable set D and $\mu(D) = 0$.

Proposition 2.1.9. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. The following conclusions hold true:*

- *if E, F are measurable sets such that E contains F, then $\mu(E) \geq \mu(F)$.*
- *If $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets, i. e. for all n in $\mathbb{N}$ it holds that E_n is contained in E_{n+1}, then*

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sup_{n\in\mathbb{N}} \mu(E_n).$$

  We say that the measure is continuous from below.
- *If $\{E_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of measurable sets, i. e. for all n in $\mathbb{N}$ it holds that E_n contains E_{n+1}, and $\mu(E_0)$ is finite then*

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \inf_{n\in\mathbb{N}} \mu(E_n).$$

  We say that the measure is continuous from above.

Proof. As for the first statement, it is enough to consider the decomposition $E = F \cup (E \setminus F)$ and the fact that the measure is additive.

As for the second statement, for all n in $\mathbb{N}^*$ we define $F_n \coloneqq E_n \setminus E_{n-1}$; we denote $F_0 \coloneqq E_0$. If we denote

$$X \coloneqq \bigcup_{n\in\mathbb{N}} E_n,$$

it's easy to see that $\{F_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint sequence of sets such that

$$X = \bigcup_{n\in\mathbb{N}} F_n.$$

If we recall that μ is σ-additive, we obtain that

$$\mu(X) = \sum_{n\in\mathbb{N}} \mu(F_n) = \sup_{n\in N} \left(\sum_{i=0}^{n} \mu(F_i) \right) = \sup_{n\in\mathbb{N}} \{\mu(E_n)\}.$$

As for the third statement, if we define

$$X \coloneqq \bigcap_{n\in\mathbb{N}} E_n,$$

we notice that

$$\mu(E_0 \setminus X) = \mu\left(E_0 \setminus \bigcap_{n\in\mathbb{N}} E_n \right) = \mu\left(\bigcup_{n\in\mathbb{N}} (E_0 \setminus E_n) \right) = \sup_{n\in\mathbb{N}} \left\{ \mu(E_0 \setminus E_n) \right\}.$$

Since $\mu(E_0)$ is finite, we can take the complementary and the following identities hold true:

$$\mu(E_0) - \mu(X) = \mu(E_0 \setminus X) = \sup_{n\in\mathbb{N}} \left\{ \mu(E_0 \setminus E_n) \right\} = \mu(E_0) - \inf_{n\in\mathbb{N}} \left\{ \mu(E_n) \right\}.$$

□

2.1.2 Carathéodory's extension theorem

Let $(\mathbb{E}; \mathscr{A}; m)$ be a set with a finite-additive algebra $\mathscr{A}$ and a finite-additive measure m. The aim of this subsection is to show that there exists a σ-algebra $\mathscr{E}$ such that $\mathscr{A}$ is completely contained in $\mathscr{E}$ and a σ-additive measure $\mu : \mathscr{E} \to [0; +\infty]$ that extends m.

Definition 2.1.10 (Outer measure)**.**
Let $\mathbb{E}$ be any set and $\varphi : \mathbb{P}(\mathbb{E}) \to [0; +\infty]$ any function with the following properties:

- $\varphi(\emptyset) = 0$;
- if A, B are in $\mathbb{P}(\mathbb{E})$ such that B contains A, then $\varphi(A) \leq \varphi(B)$;
- if $\{A_n\}_{n\in\mathbb{N}}$ is any sequence of sets (pairwise disjoint or not) in $\mathbb{P}(\mathbb{E})$, then it holds that
$$\varphi\left(\bigcup_{n\in\mathbb{N}} A_n \right) \leq \sum_{n\in\mathbb{N}} \varphi(A_n).$$

We say that φ is an outer measure over $\mathscr{E}$.

Proposition 2.1.11. *Let $\mathbb{E}$ be any set; let $\mathscr{A}$ be a finite-additive algebra of sets; let $m : \mathscr{A} \to [0; +\infty]$ be any finite-additive measure on $\mathbb{E}$. For all A in $\mathbb{P}(\mathbb{E})$ we define*

$$m^*(A) := \inf \left\{ \sum_{n\in\mathbb{N}} m(A_n) \;\middle|\; A \subseteq \bigcup_{n\in\mathbb{N}} A_n,\ \{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A} \right\}.$$

If we assume that the $\inf\{\emptyset\} = +\infty$, *the function m^* is well-defined and it is an outer measure over $\mathbb{E}$.*

Definition 2.1.12 (pre-measure)**.**
Let $\mathbb{E}$ be any set; let $\mathscr{A}$ be a finite-additive algebra of sets; let $m : \mathscr{A} \to [0; +\infty]$ be any finite-additive measure on $\mathbb{E}$. Let us assume that if $\{A_n\}_{n\in\mathbb{N}}$ is any sequence of pairwise disjoint set in $\mathscr{A}$ such that

$$A = \bigcup_{n\in\mathbb{N}} A_n$$

is in $\mathscr{A}$, then it holds that

$$\sum_{n\in\mathbb{N}} m(A_n) = m\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

We say that m is a pre-measure.

Definition 2.1.13 (Carathéodory's criterion)**.**
Let $\mathbb{E}$ be any set; let φ be an outer measure over $\mathbb{E}$. Let A be any subset of $\mathbb{E}$. We say that A satisfies the Carathéodory's criterion if the following property holds for all C in $\mathbb{P}(\mathbb{E})$:

$$\varphi(A) = \varphi(A \cap C) + \varphi(A \cap C^c).$$

Theorem 2.1.14 (Carathéodory's extension theorem)**.**
Let $\mathbb{E}$ be a set; let $\mathscr{A}$ be a finite-additive algebra of sets in $\mathbb{P}(\mathbb{E})$; let $m : \mathscr{A} \to [0; +\infty]$ be a finite-additive measure. Let us assume that m is also a pre-measure. Let us define the outer measure m^* as in 2.1.11; let us denote with $\mathscr{E}$ the collection of the sets that satisfy the Carathéodory's criterion. Then, the following conclusions hold true:

- *$\mathscr{E}$ is a σ-algebra that contains $\mathscr{A}$;*
- *the restriction of the outer measure m^* to $\mathscr{E}$ defines a σ-additive measure μ that extends m. In other words, we define $\mu : \mathscr{E} \to [0; +\infty]$ such that if A is in $\mathscr{E}$, then $\mu(A) = m^*(A)$; moreover, if A is in $\mathscr{A}$, it holds that $\mu(A) = m(A)$.*

Lebesgue measure in $\mathbb{R}^n$

The aim of this subsection is to show how the construction of the Lebesgue measure in $\mathbb{R}$ follows from the Carathéodory's extension theorem. We will introduce the main definitions and we will state the most important results.

Definition 2.1.15 (Borel σ-algebra)**.**
We define $\mathbb{B}^n$ as the smallest σ-algebra (with respect to the inclusion) that contains the open sets in $\mathbb{R}^n$.

Definition 2.1.16. We define $\mathscr{A}^d$ the collection of the boxes in $\mathbb{R}^d$.

Remark 2.1.17. It's easy to see that $\mathscr{A}^d$ is a finite-additive algebra of sets.

Definition 2.1.18 (Lebesgue measure in $\mathbb{R}^n$)**.**
Let I be a box in $\mathbb{R}^n$, namely

$$I = I_1 \times \cdots \times I_n,$$

where I_i is a interval in $\mathbb{R}$. For all interval J in $\mathbb{R}$ we define

$$\mathscr{P}^1 := \sup J - \inf J,$$

assuming that the sum is well defined in $\overline{\mathbb{R}}$. We define

$$\mathscr{P}(I)^n := \prod_{i=1}^{n} \mathscr{P}^1(I_i),$$

with the assumption that $0 \cdot (+\infty)$ equals 0.

Proposition 2.1.19. *The function $\mathscr{P}^n : \mathscr{A}^n \to [0; +\infty]$ is well define and it is a finite-additive measure; moreover, it is a pre-measure. It is called Peano-Jordan measure in $\mathbb{R}^n$.*

Corollary 2.1.20. *We define the outer measure $\mathscr{P}^{n*}$ associated to the pre-measure $\mathscr{P}^n$ as in 2.1.11. Let us denote with $\mathscr{M}^n$ the collection of the sets that satisfy the Carathéodory's criterion. Then, the following conclusions hols true:*

- *$\mathscr{M}^n$ is a σ-algebra that contains $\mathbb{B}^n$, also know as the collection of the Lebesgue measurable sets;*
- *the restriction of the outer measure $\mathscr{P}^{n*}$ to $\mathscr{M}^n$ defines a σ-additive measure $\mathscr{L}^n$ that extends $\mathscr{P}^n$; it is called Lebesgue measure in $\mathbb{R}^n$.*

Proof. It is an immediate consequence of theorem 2.1.14. □

Remark 2.1.21. Unless otherwise specified, we will always consider $\mathbb{R}^n$ equipped with the σ-algebra $\mathscr{M}^n$ and the σ-additive measure $\mathscr{L}^n$.

The following approximation result can be proved.

Proposition 2.1.22. *Let M be a measurable set in $\mathbb{R}^n$; let ε be a positive real number. There exist an open set A and a closed set C such that*

$$C \subseteq M \subseteq A, \quad \mathscr{L}^n(A \setminus C) \leq \varepsilon.$$

Remark 2.1.23. Let E be a measurable subset in $\mathbb{R}^d$ such that $\mathscr{L}^d(E) = 0$. We claim that if A is completely contained in E, then A is a measurable subset and $\mathscr{L}^d(A) = 0$. Thanks to 2.1.22, we have to show that for all positive real number ε there exists an open set A_ε such that A is completely contained in A_ε and $\mathscr{L}^d(A_\varepsilon) < \varepsilon$. As a matter of fact, there exists an open set A_ε such that $\mathscr{L}^d(A_\varepsilon) < \varepsilon$ and E is completely contained in A_ε. So A is measurable; having said that, it holds that $\mathscr{L}^d(A) = 0$ obviously. We say that the Lebesgue Measure is complete.

Example 2.1.24. Let us define the following sequence of subset in $[0;1]$:

$$\begin{cases} \mathscr{C}_0 := [0;1]; \\ \mathscr{C}_+ := \frac{1}{3}\mathscr{C}_n \cup \left(\frac{2}{3} + \frac{1}{3}\mathscr{C}_n\right). \end{cases}$$

For all n in $\mathbb{N}$ we notice that $\mathscr{C}_n$ is a closed set and

$$\mathscr{L}^1(\mathscr{C}_{n+1}) = \frac{2}{3}\mathscr{L}^1(\mathscr{C}_n) = \left(\frac{2}{3}\right)^{n+1}.$$

We define the Cantor set

$$\mathscr{C} := \bigcap_{n\in\mathbb{N}} \mathscr{C}_n.$$

It's immediate to see that $\mathscr{C}$ is a closed set; in particular, it is measurable and $\mathscr{L}^1(\mathscr{C}) = 0$. It is easy to see that $\mathscr{C}$ is in bijection with the set of the binary sequences. Hence, the cardinality of $\mathscr{C}$ is c and the cardinality of $\mathbb{P}(\mathscr{C})$ is 2^c. Thanks to 2.1.23, we can state that the cardinality $\mathscr{M}^1$ is exactly 2^c. It can be also shown that the cardinality of the Borel σ-algebra is exactly c; this is enough to conclude that $\mathscr{B}^1$ is strictly contained in $\mathscr{M}^1$. As a matter of fact, we can similarly define a Cantor set in $[0;1]^n$ and we obtain that $\mathscr{B}^n$ is strictly contained in $\mathscr{M}^n$ for all n in $\mathbb{N}$.

Proposition 2.1.25. *Let A be a measurable set in $\mathbb{R}^d$. Let us assume that $\mathscr{L}^d(A)$ is finite. For all t in $[0;\mathscr{L}^d(A)]$ there exists a measurable set E_t in A such that $\mathscr{L}^d(E_t) = t$.*

Proof. Let us define the function $\psi : \mathbb{R} \to [0;\mathscr{L}^d(A)]$ such that

$$\psi(x) := \mathscr{L}^d(A \cap \{(x_1;\ldots;x_d) \in \mathbb{R}^d \mid x_1 \le x)\}.$$

It's immediate to see that ψ is a well defined increasing function. As $\mathscr{L}^d$ is continuous from below, we have that

$$\lim_{x\to+\infty} \psi(x) = \mathscr{L}^d(A).$$

As A is a finite measure set, $\mathscr{L}^d$ is continuous from above and we have that

$$\lim_{x\to-\infty} \psi(x) = \mathscr{L}^d(\emptyset) = 0.$$

If we show that ψ is a continuous function, the thesis follows immediately. Let x be any point in $\mathbb{R}$; let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence that approaches toward n. We claim that

$$\lim_{n\to+\infty} \psi(y_n) = \psi(x).$$

Let us assume that $\{y_n\}_{n\in\mathbb{N}}$ is monotonically increasing. As $\mathscr{L}^d$ is continuous from below, we have that

$$\begin{aligned} \lim_{n\to+\infty} \psi(y_n) &= \lim_{n\to+\infty} \mathscr{L}^d(A \cap \{(x_1;\ldots;x_d) \in \mathbb{R}^d \mid x_1 \le y_n)\}) \\ &= \mathscr{L}^d((A \cap \{(x_1;\ldots;x_d) \in \mathbb{R}^d \mid x_1 < x)\}). \end{aligned}$$

As $\mathscr{L}((A \cap \{(x_1;\ldots;x_d) \in \mathbb{R}^d \mid x_1 = x)\})$ is equal to 0, we can easily conclude that

$$\begin{aligned} \lim_{n\to+\infty} \psi(y_n) &= \mathscr{L}^d((A \cap \{(x_1;\ldots;x_d) \in \mathbb{R}^d \mid x_1 < x)\}) \\ &= \mathscr{L}^d((A \cap \{(x_1;\ldots;x_d) \in \mathbb{R}^d \mid x_1 \le x)\}) \\ &= \psi(x). \end{aligned}$$

If $\{x_n\}_{n\in\mathbb{N}}$ is monotonically decreasing, the proof is completely similar. □

Example 2.1.26. If we assume the choice axiom, we show that if μ is a measure invariant by translation in $\mathbb{R}$ such that the measure of any non-empty interval is well-defined and it is a positive real number, then μ cannot be defined in $\mathbb{P}(\mathbb{R})$. Let x, y be points in $\mathbb{R}$. We say that x and y are equivalent if and only if $x - y$ is in $\mathbb{Q}$. Thanks to the choice axiom, there exists a set $\mathscr{V}$ completely contained in $[0;1]$ such that for all real number r there exists exactly an element x in $\mathscr{V}$ such that $x - r$ is in $\mathbb{Q}$. We notice that if q_1 and q_2 are rational numbers, then it holds that

$$(q_1 + \mathscr{V}) \cap (q_2 + \mathscr{V}) = \emptyset.$$

It's immediate to see that

$$[0;1] \subseteq \bigcup_{q \in \mathbb{Q} \cap [-1;1]} (\mathscr{V} + q) \subseteq [-1;2].$$

Let us assume that $\mathscr{V}$ is a measurable set, i. e. $\mathscr{V}$ is in the domain of μ. Let us denote $m := \mu(\mathscr{V})$. Since μ is invariant under translation, we obtain that for all q in $\mathbb{Q}$ it holds that $\mu(q + \mathscr{V}) = m$. Let us assume that m is 0. By definition of measure, it holds that

$$\mu\left(\bigcup_{q \in \mathbb{Q} \cap [-1;1]} (\mathscr{V} + q)\right) = 0;$$

by monotonicity, we obtain that $\mu([0;1]) = 0$. If m is a positive real number, we obtain similarly that $\mu([-1;2]) = +\infty$. Hence, we find the absurd. $\mathscr{V}$ is called the Vitali set.

Remark 2.1.27. By definition 2.1.18 it's immediate to see that the Lebesgue measure is invariant under translation. As shown in the 2.1.26, the collection of the Lebesgue measurable sets in $\mathbb{R}$ is strictly contained in $\mathbb{P}(\mathbb{R})$.

2.2 Introduction to integration theory

2.2.1 Measurable functions

Definition 2.2.1 (Measurable function).
Let $(\mathbb{E}; \mathscr{E})$ and $(\mathbb{F}; \mathscr{F})$ be measurable spaces. Let $f : \mathbb{E} \to \mathbb{F}$ be any function such that for all A in $\mathscr{F}$ it holds that $f^{-1}(A)$ is in $\mathscr{E}$. We say that f is a measurable function.

Remark 2.2.2. In the setting of definition 2.2.1, let $\mathscr{G}$ be a collection of subsets in $\mathbb{P}(\mathbb{F})$ such that $\mathscr{F} = \sigma(\mathscr{G})$. Let f be a function between $\mathbb{E}$ and $\mathbb{F}$. It's easy to see that f is measurable if and only if for all A in $\mathbb{G}$, then $f^{-1}(A)$ is in $\mathscr{E}$.

Example 2.2.3. Let $(\mathbb{E}_1; \mathscr{E}_1)$, $(\mathbb{E}_2; \mathscr{E}_2)$, $(\mathbb{E}_3; \mathscr{E}_3)$ be measurable spaces; let $f : \mathbb{E}_1 \to \mathbb{E}_2$ and $g : \mathbb{E}_2 \to \mathbb{E}_3$ be measurable functions. Then $g \circ f : \mathbb{E}_1 \to \mathbb{E}_3$ is a measurable function.

Definition 2.2.4. Let $(\mathbb{E}; \mathscr{E})$ be a measurable space. Let $f : \mathscr{E} \to \overline{\mathbb{R}}$ be a function. Let us assume that $f^{-1}(A \cap \mathbb{R})$ is in $\mathscr{E}$ for all A contained in $\overline{\mathbb{R}}$ such that $A \cap \mathbb{R}$ is in $\mathcal{M}^1$. We will say that it f is measurable.

Example 2.2.5. Let $(\mathbb{E};\mathscr{E})$ be a measurable space. Let $f,g:\mathscr{E}\to\overline{\mathbb{R}}$ be measurable functions. It's easy to see that the pointwise maximum $f\vee g$, the pointwise minimum $f\wedge g$, the sum $f+g$, the product $f\cdot g$ and the quotient f/g are measurable functions between $\mathbb{E}$ and $\overline{\mathbb{R}}$ (defined where they make sense).

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued measurable function. It's immediate to see that the pointwise supremum $\sup\limits_{n\in\mathbb{N}} f_n$, the pointwise infimum $\inf\limits_{n\in\mathbb{N}} f_n$, the pointwise limit superior $\limsup\limits_{n\to+\infty} f_n$ and the pointwise limit inferior $\liminf\limits_{n\to+\infty} f_n$ are well defined measurable functions between $\mathbb{E}$ and $\overline{\mathbb{R}}$.

2.2.2 Integration of nonnegative measurable functions

We will always assume that $0\cdot(+\infty)=0$, for all a in $\mathbb{R}$ it holds that $a+(+\infty)=+\infty$ and for all positive real number b it holds that $b\cdot(+\infty)=+\infty$.

Definition 2.2.6 (Step function)**.**
Let $(\mathbb{E};\mathscr{E})$ be a measurable space. Let $\{E_1;\dots;E_n\}$ be pairwise disjoint measurable sets in $\mathscr{E}$ such that

$$\mathbb{E}=\bigcup_{i=1}^{n}E_i.$$

Let $\{\alpha_1;\dots;\alpha_n\}$ be in $[0;+\infty]$. Let us define $f:\mathbb{E}\to[0;+\infty]$ such that

$$f(x)\coloneqq\sum_{i=1}^{n}\alpha_i\mathbb{1}_{E_i}(x).$$

We say that f is a step function. We denote as $\mathscr{S}(\mathbb{E})$ the set of the step functions between $\mathbb{E}$ and $[0;+\infty)$.

Remark 2.2.7. It's immediate to see if f is a step function, then it is a measurable function.

Definition 2.2.8 (Integration of positive step function)**.**
Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let $f:\mathbb{E}\to[0;+\infty]$ be a step function, i.e. there exist $\{E_1;\dots;E_n\}$ pairwise disjoint measurable sets in $\mathscr{E}$ and $\{\alpha_1;\dots;\alpha_n\}$ in $[0;+\infty]$ such that

$$f\coloneqq\sum_{i=1}^{n}\alpha_i\mathbb{1}_{E_i}.$$

We define the integral of f with respect to the measure μ as follows:

$$\int_{\mathbb{E}}f(x)d\mu(x)\coloneqq\sum_{i=1}^{n}\alpha_i\mu(E_1).$$

Remark 2.2.9. The definition 2.2.8 is well posed, in the sense that it does not depend on the specific representation of the step function. Let us assume that there exists $\{E_1;\dots;E_n\}$ measurable pairwise disjoint sets, $\{F_1;\dots;F_d\}$ measurable pairwise disjoint set, $\{\alpha_1;\dots;\alpha_n\}$ and $\{\beta_1;\dots;\beta_d\}$ sets contained in $[0;+\infty]$ such that

$$\sum_{i=1}^{n}\alpha_i\mathbb{1}_{E_i}=f=\sum_{j=1}^{d}\beta_j\mathbb{1}_{F_j}.$$

Let i be an integer in $\{1;\ldots;n\}$, let j be an integer in $\{1;\ldots;d\}$. If $E_i \cap F_j$ is not empty, it holds that $\alpha_i = \beta_j$. However, for all i in $\{1;\ldots;n\}$ for all j in $\{1;\ldots;d\}$, it holds that

$$\alpha_i\mu(E_i \cap F_j) = \beta_j\mu(E_i \cap F_j).$$

By definition of measure, the following identities hold true:

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i\mu(E_i) &= \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{d} \mu(E_i \cap F_j)\right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{d} \alpha_i\mu(E_i \cap F_j)\right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{d} \beta_j\mu(E_i \cap F_j)\right) \\ &= \sum_{j=1}^{d} \left(\sum_{i=1}^{n} \beta_j\mu(E_i \cap F_j)\right) \\ &= \sum_{j=1}^{d} \beta_j\mu(F_j).\end{aligned}$$

Definition 2.2.10 (Integration of nonnegative measurable function)**.**
Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let $f :\to [0;+\infty]$ be a measurable function. We define the integral of f with respect to the measure μ as follows:

$$\int_{\mathbb{E}} f(x)d\mu(x) \coloneqq \sup\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g \in \mathscr{S}(\mathbb{E}),\ \forall x \in \mathscr{E}\ g(x) \leq f(x)\right\}.$$

Definition 2.2.11 (Generalized step function)**.**
Let $(\mathbb{E};\mathscr{E})$ be a measurable space. Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint measurable set such that

$$\mathbb{E} = \bigcup_{n\in\mathbb{N}} E_n.$$

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $[0;+\infty]$. Let us define $f : \mathbb{E} \to [0;+\infty]$ such that

$$f(x) \coloneqq \sum_{n\in\mathbb{N}} \alpha_n \mathbb{1}_{E_n}(x).$$

We say that f is a generalized step function. We also denote as $\mathscr{S}'(\mathbb{E})$ the set of the generalized step functions between $\mathbb{E}$ and $[0;+\infty]$.

Remark 2.2.12. In the setting of definition 2.2.11, let f be in $\mathscr{S}'(\mathbb{E})$ such that

$$f \coloneqq \sum_{n\in\mathbb{N}} \alpha_n \mathbb{1}_{E_n}.$$

By definition 2.2.10, it immediately follows that

$$\int_{\mathbb{E}} f(x)d\mu(x) = \sum_{n\in\mathbb{N}} \alpha_n\mu(E_n).$$

Lemma 2.2.13. *Let* $(\mathbb{E};\mathscr{E};\mu)$ *be a measurable space with a measure* μ. *Let* $f:\mathbb{E}\to[0;+\infty]$ *be a measurable function. Then, the following conclusions hold true:*

$$\int_{\mathbb{E}} f(x)d\mu(x) = \inf\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g\in\mathscr{S}'(\mathbb{E}),\ \forall x\in\mathscr{E}\ g(x)\geq f(x)\right\};$$

$$\int_{\mathbb{E}} f(x)d\mu(x) = \sup\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g\in\mathscr{S}'(\mathbb{E}),\ \forall x\in\mathscr{E}\ g(x)\leq f(x)\right\}.$$

Proof. By definition 2.2.10, it follows that if $f,g:\mathbb{E}\to[0;+\infty]$ are measurable functions such that $f(x)\leq g(x)$ for all x in $\mathbb{E}$, then

$$\int_{\mathbb{E}} f(x)d\mu(x) \leq \int_{\mathbb{E}} g(x)d\mu(x).$$

This is enough to conclude that

$$\int_{\mathbb{E}} f(x)d\mu(x) \leq \inf\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g\in\mathscr{S}'(\mathbb{E}),\ \forall x\in\mathscr{E}\ g(x)\geq f(x)\right\},$$

$$\int_{\mathbb{E}} f(x)d\mu(x) \geq \sup\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g\in\mathscr{S}'(\mathbb{E}),\ \forall x\in\mathscr{E}\ g(x)\leq f(x)\right\}.$$

By definition 2.2.10, it immediately follows that

$$\int_{\mathbb{E}} f(x)d\mu(x) \leq \sup\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g\in\mathscr{S}'(\mathbb{E}),\ \forall x\in\mathscr{E}\ g(x)\leq f(x)\right\}.$$

We complete the proof assuming that $\mu(\mathbb{E})$ is a real number. We notice that it is not restrictive to assume that

$$\int_{\mathbb{E}} f(x)d\mu(x) < +\infty,$$

otherwise the conclusion is trivial. Let δ be a positive real number; for all n in $\mathbb{N}$ we define $E_n := f^{-1}([n\delta;(n+1)\delta)$; we also define $E_\infty := f^{-1}(\{+\infty\})$. Under our assumption, wa have that $\mu(E_\infty)=0$. Since f is measurable, the sets $\{E_n \mid n\in\mathbb{N}\cup\{+\infty\}\}$ are measurable. We define the generalized step function g_δ as follows:

$$g_\delta := \left(\sum_{n\in\mathbb{N}} n\delta\mathbb{1}_{E_n}\right) + \infty\mathbb{1}_{E_\infty}.$$

We notice that for all x in $\mathbb{E}$ it holds that $g_\delta(x)\leq f(x)\leq g_\delta(+)+\delta$. Hence, we obtain that

$$\int_{\mathbb{E}} g_\delta(x)d\mu(x) \leq \int_{\mathbb{E}} f(x)d\mu(x) \leq \int_{\mathbb{E}} [g_\delta(x)+\delta]d\mu(x).$$

We notice that

$$\begin{aligned}\int_{\mathbb{E}} [g_\delta(x)+\delta]d\mu(x) &= \left(\sum_{n\in\mathbb{N}}(n+1)\delta\mu(E_n)\right) + (\infty+\delta)\mu(E_\infty)\\ &= \sum_{n\in\mathbb{N}}(n+1)\delta\mu(E_n)\\ &= \left(\sum_{n\in\mathbb{N}} n\delta\mu(E_n)\right) + \left(\sum_{n\in\mathbb{N}}\delta\mu(E_n)\right)\\ &= \left(\int_{\mathbb{E}} g_\delta(x)d\mu(x)\right) + \delta\mu(\mathbb{E}).\end{aligned}$$

In other words, for all δ in $[0;+\infty]$ there exists g_δ in $\mathscr{S}'(\mathbb{E})$ such that for all x in $\mathbb{E}$ it holds that $g_\delta(x) \leq f(x)$ and

$$\left|\int_{\mathbb{E}} f(x)d\mu(x) - \int_{\mathbb{E}} g_\delta(x)d\mu(x)\right| \leq \delta\mu(E).$$

This is enough to conclude that

$$\int_{\mathbb{E}} f(x)d\mu(x) \geq \inf\left\{\int_{\mathbb{E}} g(x)d\mu(x) \;\middle|\; g \in \mathscr{S}'(\mathbb{E}),\ \forall x \in \mathscr{E}\ g(x) \geq f(x)\right\}.$$

□

Proposition 2.2.14. *Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let $f_1, f_2 : \mathbb{E} \to [0;+\infty]$ be measurable functions. Let α be a real number in $[0;+\infty)$. The following conclusions hold true:*

- $\displaystyle\int_{\mathbb{E}} \alpha f_1(x)d\mu(x) = \alpha \int_{\mathbb{E}} f_1(x)d\mu(x)$;
- $\displaystyle\int_{\mathbb{E}} [f_1(x) + f_2(x)]d\mu(x) = \int_{\mathbb{E}} f_1(x)d\mu(x) + \int_{\mathbb{E}} f_2(x)d\mu(x)$.

Proof. The first statement is an immediate consequence of definition 2.2.10. As for the second one, let us suppose that f_1 and f_2 are in $\mathscr{S}'(\mathbb{E})$, namely

$$f_1 := \sum_{n\in\mathbb{N}} \alpha_n \mathbb{1}_{E_n},$$

$$f_2 := \sum_{n\in\mathbb{N}} \beta_n \mathbb{1}_{F_n}.$$

We notice that

$$f_1 + f_2 = \sum_{(n;m)\in\mathbb{N}^2} (\alpha_n + \beta_m)\mathbb{1}_{E_n\cap F_m}.$$

Then, it holds that

$$\begin{aligned}
\int_{\mathbb{E}} (f_1(x) + f_2(x))d\mu(x) &= \sum_{(n;m)\in\mathbb{N}^2} [\alpha_n + \beta_m]\mu(E_n \cap F_m) \\
&= \sum_{n\in\mathbb{N}} \left(\sum_{m\in\mathbb{N}} \alpha_n\mu(E_n \cap F_m)\right) + \sum_{m\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} \beta_m\mu(E_n \cap F_m)\right) \\
&= \sum_{n\in\mathbb{N}} \alpha_n\mu(E_n) + \sum_{m\in\mathbb{M}} \beta_m\mu(F_m) \\
&= \int_{\mathbb{E}} f_1(x)d\mu(x) + \int_{\mathbb{E}} f_2(x)d\mu(x).
\end{aligned}$$

Let f_1, f_2 be measurable functions between $\mathbb{E}$ and $[0;+\infty]$; let g_1, g_2 be step functions such that for all x in $\mathbb{E}$ it holds that $f_1(x) \geq g_1(x)$ and $f_2(x) \geq g_2(x)$. Hence, we obtain that

$$\int_{\mathbb{E}} [f_1(x) + f_2(x)]d\mu(x) \geq \int_{\mathbb{E}} [g_1(x) + g_2(x)]d\mu(x) = \int_{\mathbb{E}} g_1(x)d\mu(x) + \int_{\mathbb{E}} g_2(x)d\mu(x).$$

By definition 2.2.10, if we take the supremum, we obtain that

$$\int_{\mathbb{E}}[f_1(x)+f_2(x)]d\mu(x) \geq \int_{\mathbb{E}} f_1(x)d\mu(x) + \int_{\mathbb{E}} f_2(x)d\mu(x).$$

Let g_1, g_2 be generalized step functions such that for all x in $\mathbb{E}$ it holds that $f_1(x) \leq g_1(x)$ and $f_2(x) \leq g_2(x)$. Hence, we obtain that

$$\int_{\mathbb{E}}[f_1(x)+f_2(x)]d\mu(x) \leq \int_{\mathbb{E}}[g_1(x)+g_2(x)]d\mu(x) = \int_{\mathbb{E}} g_1(x)d\mu(x) + \int_{\mathbb{E}} g_2(x)d\mu(x).$$

Thanks to lemma 2.2.13, if we take the infimum, we obtain that

$$\int_{\mathbb{E}}[f_1(x)+f_2(x)]d\mu(x) \leq \int_{\mathbb{E}} f_1(x)d\mu(x) + \int_{\mathbb{E}} f_2(x)d\mu(x).$$

□

Theorem 2.2.15 (Beppo Levi's theorem)**.**
Let $(\mathbb{E};\mathscr{E};\mu)$ *be a measurable space with a measure* μ*. Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable functions with the following properties:*

- *for all* n *in* $\mathbb{N}$ *for almost every* x *in* $\mathbb{E}$ *it holds that* $f_n(x) \geq 0$*;*
- *for all* n *in* $\mathbb{N}$ *for almost every* x *in* $\mathbb{E}$ *it holds that* $f_n(x) \leq f_{n+1}(x)$*.*

Let us define the pointwise supremum $f := \sup_{n\in\mathbb{N}}\{f_n\}$*.* f *is a measurable function and*

$$\int_{\mathbb{E}} f(x)d\mu(x) = \sup_{n\in\mathbb{N}}\left\{\int_{\mathbb{E}} f_n(x)d\mu(x)\right\}.$$

Proof. We have already discussed the measurability of f. Let n be a positive integer. Since $f_n(x) \leq f(x)$ for almost every x in $\mathbb{E}$, we have that

$$\int_{\mathbb{E}} f_n(x)d\mu(x) \leq \int_{\mathbb{E}} f(x)d\mu(x).$$

Hence, we obtain that

$$\sup_{n\in\mathbb{N}}\left\{\int_{\mathbb{E}} f_n(x)d\mu(x)\right\} \leq \int_{\mathbb{E}} f(x)d\mu(x).$$

If the right hand side is equal to 0, the conclusion is trivial. Hence, we can assume that

$$\int_{\mathbb{E}} f(x)d\mu(x) > 0$$

and that there exists a positive real number M such that for all n in $\mathbb{N}$ it holds that

$$\int_{\mathbb{E}} f_n(x)d\mu(x) \leq M.$$

Let m be any positive real number such that

$$m < \int_{\mathbb{E}} f(x)d\mu(x).$$

We claim that there exists n_0 in $\mathbb{N}$ such that if n is a positive integer greater than n_0, then it holds that

$$\int_{\mathbb{E}} f_n(x)d\mu(x) \geq m.$$

Let g be a step function such that $g(x) \leq f(x)$ for almost every x in $\mathbb{E}$ and

$$\int_{\mathbb{E}} g(x)d\mu(x) > m.$$

In other words, as declared in definitions 2.2.6 and 2.2.8, we are assuming that

$$g \coloneqq \sum_{i=1}^{d} \alpha_i \mathbb{1}_{\mathbb{E}}$$

$$\int_{\mathbb{E}} g(x)d\mu(x) \coloneqq \sum_{i=1}^{d} \alpha_i \mu(E_i),$$

with the convention that $0 \cdot \infty = 0$. For all i in $\{1; \ldots; d\}$ we define ε_i with the following properties:

- if $\alpha_i = 0$, then $\varepsilon_i = 0$;
- if $\alpha_i > 0$, then ε_i is a real number in $(0; \alpha_i)$ such that

$$\sum_{i=1}^{d} (\alpha_i - \varepsilon_i)\mu(E_i) > m.$$

We notice that the choice is possible: if

$$\int_{\mathbb{E}} g(x)d\mu(x) = +\infty$$

the choice is trivial; if

$$\int_{\mathbb{E}} g(x)d\mu(x) < +\infty,$$

we notice that if $\mu(E_i) = +\infty$, then $\alpha_i = 0$; hence, we can choose $\{\varepsilon_1; \ldots; \varepsilon_d\}$ as declared. Let us denote $\tilde{g}$ the step function such that

$$\tilde{g} \coloneqq \sum_{i=1}^{d} (\alpha_i - \varepsilon_i)\mathbb{1}_{E_i}.$$

We have that $\tilde{g}(x) < f(x)$ for almost every x such that $f(x) \neq 0$ and

$$\int_{\mathbb{E}} \tilde{g}(x)d\mu(x) > m.$$

For all positive integer n we define

$$D_n \coloneqq \{x \in \mathbb{E} \mid f_n(x) \geq \tilde{g}(x)\}.$$

Under our hypothesis on $\{f_n\}_{n\in\mathbb{N}}$, it's easy to see that $\{D_n\}_{n\in\mathbb{N}}$ is a increasing sequence of measurable set such that

$$\mu\left(\bigcap_{n\in\mathbb{N}} D_n^c\right) = 0.$$

For all n in $\mathbb{N}$ we have that

$$\begin{aligned}\int_{\mathbb{E}} f(x)d\mu(x) &\geq \int_{\mathbb{E}} f_n(x)\mathbb{1}_{D_n}(x)d\mu(x)\\ &\geq \int_{\mathbb{E}} \tilde{g}_n(x)\mathbb{1}_{D_n}(x)d\mu(x)\\ &= \sum_{i=1}^{d}(\alpha_i - \varepsilon_i)\mu(E_i\cap D_n).\end{aligned}$$

We claim that for all i in $\{1;\ldots;d\}$ it holds that

$$\mu(E_i) = \lim_{n\to+\infty}\mu(E_i\cap D_n).$$

This is a consequence of the fact that

$$\begin{aligned}\mu(E_i) &= \lim_{n\to+\infty}\mu\left(E_i\cap\left(D_n\cup\left(\bigcup_{n\in N} D_n\right)^c\right)\right)\\ &= \lim_{n\to+\infty}\mu\left(E_i\cap D_n\right) + \mu\left(E_i\cap\left(\bigcup_{n\in N} D_n\right)^c\right)\\ &= \lim_{n\to+\infty}\mu(E_i\cap D_n).\end{aligned}$$

Hence, we have shown that

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbb{E}} f_n(x)d\mu(x) &\geq \lim_{n\to+\infty}\sum_{i=1}^{d}(\alpha_i-\varepsilon_i)\mu(E_i\cap D_n)\\ &= \sum_{i=1}^{d}(\alpha_i-\varepsilon_i)\mu(E_i)\\ &= \int_{\mathbb{E}}\tilde{g}(x)d\mu(x)\\ &geqm.\end{aligned}$$

Then, the theorem is completely proved. □

Lemma 2.2.16 (Fatou's lemma)**.**
Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions such that for all n in $\mathbb{N}$ for almost every x in $\mathbb{E}$ it holds that $f_n(x)\geq 0$. Then, the pointwise limit inferior is measurable and it holds that

$$\int_{\mathbb{E}}\left(\liminf_{n\to+\infty} f_n(x)\right)d\mu(x) \leq \liminf_{n\to+\infty}\int_{\mathbb{E}} f_n(x)d\mu(x).$$

Proof. We have already discussed the measurability of both the pointwise limit inferior and the infimum. Let n, m be positive integers such that $n \geq m$. We notice that

$$\int_{\mathbb{E}} f_n(x)d\mu(x) \geq \int_{\mathbb{E}} \inf_{n\geq m} \{f_n(x)\}d\mu(x);$$

hence, we can state that

$$\inf_{n\geq m} \left\{ \int_{\mathbb{E}} f_n(x)d\mu(x) \right\} \geq \int_{\mathbb{E}} \inf_{n\geq m} \{f_n(x)\}d\mu(x).$$

If we join the definition of limit inferior and theorem 2.2.15, we obtain that

$$\begin{aligned}
\liminf_{n\to+\infty} \int_{\mathbb{E}} f_n(x)d\mu(x) &= \sup_{m\in\mathbb{N}} \left\{ \inf_{n\geq m} \left\{ \int_{\mathbb{E}} f_n(x)d\mu(x) \right\} \right\} \\
&\geq \sup_{m\in\mathbb{N}} \left\{ \int_{\mathbb{E}} \inf_{n\geq m} \{f_n(x)\}d\mu(x) \right\} \\
&= \int_{\mathbb{E}} \sup_{m\in\mathbb{N}} \left\{ \inf_{n\geq m} \{f_n(x)\} \right\} d\mu(x) \\
&= \int_{\mathbb{E}} \left\{ \liminf_{n\to+\infty} f_n(x) \right\} d\mu(x).
\end{aligned}$$

□

2.2.3 Integration of variable sign measurable functions

Definition 2.2.17 (Integration of variable sign measurable function)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $f :\to [-\infty; +\infty]$ be a measurable function. Let us consider the usual decomposition in of f in positive part f^+ and negative part f^-, i. e. $f = f^+ - f^-$. Let us assume that

$$\int_{\mathbb{E}} f^+(x)d\mu(x) < +\infty$$

or

$$\int_{\mathbb{E}} f^-(x)d\mu(x) < +\infty.$$

We define the integral of f with respect to the measure μ as follows:

$$\int_{\mathbb{E}} f(x)d\mu(x) := \int_{\mathbb{E}} f^+(x)d\mu(x) - \int_{\mathbb{E}} f^-(x)d\mu(x).$$

Proposition 2.2.18. *Let* $(\mathbb{E}; \mathscr{E}; \mu)$ *be a measurable space with a measure* μ*. Let* $f_1, f_2 :\to \mathbb{R}$ *be a measurable functions such that*

$$\int_{\mathbb{E}} |f_1(x)|\, d\mu(x) = \int_{\mathbb{E}} f_1^+(x)d\mu(x) + \int_{\mathbb{E}} f_1^-(x)d\mu(x) < +\infty,$$

$$\int_{\mathbb{E}} |f_2(x)|\, d\mu(x) = \int_{\mathbb{E}} f_2^+(x)d\mu(x) + \int_{\mathbb{E}} f_2^-(x)d\mu(x) < +\infty.$$

Then, we can define the integral of $f_1 + f_2$ *as in definition 2.2.17 and it holds that*

$$\int_{\mathbb{E}} [f_1(x) + f_2(x)]d\mu(x) = \int_{\mathbb{E}} f_1(x)d\mu(x) + \int_{\mathbb{E}} f_2(x)d\mu(x).$$

If f_1 is such that

$$\int_{\mathbb{E}} f^+(x)d\mu(x) < +\infty$$

or

$$\int_{\mathbb{E}} f^-(x)d\mu(x) < +\infty$$

and α is any real number, then we can define the integral for αf_1 as in definition 2.2.17 and it holds that

$$\int_{\mathbb{E}} \alpha f_1(x)d\mu(x) = \alpha \int_{\mathbb{E}} f_1(x)d\mu(x).$$

Proof. As for the second part of the statement, it is an immediate consequence of definition 2.2.17.

As for the first statement, we notice that

$$|f_1 + f_2| \leq |f_1| + |f_2|\,;$$

hence, it holds that

$$\int_{\mathbb{E}} |f_1(x) + f_2(x)|\, d\mu(x) \leq \int_{\mathbb{E}} |f_1(x)|\, d\mu(x) + \int_{\mathbb{E}} |f_2(x)|\, d\mu(x).$$

Therefore, we can define the integral of $f_1 + f_2$ with respect to the measure μ as in 2.2.17. We notice that

$$(f+g)^+ - (f+g)^- = f + g = f^+ - f^- + g^+ - g^-,$$

namely

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

If we integrate both sides and we use 2.2.14, we obtain that

$$\int_{\mathbb{E}} (f+g)^+\, d\mu + \int_{\mathbb{E}} f^-\, d\mu + \int_{\mathbb{E}} g^-\, d\mu = \int_{\mathbb{E}} (f+g)^-\, d\mu + \int_{\mathbb{E}} f^+\, d\mu + \int_{\mathbb{E}} g^+\, d\mu;$$

if we rearrange terms and use the definition 2.2.17, the conclusion is immediate. □

Theorem 2.2.19 (Dominated convergence theorem).
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions between $\mathbb{E}$ and $\overline{\mathbb{R}}$. Let us assume that there exists a measurable function $g : \mathbb{E} \to \mathbb{R}$ (usually called domination) with the following properties:

- *for all n in $\mathbb{N}$ for almost every x in $\mathbb{E}$ it holds that $|f_n(x)| \leq g(x)$;*
- $\displaystyle\int_{\mathbb{E}} |g(x)|\, d\mu(x) < +\infty.$

Let us assume that there exists a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ such that for almost every x in $\mathbb{E}$ it holds that

$$\lim_{n\to+\infty} f_n(x) = f(x).$$

Then, f is a measurable function and

$$\lim_{n\to+\infty} \int_{\mathbb{E}} f_n(x)d\mu(x) = \int_{\mathbb{E}} f(x)d\mu(x).$$

Proof. We have already discussed the measurability of f; obviously, $|f(x)| \leq g(x)$ for almost every x in $\mathbb{E}$; hence, we have that

$$\int_{\mathbb{E}} f(x)d\mu(x) < +\infty.$$

Under our hypothesis, we can apply lemma 2.2.16 to the sequence $\{g - f_n\}_{n\in\mathbb{N}}$. We obtain that

$$\liminf_{n\to+\infty} \int_{\mathbb{E}} (g(x) - f_n(x))d\mu(x) \geq \int_{\mathbb{E}} \left\{ \liminf_{n\to+\infty} (g(x) - f_n(x)) \right\} d\mu(x).$$

Since g has finite integral, we can split the integral; hence, we obtain that

$$\int_{\mathbb{E}} g(x)d\mu(x) - \limsup_{n\to+\infty} f_n(x)d\mu(x) \geq \int_{\mathbb{E}} g(x)d\mu(x) - \int_{\mathbb{E}} \left\{ \limsup_{n\to+\infty} f_n(x) \right\} d\mu(x).$$

In other words, we have that

$$\limsup_{n\to+\infty} f_n(x)d\mu(x) \leq \int_{\mathbb{E}} \left\{ \limsup_{n\to+\infty} f_n(x) \right\} d\mu(x).$$

If we apply the Fatou's lemma to $\{f_n + g\}_{n\in N}$, we obtain that

$$\liminf_{n\to+\infty} \int_{\mathbb{E}} f_n(x)d\mu(x) \geq \int_{\mathbb{E}} \left\{ \liminf_{n\to+\infty} f_n(x) \right\} d\mu(x).$$

Since f is the pointwise limit for almost every x in $\mathbb{E}$, the thesis follows immediately. □

2.2.4 Product measure

Definition 2.2.20. Let $(\mathbb{E}_1; \mathscr{E}_1), \dots (\mathbb{E}_n; \mathscr{E}_n)$ be measurable spaces. We define $\mathbb{E}$ the Cartesian product of $\mathbb{E}_1, \dots \mathbb{E}_n$, namely

$$\mathbb{E} = \prod_{i=1}^{n} \mathbb{E}_i.$$

Let A be any subset in $\mathbb{E}$. We define the tensor product σ-algebra $\mathscr{E}$ as the σ-algebra generated by the subset of the form $B_1 \times \cdots \times B_n$, where B_i is $\mathscr{E}_i$ for all integer i in $\{1; \dots; n\}$. We denote the tensor-product σ-algebra as

$$\mathscr{E} := \bigotimes_{i=1}^{n} \mathscr{E}_i.$$

In particular, $(\mathbb{E}; \mathscr{E})$ is a measurable space called product measurable space.

Remark 2.2.21. We immediately notice that the construction of the tensor product σ-algebra is associative, namely

$$(\mathscr{E}_1 \otimes \mathscr{E}_2) \otimes \mathscr{E}_3 = \mathscr{E}_1 \otimes (\mathscr{E}_2 \otimes \mathscr{E}_3).$$

Hence, the theory will be developed for two measurable spaces: the generalization to finite measurable spaces is an immediate consequence of the induction principle.

Lemma 2.2.22. *Let* $(\mathbb{E}_1; \mathscr{E}_1), (\mathbb{E}_2; \mathscr{E}_2)$ *be measurable spaces. Let* C *be in* $\mathscr{E}_1 \otimes \mathscr{E}_2$*. For all* x *in* $\mathbb{E}_1$ *we define*

$$C^1(x) := \{y \in \mathbb{E}_2 \mid (x; y) \in C\}.$$

Then, $C^1(x)$ *is in* $\mathscr{E}_2$*. Similarly, for all* y *in* $\mathbb{E}_2$ *we define*

$$C^2(y) := \{x \in \mathbb{E}_1 \mid (x; y) \in C\}.$$

Then $C^2(y)$ *is in* $\mathscr{E}_1$*.*

Proposition 2.2.23. *Let* $(\mathbb{E}_1; \mathscr{E}_1; \mu_1), (\mathbb{E}_2; \mathscr{E}_2; \mu_2)$ *be measurable spaces with measure* μ_1, μ_2*. Let* $f : \mathbb{E}_1 \times \mathbb{E}_2 \to [0; +\infty]$ *any* $\mathscr{E}_1 \otimes \mathscr{E}_2$*-measurable function. For all* x *in* $\mathbb{E}_1$ *we define the function* $f_x : \mathbb{E}_2 \to [0; +\infty]$ *such that* $f_x(y) := f(x; y)$*; then, the following conclusions hold true:*

- f_x *is* $\mathscr{E}_2$*-measurable;*
- *the function* $\varphi_1 : \mathbb{E}_1 \to [0; +\infty]$ *such that*

$$\varphi_1(x) := \int_{\mathbb{E}_2} f_x(y) d\mu_2$$

  is well defined and it is $\mathscr{E}_1$*-measurable.*

Similarly, For all y *in* $\mathbb{E}_2$ *we define the function* $f_y : \mathbb{E}_1 \to [0; +\infty]$ *such that* $f_y(x) := f(x; y)$*; then, the following conclusions hold true:*

- f_y *is* $\mathscr{E}_1$*-measurable;*
- *the function* $\varphi_2 : \mathbb{E}_2 \to [0; +\infty]$ *such that*

$$\varphi_2(y) := \int_{\mathbb{E}_1} f_y(x) d\mu_2$$

  is well defined and it is $\mathscr{E}_1$*-measurable.*

Theorem 2.2.24 (Fubini-Tonelli' theorem)**.** *Let* $(\mathbb{E}_1; \mathscr{E}_1; \mu_1), (\mathbb{E}_2; \mathscr{E}_2; \mu_2)$ *be measurable spaces with measure* μ_1, μ_2*. Let* C *be in* $\mathscr{E}_1 \otimes \mathscr{E}_2$*. We define*

$$\mu_1 \otimes \mu_2(C) := \int_{\mathbb{E}_1} \mu_2(C^1(x)) d\mu_1(x).$$

The function $\mu_1 \otimes \mu_2 : \mathscr{E}_1 \otimes \mathscr{E}_2 \to [0; +\infty]$ *is a measure on the measurable space* $(\mathbb{E}_1 \times \mathbb{E}_2; \mathscr{E}_1 \otimes \mathscr{E}_2)$ *with the following properties:*

- *for all* A_1 *in* $\mathscr{E}_1$*, for all* A_2 *in* $\mathscr{E}_2$ *it holds that*

$$\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2);$$

- *if* $f : \mathbb{E}_1 \times \mathbb{E}_2 \to [0; +\infty]$ *is a measurable function, it holds that*

$$\int_{\mathbb{E}_1 \times \mathbb{E}_2} f(x; y) d(\mu_1 \otimes \mu_2)(x; y) = \int_{E_1} \left(\int_{\mathbb{E}_2} f(x; y) d\mu_2(y) \right) d\mu_1(x).$$

Corollary 2.2.25. *Let $(\mathbb{E}_1; \mathscr{E}_1; \mu_1), (\mathbb{E}_2; \mathscr{E}_2; \mu_2)$ be measurable spaces with measure μ_1, μ_2. Let $f : \mathbb{E}_1 \times \mathbb{E}_2 \to \overline{\mathbb{R}}$ be a measurable function such that*

$$\int_{\mathbb{E}_1 \times \mathbb{E}_2} |f(x; y)| \, d(\mu_1 \otimes \mu_2)(x; y) < +\infty.$$

For all x in $\mathbb{E}_1$ we define $f_x : \mathbb{E}_2 \to \overline{\mathbb{R}}$ such that $f_x(y) := f(x; y)$. Then the following conclusions hold true:

- *the function f_x is $\mathscr{E}_2$-measurable;*
- *the function $\varphi_1 : \mathbb{E}_1 \to \overline{\mathbb{R}}$ such that*

$$\varphi_1(x) := \int_{\mathbb{E}_2} f_x(y) d\mu_2(y)$$

  is well defined for almost every x in $\mathbb{E}_1$ and it is $\mathscr{E}_2$-measurable;
- *it holds that*

$$\int_{\mathbb{E}_1 \times \mathbb{E}_2} f(x; y) d(\mu_1 \otimes \mu_2)(x; y) = \int_{\mathbb{E}_1} \left(\int_{\mathbb{E}_2} f(x; y) d\mu_2(y) \right) d\mu_1(x).$$

Remark 2.2.26. It's easy to see that the for all C in $\mathscr{E}_1 \otimes \mathscr{E}_2$ it holds that

$$\int_{\mathbb{E}_1} C^1(x) d\mu_1(x) = \int_{\mathbb{E}_2} C^2(y) d\mu_2(y).$$

Let $f : \mathbb{E}_1 \times \mathbb{E}_2 \to [0; +\infty]$ be a measurable function; then, it holds that

$$\int_{\mathbb{E}_1 \times \mathbb{E}_2} f(x; y) d(\mu_1 \otimes \mu_2)(x; y) = \int_{E_2} \left(\int_{\mathbb{E}_1} f(x; y) d\mu_1(x) \right) d\mu_2(y).$$

Similarly, if $f : \mathbb{E}_1 \times \mathbb{E}_2 \to \overline{\mathbb{R}}$ is a measurable function such that

$$\int_{\mathbb{E}_1 \times \mathbb{E}_2} |f(x; y)| \, d(\mu_1 \otimes \mu_2)(x; y) < +\infty,$$

it holds that

$$\int_{\mathbb{E}_1 \times \mathbb{E}_2} f(x; y) d(\mu_1 \otimes \mu_2)(x; y) = \int_{E_2} \left(\int_{\mathbb{E}_1} f(x; y) d\mu_1(x) \right) d\mu_2(y).$$

Lebesgue measure in $\mathbb{R}^d$ as product measure

Proposition 2.2.27. *Let d be a positive integer. Let $\mathscr{M}$ be the σ-algebra of the Lebesgue measurable set in $\mathbb{R}$; we denote as $\mathscr{M}^d$ the product σ-algebra, i.e.*

$$\mathscr{M}^d := \bigotimes_{i=1}^{d} \mathscr{M}.$$

There exists a measure $\mathscr{L}^d$ on $(\mathbb{R}^d; \mathscr{M}^d)$ with the following property: if A is a box, namely there exists $\{I_1; \ldots; I_d\}$ measurable sets in $\mathbb{R}$ such that

$$A := \prod_{i=1}^{d} I_i,$$

then it holds that

$$\mathscr{L}^d(A) = \prod_{i=1}^{d} \mathscr{L}^1(I_i).$$

Moreover, if $f : \mathbb{R}^d \to [0; +\infty]$ *is a measurable function, the following conclusions hold true:*

- *for all* x *in* $\mathbb{R}$ *the function* $f_d : \mathbb{R}^{d-1} \to [0; +\infty]$ *such that*

$$f_d(x_1; \ldots; x_{d-1}) \coloneqq f(x_1; \ldots; x_{d-1}; x)$$

  is measurable;

- *the function* $\varphi_d : \mathbb{R} \to [0; +\infty]$ *such that*

$$\varphi_d(x) \coloneqq \int_{\mathbb{R}^{d-1}} f_d(x_1; \ldots; x_{d-1}) dx_1 \cdots dx_{d-1}$$

  is measurable;

- *it holds that*

$$\int_{\mathbb{R}^d} f(x_1; \ldots x_d) dx_1 \cdots dx_d = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} f(x_1; \ldots; x_{d-1}) dx_1 \cdots dx_{d-1} \right) dx_d.$$

Proof. It is an immediate consequence of Fubini-Tonelli's theorem (see 2.2.24). □

Corollary 2.2.28. *Let* d *be a positive integer. Let* $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ *be a measurable function such that*

$$\int_{\mathbb{R}^d} |f(x_1; \ldots x_d)| \, dx_1 \cdots dx_d < +\infty.$$

Then, the following conclusions hold true:

- *for all* x *in* $\mathbb{R}$ *the function* $f_d : \mathbb{R}^{d-1} \to \bar{\mathbb{R}}$ *such that*

$$f_d(x_1; \ldots; x_{d-1}) \coloneqq f(x_1; \ldots; x_{d-1}; x)$$

  is measurable;

- *the function* $\varphi_d : \mathbb{R} \to \bar{\mathbb{R}}$ *such that*

$$\varphi_d(x) \coloneqq \int_{\mathbb{R}^{d-1}} f_d(x_1; \ldots; x_{d-1}) dx_1 \cdots dx_{d-1}$$

  is well defined for almost every x *in* $\mathbb{R}$ *and it is measurable;*

- *it holds that*

$$\int_{\mathbb{R}^d} f(x_1; \ldots x_d) dx_1 \cdots dx_d = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} f(x_1; \ldots; x_{d-1}) dx_1 \cdots dx_{d-1} \right) dx_d.$$

Proof. It is an immediate consequence of 2.2.25. □

2.2.5 Lebesgue integral vs Riemann integral

Let A in $\mathbb{R}$ be a closed interval; let $f : A \to \mathbb{R}$ be a continuous function. In particular, if we extend f at 0 out of A, we have that f is a measurable function between $\mathbb{R}^d$ and $\mathbb{R}$. We can define the Lebesgue integral and the Riemann integral. As a matter of facts, they coincide.

In deed the following theorem holds true.

Theorem 2.2.29 (Vitali-Lebesgue).
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then it is Riemann-integrable if and only if it is measurable, Lebesgue-integrable and the set of discontinuity points has Lebesgue measure 0. Moreover, the two integral coincide.

Example 2.2.30. The function $f(x) := \frac{\sin(x)}{x}$ is continuous and bounded in $(0, +\infty)$; it can be proved that

$$\lim_{n\to+\infty} \int_0^n \frac{\sin(x)}{x}\, dx = \int_0^{+\infty} \frac{\sin(x)}{x}\, dx = \frac{\pi}{2},$$

namely the sequence of the Riemann integrals converges to $\frac{\pi}{2}$; unfortunately it is also true that

$$\int_0^{+\infty} \frac{|\sin(x)|}{|x|}\, dx = +\infty$$

and it is not possible to define the Lebesgue integral of f in $(0, +\infty)$.

We state the change of variable formula.

Theorem 2.2.31 (Change of variable).
Let A, B be bounded open sets in $\mathbb{R}^n$; let $f : A \to B$ be a diffeomorphism (namely a bijective C^1-function such that f^{-1} is a C^1-function). For all $x \in A$ we define

$$J_f(x) := \sqrt{\det\left(\left[\frac{\partial f}{\partial x}(x)\right]^T \cdot \left[\frac{\partial f}{\partial x}(x)\right]\right)},$$

where $\frac{\partial f}{\partial x}(x)$ is the jacobian of f at the point x (it is an $n \times n$-matrix). Assume that J_f is bounded in A. For all Borel function $h : A \to \mathbb{R}$ (measurable function in $\mathbb{R}^n$ equipped with the Borel sets σ-algebra) it holds that $h \circ f^{-1} : B \to \mathbb{R}$ is measurable and

$$\int_A h(x) J_f(x)\, dx = \int_B h(f^{-1}(y))\, dy.$$

2.3 Appendix

Theorem 2.3.1 (Continuity of integral).
Let Ω be any open set in $\mathbb{R}^d$. Let $f : [a; b] \times \Omega \to \mathbb{R}$ be any function with the following properties:

- *for almost every x in Ω for all t in $[a; b]$ it holds that*

$$\lim_{h\to 0} f(t + h; x) - f(x) = 0;$$

- *for all t in $[a;b]$ the function $\varphi_t : \Omega \to \mathbb{R}$ such that $\varphi_t(x) = f(t;x)$ is measurable;*
- *there exists a function α in $L^1(\mathbb{R}^d)$ such that for all $(t;x)$ in $[a;b] \times \Omega$ it holds that*
$$|f(t;x)| \leq \alpha(x).$$

For all t in $[a;b]$, we denote
$$F(t) \coloneqq \int_\Omega f(t;x)dx.$$
Then, $F : [a;b] \to \mathbb{R}$ is well defined and it is continuous.

Proof. Under our hypothesis, F is well defined, obviously. Let t_0 be any point in $[a;b]$; we claim that
$$\lim_{h\to 0} \int_\Omega f(t_0 + h;x)dx = \int_\Omega f(t_0;x)dx.$$
We notice that for almost every x in Ω it holds that
$$\lim_{h\to 0} f(t_0 + h;x) = f(t_0;x);$$
moreover, α is a suitable domination in $L^1(\Omega)$. Hence, the conclusion is an immediate consequence of the dominated convergence theorem. □

Theorem 2.3.2 (Derivation under integral)**.**
Let Ω be any open set in $\mathbb{R}^d$. Let $f : [a;b] \times \Omega \to \mathbb{R}$ be any function with the following properties:

- *for all t in $[a;b]$ the function $\varphi_t : \Omega \to \mathbb{R}$ such that $\varphi_t(x) = f(t;x)$ is measurable;*
- *for all t in $[a;b]$ for almost every x in Ω there exists $\frac{\partial f}{\partial t}(t;x)$;*
- *for all t in $[a;b]$ there exists a measurable function $\psi_t : \Omega \to \mathbb{R}$ such that for almost every x in Ω it holds that*
$$\psi_t(x) = \frac{\partial f}{\partial t}(t;x);$$
- *for all t in $[a;b]$ for almost every x in Ω it holds that*
$$\lim_{h\to 0} \psi_{t+h}(x) = \psi_t(x);$$
- *there exists a function α in $L^1(\mathbb{R}^d)$ such that for all t in $[a;b]$ for almost every x in Ω it holds that*
$$\max\left\{|f(t;x)|\,;\left|\frac{\partial f}{\partial t}(t;x)\right|\right\} \leq \alpha(x).$$

Then, the following conclusions hold true:

- *$F : [a;b] \to \mathbb{R}$ and $G : [a;b] \to \mathbb{R}$ are well defined;*
- *G is continuous in $[a;b]$;*
- *F is in $C^1((a;b))$ and for all t in $(a;b)$ it holds that $F'(t) = G(t)$.*

Proof. We notice that the function F is well defined, obviously. Thanks to theorem 2.3.1, the function G is well defined and continuous. Let t_0 be any point in $[a; b]$; if we show that

$$F(t_0) - F(a) = \int_a^{t_0} G(t)dt,$$

then the thesis follows immediately from the fundamental theorem of calculus. By definition of G, we have that

$$\int_a^{t_0} G(t)dt = \int_a^{t_0} \left(\int_\Omega \frac{\partial f}{\partial t}(t; x)dx \right) dt. \tag{2.1}$$

We notice that

$$\int_{\Omega\times[a;t_0]} \left| \frac{\partial f}{\partial t}(t; x) \right| dtdx \leq \int_{\Omega\times[a;t_0]} \alpha(x)dtdx = \|\alpha\|_{L^1(\Omega)} (t_0 - a).$$

Having said that, we can use Fubini's theorem and switch the order of integration at the right hand side of (2.1); thanks to the fundamental theorem of calculus, we obtain that

$$\begin{aligned} \int_a^{t_0} G(t)dt &= \int_\Omega \left(\int_a^{t_0} \frac{\partial f}{\partial t}(t; x)dt \right) dx \\ &= \int_\Omega [f(t_0; x) - f(a; x)]dx. \end{aligned}$$

To conclude, since α is a suitable domination for φ_t in $L^1(\Omega)$, we notice that we can split the integral and the following identity holds true:

$$\int_a^{t_0} G(t)dt = \int_\Omega f(t_0; x)dx - \int_\Omega f(a; x)dx,$$

that is equivalent to the thesis. □

Chapter 3

L^p space

3.1 Definitions and main properties

Definition 3.1.1. Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let p be a real number in $[1;+\infty)$. Let $f:\mathbb{E}\to\overline{\mathbb{R}}$ be a measurable function. We denote

$$\|f\|_{L^p(\mathbb{E})} := \left(\int_{\mathbb{E}} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}.$$

We denote

$$\|f\|_{L^\infty(\mathbb{E})} := \inf\left\{c\in\overline{R} \mid c\geq |f(x)| \text{ for almost every } x\in\mathbb{E}\right\}.$$

Remark 3.1.2. Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let $f:\mathbb{E}\to\mathbb{R}$ be a measurable function. We claim that the infimum in the definition of $\|f\|_{L^\infty(\mathbb{E})}$ is actually a minimum. We have to show that for almost every x in $\mathbb{E}$ it holds that $|f(x)|\leq\|f\|_{L^\infty(\mathbb{E})}$. We notice that $|f(x)|>\|f\|_{L^\infty(\mathbb{E})}$ if and only if there exists n in $\mathbb{N}$ such that

$$|f(x)|\geq\frac{1}{n}+\|f\|_{L^\infty(\mathbb{E})}.$$

For all n in $\mathbb{N}$ we define

$$A_n := f^{-1}\left(\left[-\frac{1}{n}-\|f\|_{L^\infty(\mathbb{E})};\frac{1}{n}+\|f\|_{L^\infty(\mathbb{E})}\right]^c\right).$$

This is enough to state that

$$f^{-1}\left(\left[-\|f\|_{L^\infty(\mathbb{E})};\|f\|_{L^\infty(\mathbb{E})}\right]^c\right)=\bigcup_{n\in\mathbb{N}}A_n.$$

Hence, we can conclude that

$$\mu\left(f^{-1}\left(\left[-\|f\|_{L^\infty(\mathbb{E})};\|f\|_{L^\infty(\mathbb{E})}\right]^c\right)\right)\leq\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right)\leq\sum_{n\in\mathbb{N}}\mu(A_n)=0.$$

Remark 3.1.3. In the setting of definition 3.1.1, if $\|f\|_{L^\infty(\mathbb{E})}$ is a real number, then f is finite for almost every x in $\mathbb{E}$. Let p be a real number in $[1;+\infty)$; let us assume that $\|f\|_{L^p(\mathbb{E})}$ is a real number. Then f is finite for almost every x in $\mathscr{E}$. In fact, if we denote

$$A := f^{-1}(\{-\infty;+\infty\}),$$

then A is measurable and we have that

$$\int_{\mathbb{E}} |f(x)|^p \, dx \geq \mu_A \cdot (+\infty).$$

Therefore, it must be that $\mu(A)$ is equal to 0.

Definition 3.1.4 (L^p space)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be a measurable function. Let p be in $[1; +\infty]$. Let us denote $\mathcal{D}^p$ the collection of the measurable functions between $\mathbb{E}$ and $\overline{\mathbb{R}}$ such that $\|f\|_{L^p(\mathbb{E})}$ is a real number. We introduce the following relation of equivalence in $\mathcal{D}^p$: we say that f and g are equivalent if and only if $f(x) = g(x)$ for almost every x in $\mathbb{E}$; we will write $f \sim g$. We define $L^p(\mathbb{E})$ as the quotient set $\mathcal{D}^p / \sim$.

Remark 3.1.5. In the setting of definition 3.1.1, it is immediate to see that if f, g are measurable functions such that $f(x) = g(x)$ for almost every x in $\mathbb{E}$, then it holds that

$$\|f\|_{L^p(\mathbb{E})} = \|g\|_{L^p(\mathbb{E})} .$$

So, if $[f]$ is an element in $L^p(\mathbb{E})$, i.e. $[f]$ is a set of functions that coincide almost everywhere, we can well define

$$\|[f]\|_{L^p(\mathbb{E})} \coloneqq \|g\|_{L^p(\mathbb{E})} ,$$

where g is any function in $[f]$. As a matter of facts, we will always refer to the classes of functions coinciding almost everywhere as functions.

3.1.1 Integral inequalities

We show the most famous integral inequalities. However, the aim of this subsection is to give L^p the structure of normed vector space.

Proposition 3.1.6 (Jensen's inequality)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $f : \mathbb{E} \to \mathbb{R}$ be a measurable function that is μ-integrable. Let us assume that $\mu(\mathbb{E})$ is a real number. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, it holds that

$$\varphi\left(\frac{1}{\mu(\mathbb{E})} \int_{\mathbb{E}} f(x) d\mu(x)\right) \leq \frac{1}{\mu(\mathbb{E})} \int_{\mathbb{E}} \varphi(f(x)) d\mu(x).$$

Proof. Let us denote

$$y_0 \coloneqq \frac{1}{\mu(\mathbb{E})} \int_{\mathbb{E}} f(x) d\mu(x).$$

Since φ is a convex function, there exists a real number m such that for all y in $\mathbb{R}$ it holds that

$$\varphi(y) \geq \varphi(y_0) + m(y - y_0).$$

In particular, for all x in $\mathbb{E}$ it holds that

$$\varphi(f(x)) \geq \varphi(y_0) + m(f(x) - y_0).$$

If we integrate, we obtain the following inequalities:

$$\begin{aligned}\int_{\mathbb{E}} \varphi(f(x))d\mu(x) &\geq \int_{\mathbb{E}} [\varphi(y_0) + m(f(x) - y_0)]d\mu(x) \\ &= \mu(\mathbb{E})\varphi\left(\frac{1}{\mu(\mathbb{E})}\int_{\mathbb{E}} f(x)d\mu(x)\right) + m\left[\int_{\mathbb{E}} f(x)d\mu(x) - \mu(\mathbb{E})y_0\right] \\ &= \mu(\mathbb{E})\varphi\left(\frac{1}{\mu(\mathbb{E})}\int_{\mathbb{E}} f(x)d\mu(x)\right).\end{aligned}$$

□

Definition 3.1.7 (Conjugate indices)**.** Let p be in $[1; +\infty]$. We say that p^* is it's conjugate index of p if it holds that

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

with the convention that $\frac{1}{\infty} = 0$.

Proposition 3.1.8 (Young's inequality)**.**
Let p, p^* be real conjugate indices in $(1; +\infty)$. Let a, b be real numbers in $[0; +\infty)$; then the following inequality holds true:

$$ab \leq \frac{a^p}{p} + \frac{b^{p^*}}{p^*}.$$

More precisely, the equal holds true if and only if $a^p = b^{p^*}$.

Proof. If a equals 0 or b equals 0, the conclusion is trivial. So, it is not restrictive to assume that either a and b are positive real numbers. Since $\log(x)$ is a concave function and p, p^* are conjugate indices, we can state that

$$\frac{1}{p}\log(a^p) + \frac{1}{p^*}\log(b^{p^*}) \leq \log\left(\frac{a^p}{p} + \frac{b^{p^*}}{p^*}\right).$$

We notice that the left hand side equals $\log(ab)$. Since $\log(x)$ is an increasing function, we obtain that

$$ab \leq \frac{a^p}{p} + \frac{b^{p^*}}{p^*}.$$

Since $\log(x)$ is a strictly concave function, equal holds true if and only if $a^p = b^q$. □

Proposition 3.1.9 (Hölder's inequality)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $f, g : \mathbb{E} \to \mathbb{R}$ be measurable functions. Let p, p^* be conjugate indices in $[1; +\infty]$. Then, the following inequality holds true:

$$\|fg\|_{L^1(\mathbb{E})} \leq \|f\|_{L^p(\mathbb{E})} \|g\|_{L^{p^*}(\mathbb{E})}.$$

Moreover, if p, p^* are real numbers in $(1; +\infty)$ and $\|f\|_{L^p(\mathbb{E})}, \|g\|_{L^{p^*}(\mathbb{E})}$ are positive real numbers, it holds that

$$\|fg\|_{L^1(\mathbb{E})} = \|f\|_{L^p(\mathbb{E})} \|g\|_{L^{p^*}(\mathbb{E})}$$

if and only if

$$|f(x)|^p = |g(x)|^{p^*} \frac{\|f\|^p_{L^p(\mathbb{E})}}{\|g\|^{p^*}_{L^{p^*}(\mathbb{E})}}$$

for almost every x in $\mathbb{E}$.

Proof. Let us assume that $p = +\infty$ and $p^* = 1$. Then, for almost every x in $\mathbb{E}$ it holds that

$$|f(x)| \leq \|f\|_{L^\infty(\mathbb{E})}\,.$$

Hence, we have that

$$\begin{aligned}\|fg\|_{L^1(\mathbb{E})} &= \int_{\mathbb{E}} |f(x)g(x)|\, d\mu(x) \\ &\leq \int_{\mathbb{E}} \|f\|_{L^\infty(\mathbb{E})} |g(x)|\, d\mu(x) \\ &= \|f\|_{L^\infty(\mathbb{E})} \|g\|_{L^1(\mathbb{E})}\,.\end{aligned}$$

If $p = 1$ and $p^* = +\infty$, the proof is completely similar. So, we can assume that p, p^* are real numbers in $(1; +\infty)$. If $\|f\|_{L^p(\mathbb{E})}$ is 0, then $f(x)$ equals 0 for almost every x in $\mathbb{E}$; then $f(x)g(x)$ is 0 for almost every x in $\mathbb{E}$ and the conclusion is trivial. If $\|g\|_{L^{p^*}}(\mathbb{E})$ is 0, the conclusion is trivial. Hence, we can assume that both $\|f\|_{L^p(E)}$ and $\|g\|_{L^{p^*}}(\mathbb{E})$ are in $(0; +\infty]$. We notice that if $\|f\|_{L^p(\mathbb{E})} = +\infty$ or $\|g\|_{L^{p^*}(\mathbb{E})} = +\infty$, the conclusion is trivial. Having said that, it is not restrictive to assume that p, p^* are real numbers in $(1; +\infty)$ and $\|f\|_{L^p(\mathbb{E})}, \|g\|_{L^{p^*}(\mathbb{E})}$ are real numbers in $(0; +\infty)$. In particular, both $f(x)$ and $g(x)$ are real numbers for almost every x in $\mathbb{E}$. Thanks to Young's inequality (see 3.1.8), for almost every x in $\mathbb{E}$ it holds that

$$\frac{|f(x)g(x)|}{\|f\|_{L^p(\mathbb{E})} \|g\|_{L^{p^*}(\mathbb{E})}} \leq \frac{|f(x)|^p}{p\|f\|^p_{L^p(\mathbb{E})}} + \frac{|g(x)|^{p^*}}{p^*\|g\|^{p^*}_{L^{p^*}(\mathbb{E})}}.$$

If we integrate, we obtain that

$$\begin{aligned}\int_{\mathbb{E}} \frac{|f(x)g(x)|}{\|f\|_{L^p(\mathbb{E})} \|g\|_{L^{p^*}(\mathbb{E})}} d\mu(x) &\leq \frac{1}{p}\int_{\mathbb{E}} \frac{|f(x)|^p}{\|f\|^p_{L^p(\mathbb{E})}} d\mu(x) + \frac{1}{p^*}\int_{\mathbb{E}} \frac{|g(x)|^{p^*}}{\|g\|^{p^*}_{L^{p^*}(\mathbb{E})}} d\mu(x) \\ &= \frac{1}{p} + \frac{1}{p^*} = 1.\end{aligned}$$

The thesis follows rearranging terms. Moreover, we notice that the equal holds true if and only for almost every x in $\mathbb{E}$ it holds that

$$\frac{|f(x)g(x)|}{\|f\|_{L^p(\mathbb{E})} \|g\|_{L^{p^*}(\mathbb{E})}} = \frac{|f(x)|^p}{p\|f\|^p_{L^p(\mathbb{E})}} + \frac{|g(x)|^{p^*}}{p^*\|g\|^{p^*}_{L^{p^*}(\mathbb{E})}}.$$

As shown in 3.1.8, this is equivalent to require that for almost every x in $\mathbb{E}$ it holds that

$$\frac{|f(x)|^p}{\|f\|^p_{L^p(\mathbb{E})}} = \frac{|g(x)|^{p^*}}{\|g\|^{p^*}_{L^{p^*}(\mathbb{E})}}.$$

□

Proposition 3.1.10 (Minkowski's inequality)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $f, g : \mathbb{E} \to \mathbb{R}$ be measurable functions. Let p be in $[1; +\infty]$. The following inequality holds true:

$$\|f + g\|_{L^p(\mathbb{E})} \leq \|f\|_{L^p(\mathbb{E})} + \|g\|_{L^p(\mathbb{E})}\,.$$

Proof. If p is equal to 1, the thesis is an immediate consequence of the triangular inequality.

If p is equal to $+\infty$, for almost every x in $\mathbb{E}$ it holds that $|f(x)| \leq \|f\|_{L^\infty(\mathbb{E})}$ and $|g(x)| \leq \|g\|_{L^\infty(\mathbb{E})}$. Hence, for almost every x in $\mathbb{E}$ it holds that

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_{L^\infty(\mathbb{E})} + \|g\|_{L^\infty(\mathbb{E})}.$$

So, the conclusion is immediate.

Let us assume that p is a real number in $(1; +\infty)$. If $\|f\|_{L^p(\mathbb{E})}$ equals $+\infty$ or $\|g\|_{L^p(\mathbb{E})}$ equals $+\infty$, the conclusion is trivial. So, we can assume that both $\|f\|_{L^p(\mathbb{E})}, \|g\|_{L^p(\mathbb{E})}$ are real numbers. In particular, $f(x)$ and $g(x)$ are real numbers for almost every x in $\mathbb{E}$. Since p is greater that 1, the function $\varphi_p(x) := |x|^p$ is convex. For all x in $\mathbb{E}$, it holds that

$$|f(x) + g(x)|^p \leq 2^p \left(\frac{|f(x)| + |g(x)|}{2} \right)^p \leq 2^{p-1} \left(|f(x)|^p + |g(x)|^p \right).$$

If we integrate, we obtain that

$$\|f + g\|_{L^p(\mathbb{E})} \leq 2^{p-1} \left(\|f\|_{L^p(\mathbb{E})} + \|g\|_{L^p(\mathbb{E})} \right) < +\infty.$$

Let p^* be the conjugate index of p; by definition 3.1.7, we have that

$$p^* = \frac{p}{p-1}.$$

Thanks to Hölder's inequality (see 3.1.9), we obtain that

$$\begin{aligned}
\int_{\mathbb{E}} |f(x) + g(x)|^p \, d\mu(x) &= \int_{\mathbb{E}} |f(x) + g(x)|^{p-1} |f(x) + g(x)| \, d\mu(x) \\
&\leq \int_{\mathbb{E}} |f(x) + g(x)|^{p-1} |f(x)| \, d\mu(x) \\
&\quad + \int_{\mathbb{E}} |f(x) + g(x)|^{p-1} |g(x)| \, d\mu(x) \\
&\leq \left(\int_{\mathbb{E}} |f(x) + g(x)|^{(p-1)p^*} \, d\mu(x) \right)^{\frac{1}{p^*}} \left(\int_{\mathbb{E}} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\
&\quad + \left(\int_{\mathbb{E}} |f(x) + g(x)|^{(p-1)p^*} \, d\mu(x) \right)^{\frac{1}{p^*}} \left(\int_{\mathbb{E}} |g(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{E}} |f(x) + g(x)|^p \, d\mu(x) \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{E}} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\
&\quad + \left(\int_{\mathbb{E}} |f(x) + g(x)|^p \, d\mu(x) \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{E}} |g(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.
\end{aligned}$$

We have shown that

$$\|f + g\|_{L^p(\mathbb{E})}^p \leq \|f + g\|_{L^p(\mathbb{E})}^{p-1} \left(\|f\|_{L^p(\mathbb{E})} + \|g\|_{L^p(\mathbb{E})} \right).$$

We notice that if $\|f + g\|_{L^p(\mathbb{E})}^p$ is 0, the conclusion is trivial; otherwise we can divide both sides by the real positive real number $\|f + g\|_{L^p(\mathbb{E})}^{p-1}$ and the thesis follows immediately. □

Theorem 3.1.11. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let p be in $[1; +\infty]$. The function*

$$\|\cdot\|_{L^p(\mathbb{E})} : L^p(\mathbb{E}) \to [0; +\infty)$$

defined in 3.1.5 is a norm. In particular, $L^p(\mathbb{E})$ is a metric space with the distance induced by the norm.

Proof. The well definition of the function

$$\|\cdot\|_{L^p(\mathbb{E})} : L^p(\mathbb{E}) \to [0; +\infty)$$

has already bee discussed in 3.1.5. We claim that it is a norm.

- Obviously, for all f in $L^p(\mathbb{E})$, it holds that $\|f\|_{L^p(\mathbb{E})}$ is a real number in $[0; +\infty)$.
- We notice that $\|f\|_{L^p(\mathbb{E})} = 0$ if and only if $f(x) = 0$ for almost every x in $\mathbb{E}$, i.e. f is the null function in the quotient set.
- If λ is any real number, it holds that

$$|\lambda| \, \|f\|_{L^p(\mathbb{E})} = \|\lambda f\|_{L^p(\mathbb{E})} .$$

- As for the triangular inequality, it is an immediate consequence of the Minkowski's inequality (see 3.1.10).

□

L^p* vs *L^q

Proposition 3.1.12. *Let $\mathbb{X}, \mathbb{Y}$ be normed vector spaces. Let $T : \mathbb{X} \to \mathbb{Y}$ a linear map. The following facts are equivalent:*

1. *T is continuous;*
2. *T is continuous in 0;*
3. *T is bounded, i. e. there exists C in $\mathbb{R}$ such that for all x in $\mathbb{X}$ it holds that*

$$\|T(x)\|_{\mathbb{Y}} \leq C \|x\|_{\mathbb{X}} ;$$

4. *there exists D in $\mathbb{R}$ such that T is D-Lipschitz.*

Proof. It is obvious that 3) implies 4), that implies 1) that implies 2). As for the the 2) implies 3), by definition of continuity in 0, there exists a positive real number δ such that if $\|x\|_{\mathbb{X}} \leq \delta$ then $\|T(x)\|_{\mathbb{Y}} \leq 1$. If x is any vector in $\mathbb{X} \setminus \{0\}$, it holds that

$$\|T(x)\|_{\mathbb{Y}} = \left\| T\left(\frac{\|x\|_{\mathbb{X}}}{\delta} \frac{\delta x}{\|x\|_{\mathbb{X}}} \right) \right\|_{\mathbb{Y}} = \frac{\|x\|_{\mathbb{X}}}{\delta} \left\| T\left(\frac{\delta x}{\|x\|_{\mathbb{X}}} \right) \right\|_{\mathbb{Y}} \leq \frac{1}{\delta} \|x\|_{\mathbb{X}} .$$

□

Proposition 3.1.13. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let us assume that $\mu(\mathbb{E})$ is finite. Let p, q such that $1 \leq p < q \leq +\infty$. Let us consider the inclusion map $i : L^q(\mathbb{E}) \hookrightarrow L^p(\mathbb{E})$. Then, it is well defined and continuous.*

Proof. If q equals $+\infty$ and f is any function in $L^\infty(\mathbb{E})$, then it is bounded almost everywhere; in particular, it is in $L^p(\mathbb{E})$ and the following inequality holds true:

$$\|f\|_{L^p} \le \mu(\mathbb{E})^{\frac{1}{q}} \|f\|_{L^\infty(\mathbb{E})};$$

so, the conclusion follows by proposition 3.1.12.

Let us assume that p, q are real numbers. We notice that the function $\varphi(x) := |x|^{\frac{q}{p}}$ is convex. Thanks to Jensen's inequality (see 3.1.6), it holds that

$$\varphi\left(\frac{1}{\mu(\mathbb{E})}\int_{\mathbb{E}} |f|^p \, d\mu(x)\right) \le \frac{1}{\mu(\mathbb{E})}\int_{\mathbb{E}} \varphi(|f(x)|^p) d\mu(x).$$

In other words, we obtain that

$$\left(\frac{1}{\mu(\mathbb{E})}\int_{\mathbb{E}} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \le \left(\frac{1}{\mu(\mathbb{E})}\int_{\mathbb{E}} |f(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}.$$

If we rearrange terms, we obtain that

$$\|f\|_{L^q(\mathbb{E})} \le \mu(\mathbb{E})^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(\mathbb{E})}$$

and the thesis is an immediate consequence of 3.1.12. □

Remark 3.1.14. The statement of the proposition 3.1.13 is generally false if $\mu(\mathbb{E})$ is not finite. In $\mathbb{R}^d$ with the Lebesgue measure, we can consider the function $f_\alpha : \mathbb{R}^d \to \mathbb{R}$ such that

$$f_\alpha(x) := \frac{1}{|x|^\alpha} \mathbb{1}_{\mathcal{B}(0;1)}.$$

Let p be a real number in $[1; +\infty)$; it's easy to see that f_α is in $L^p(\mathbb{R}^d)$ if and only if $\alpha p < d$; if we consider the function $g_\alpha : \mathbb{R}^d \to \mathbb{R}$ such that

$$g_\alpha(x) := \frac{1}{|x|^\alpha} \mathbb{1}_{\mathcal{B}(0;1)^c},$$

it's easy to see that g_α is in $L^p(\mathbb{R}^d)$ if and only if $\alpha p > d$. Then, we obtain all the counterexamples requested.

3.1.2 Completeness

The aim of this subsection is to show that $L^P(\mathbb{E})$ is a complete metric space with respect to the distance induced by the norm. In other words, $(L^p(\mathbb{E}; \|\cdot\|_{L^p(\mathbb{E})})$ is a Banach space. However, we have to show some preliminary lemmas.

Lemma 3.1.15 (Chebyshev's inequality).
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $g : \mathbb{E} \to [0; +\infty]$ be a measurable function. Let δ be any positive real number. If we define

$$E_\delta := \{x \mid g(x) \ge \delta\},$$

then, the following inequality holds true:

$$\mu(E_\delta) \le \frac{1}{\delta}\int_{\mathbb{E}} g(x) d\mu(x).$$

Proof. We notice that for all x in $\mathbb{E}$ it holds that $\delta \mathbb{1}_{E_\delta}(x) \leq g(x)$. If we integrate, we obtain that

$$\delta \mu(E_\delta) = \int_{\mathbb{E}} \delta \mathbb{1}_{E_\delta}(x) d\mu(x) \leq \int_{\mathbb{E}} g(x) d\mu(x).$$

□

Lemma 3.1.16 (Borel-Cantelli' lemma)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ *be a measurable space with a measure* μ*. Let* $\{A_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable sets. We define*

$$A := \{x \in \mathbb{E} \mid x \in A_n \text{ for infinite indices } n\}.$$

Let us assume that

$$\sum_{n\in\mathbb{N}} \mu(A_n) < +\infty.$$

Then, A *is a measurable set and* $\mu(A) = 0$*.*

Proof. Let m be any positive integer. We define

$$E_m := \bigcup_{n\geq m} A_n.$$

We notice that

$$A = \bigcap_{m\in\mathbb{N}} \left(\bigcup_{n\geq m} A_n \right) = \bigcap_{m\in\mathbb{N}} E_m.$$

So, A is measurable. It's easy to see that

$$\mu(A) \leq \inf_{m\in\mathbb{N}} \mu(E_m) \leq \inf_{m\in\mathbb{N}} \left(\sum_{n\geq m} \mu(A_n) \right);$$

in conclusion, we notice that the right hand side is 0 because we are assuming that

$$\sum_{n\in\mathbb{N}} \mu(A_n) < +\infty.$$

□

Lemma 3.1.17. *Let* $(\mathbb{X}; d)$ *be a metric space; let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{X}$*. Let us assume that*

$$\sum_{n\in\mathbb{N}} d(x_n; x_{n+1}) < +\infty.$$

Then $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy's sequence with respect to the distance in* $\mathbb{X}$*.*

Proof. Let ε be any positive real number; let n_0 in $\mathbb{N}$ such that

$$\sum_{n\geq n_0} d(x_n; x_{n+1}) \leq \varepsilon.$$

If n, m are positive integer such that $m > n > n_0$, we can use the triangular inequality and we obtain that:

$$d(x_m; x_n) \leq \sum_{k=n}^{m-1} d(x_k; x_{k+1}) \leq \sum_{k\geq n_0} d(x_k; x_{k+1}) \leq \varepsilon.$$

□

Lemma 3.1.18. *Let $(\mathbb{X}; d)$ be a metric space; let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy's sequence in $\mathbb{X}$. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be any infinitesimal sequence of positive real numbers. There exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that for all k in $\mathbb{N}$ it holds that $d(x_{n_k}; x_{n_{k+1}}) \leq \delta_k$.*

Proof. The sequence $\{n_k\}_{k\in\mathbb{N}}$ can be defined by recursion. Thanks to our hypothesis, there exists a natural number n_0 such that for all n greater than or equal to n_0 it holds that $d(x_{n_0}; x_n) \leq \delta_0$. Hence, we have defined x_{n_0}. Let k any positive integer. Let us assume that $\{x_{n_0}; \dots ; x_{n_k}\}$ have already been defined. There exists an integer n_{k+1} greater than n_k such that for all n greater than or equal to n_{k+1} it holds that $d(x_{n_{k+1}}; x_n) \leq \delta_{k+1}$. Hence, we have defined $x_{n_{k+1}}$. □

Theorem 3.1.19. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let p be in $[1; +\infty]$. Then $\left(L^p(\mathbb{E}); \|\cdot\|_{L^p(\mathbb{E})}\right)$ is a complete metric space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy's sequence in $L^p(\mathbb{E})$. Let us denote $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ the correspondent sequence of functions in $\mathcal{D}^p$. If we show that there exists a measurable function $\tilde{f} : \mathbb{E} \to \overline{\mathbb{R}}$ such that $\tilde{f}$ is in $\mathcal{D}^p$ and

$$\lim_{n\to+\infty} \left\| \tilde{f}_n - \tilde{f} \right\|_{L^p(\mathbb{E})} = 0$$

and we denote f the corresponding class of $\tilde{f}$ in $L^p(\mathbb{E})$, it is immediate to see that $\{f_n\}_{n\in\mathbb{N}}$ converges toward f with respect to the L^p norm in the quotient set. In other words, we can assume that $\{f_n\}_{n\in\mathbb{N}}$ is a well defined sequence of functions.

Step 1: Let us assume that p is $+\infty$. By definition of Cauchy's sequence, if ε is a positive real number, there exists a positive integer n_0 such that for all integer n, m greater than or equal to n_0 it holds that

$$\|f_n - f_m\|_{L^\infty(\mathbb{E})} < \varepsilon.$$

Be definition 3.1.1, there exists a measurable set C in $\mathbb{E}$ such that $\mu(C^c) = 0$ and for all x in C we have that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy's sequence. Hence, for all x in C we define $f(x)$ as the pointwise limit of $\{f_n(x)\}_{n\in\mathbb{N}}$; if x is in C^c, we define $f(x) := 0$. Hence, f is a well defined function between $\mathbb{E}$ and $\mathbb{R}$. Moreover, we can also assume that for all n in $\mathbb{N}$ for all x in C it holds that $|f_n(x)| \leq \|f_n\|_{L^\infty(E)}$. Since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy's sequence in $L^\infty(\mathbb{E})$, it's easy to see that there exists a positive real number M such that for all n in $\mathbb{N}$ it holds that $\|f_n\|_{L^\infty(\mathbb{E})} \leq M$. In particular, for all x in C, we have that $|f(x)| \leq M$. As f is the pointwise limit of $\{f_n\}_{n\in\mathbb{N}}$ in C and f is 0 in C^c, it is a measurable function. Hence, f in $L^\infty(\mathbb{E})$. Let ε be any positive real number; let n_0 be a positive integer such that for all integers n, m greater than or equal to n_0 it holds that

$$\|f_n - f_m\|_{L^\infty(\mathbb{E})} \leq \varepsilon.$$

By definition of C, we have that for all x in C for all integers n, m greater than or equal to n_0 it holds that $|f_n(x) - f_m(x)| \leq \varepsilon$. In particular, we can state that

$$|f_n(x) - f(x)| = \lim_{m\to+\infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

In other words, $\{f_n\}_{n\in\mathbb{N}}$ converges toward f with respect to L^∞ norm.

Step 2: Let us assume that p is a real number in $[1;+\infty)$. Thanks to lemma 3.1.18, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that for all k in $\mathbb{N}$ it holds that

$$\left\|f_{n_{k-1}} - f_{n_k}\right\|_{L^p(\mathbb{E})} \leq 4^{-k}.$$

For all k in $\mathbb{N}$ we define

$$g_k := \left|f_{n_{k+1}} - f_{n_k}\right|,$$
$$A_k := \left\{x \in \mathbb{E} \mid g_k(x) \geq 2^{-k}\right\}.$$

It's immediate to see that g_k is a measurable function between $\mathbb{E}$ and $\mathbb{R}$ and A_k is a measurable set. Thanks to Chebyshev's inequality (see 3.1.15), we obtain that

$$\mu(A_k) = \mu(\{x \in \mathbb{E} \mid g_k(x)^p \geq 2^{-kp}\}) \leq \frac{1}{2^{-kp}} \int_{\mathbb{E}} g_k(x)^p d\mu(x) \leq 2^{-kp}.$$

We define the measurable set

$$A := \{x \in \mathbb{E} \mid x \in A_k \text{ for infinite indices } k\}.$$

Thanks to Borel-Cantelli' lemma (see 3.1.16), we obtain that $\mu(A) = 0$. Hence, for all x in A^c we have that $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$ is a Cauchy's sequence (see lemma 3.1.17). For all x in A^c we define $f(x)$ the pointwise limit of $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$; for all x in A we define $f(x) = 0$. As shown in the previous step, we have that f is a well defined measurable function between $\mathbb{E}$ and $\mathbb{R}$. As described in the previous step, there exists a positive real number M such that for all n in $\mathbb{N}$ it holds that $\|f_n\|_{L^p(\mathbb{E})} \leq M$; thanks to Fatou's lemma, we have that

$$\int_{\mathbb{E}} |f(x)|^p \, d\mu(x) \leq \liminf_{k\to+\infty} \int_{\mathbb{E}} |f_{n_k}(x)|^p \, d\mu(x) \leq M^p.$$

In particular, f is in $L^p(\mathbb{E})$. Let k be any positive integer. Thanks to Fatou's lemma, we have that

$$\begin{aligned}
\|f - f_{n_k}\|^p_{L^p(\mathbb{E})} &= \int_{\mathbb{E}} |f_{n_k}(x) - f(x)|^p \, d\mu(x) \\
&= \int_{\mathbb{E}} \left(\lim_{h\to+\infty} |f_{n_k}(x) - f_{n_h}(x)|^p\right) d\mu(x) \\
&\leq \liminf_{h\to+\infty} \int_{\mathbb{E}} |f_{n_k}(x) - f_{n_h}(x)|^p \, d\mu(x) \\
&\leq 4^{-kp}.
\end{aligned}$$

Hence, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges toward f with respect to L^p norm. To conclude, we notice that the whole sequence $\{f_n\}_{n\in\mathbb{N}}$ converges toward f with respect to L^p norm, because it is a Cauchy's sequence. □

Example 3.1.20 (ℓ^p space).
Let μ be the measure in $\mathbb{N}$ that counts point, i.e. if A is any subset in $\mathbb{N}$ we define $\mu(A)$ as it's cardinality. We notice that $(\mathbb{N}; \mathbb{P}(\mathbb{N}); \mu)$ is a measurable space with a measure μ. We define $l^p := L^p(\mathbb{N})$. In other words, if p is a real number, l^p is the collection of the function $f : \mathbb{N} \to \mathbb{R}$ such that

$$\int_{\mathbb{N}} |f(n)|^p \, d\mu(n) < +\infty.$$

More explicitly, we notice that

$$\int_{\mathbb{N}} |f(n)|^p \, d\mu(n) = \sum_{n\in\mathbb{N}} |f(n)|^p \, .$$

Obviously, if we define $\|\cdot\|_{\ell^p} : \ell^p \to [0;+\infty)$ such that

$$\|f\|_{\ell^p} := \left(\sum_{n\in\mathbb{N}} |f(n)|^p \right)^{\frac{1}{p}},$$

then $\|\cdot\|_{\ell^p}$ is a norm and $(\ell^p; \|\cdot\|_{\ell^p})$ is a a complete metric space. We claim that ℓ^p is separable. If n is any positive integer, we define $f_n : \mathbb{N} \to \mathbb{R}$ such that $f_n(n) = 1$ and $f_n(k) = 0$ for all natural number $k \neq n$. It's easy to see that $\mathrm{Span}_{\mathbb{Q}} \{f_n \mid n \in \mathbb{N}\}$ is a countable dense subset in ℓ^p.

If p is equal to $+\infty$, we define $\ell^\infty := L^\infty(\mathbb{N})$. In other words, ℓ^∞ is the collection of the bounded-valued function between $\mathbb{N}$ and $\mathbb{R}$. We notice that

$$\|f\|_{L^\infty(\mathbb{N})} = \sup_{n\in\mathbb{N}} \{|f(n)|\}$$

and it is always denoted as $\|f\|_{\ell^\infty(\mathbb{N})}$. Obviously, $(\ell^\infty; \|\cdot\|_{\ell^\infty})$ is a complete metric space.

3.1.3 Convergence of measurable functions

Definition 3.1.21 (Convergence in measure)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $\{f_n\}_{n\in N}$ be a sequence of measurable functions between $\mathbb{E}$ and $\mathbb{R}$; let f be a measurable function between $\mathbb{E}$ and $\mathbb{R}$. Let us assume that if ε is any positive real number it holds that

$$\lim_{n\to+\infty} \mu\left(\{x \in \mathbb{E} \mid \ |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

We say that $\{f_n\}_{n\in\mathbb{N}}$ converges toward f in measure.

Proposition 3.1.22. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let us assume that $\mu(\mathbb{E}) < +\infty$. Let $\{f_n\}_{n\in N}$ be a sequence of measurable functions between $\mathbb{E}$ and $\mathbb{R}$; let f be a measurable function between $\mathbb{E}$ and $\mathbb{R}$. If $\{f_n(x)\}_{n\in\mathbb{N}}$ converges toward $f(x)$ for almost every x in $\mathbb{E}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges toward f in measure.*

Proof. Let ε be any positive real number. For all n in $\mathbb{N}$ we define

$$B_n^\varepsilon := \{x \in \mathbb{E} \mid \exists m \geq n : \ |f_m(x) - f(x)| \geq \varepsilon\}.$$

If we define

$$B^\varepsilon := \bigcap_{n\in\mathbb{N}} B_n^\varepsilon,$$

$$B := \left\{ x \in \mathbb{E} \ \middle| \ \liminf_{n\to+\infty} |f_n(x) - f(x)| > 0 \right\},$$

we notice that B^ε is completely contained in B. We know that $\mu(B) = 0$; since $\mathbb{E}$ is a finite measure space, we have that

$$\lim_{n\to+\infty} \mu(B_n^\varepsilon) = \mu(B^\varepsilon) \leq \mu(B) = 0.$$

If we notice that for all n in $\mathbb{N}$ it holds that

$$\{x \in \mathbb{E} \mid |f_n(x) - f(x)| \geq \varepsilon\} \subseteq B_n^\varepsilon,$$

the conclusion follows immediately. □

Example 3.1.23. In 3.1.22, it is necessary to assume that $\mathbb{E}$ is a finite measure space. In fact, if $\mathbb{E} = \mathbb{R}$ with the Lebesgue measure, we notice that $\{\mathbb{1}_{[n;+\infty)}\}_{n\in\mathbb{N}}$ is a sequence of measurable functions whose pointwise limit is the null function; however, it does not converge toward 0 in measure.

Proposition 3.1.24. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $\{f_n\}_{n\in N}$ be a sequence of measurable functions between $\mathbb{E}$ and $\mathbb{R}$; let f be a measurable function between $\mathbb{E}$ and $\mathbb{R}$. Let p be a real number in $[1; +\infty)$. If $\{f_n\}_{n\in\mathbb{N}}$ converges toward f with respect to L^p norm, then $\{f_n\}_{n\in\mathbb{N}}$ converges toward f in measure.*

Proof. Let us fix a positive real number ε. Thanks to Chebyshev's inequality (see 3.1.15), we have that

$$\lim_{n\to+\infty} \mu\left(x \in \mathbb{E} \mid |f_n(x) - f(x)| \geq \varepsilon\right) \leq \lim_{n\to+\infty} \frac{1}{\varepsilon^p} \int_{\mathbb{E}} |f_n(x) - f(x)|^p \, d\mu(x) = 0.$$

□

Proposition 3.1.25. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let $\{f_n\}_{n\in N}$ be a sequence of measurable functions between $\mathbb{E}$ and $\mathbb{R}$; let f be a measurable function between $\mathbb{E}$ and $\mathbb{R}$. Let us assume that $\{f_n\}_{n\in\mathbb{N}}$ converges toward f in measure. There exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$ converges toward $f(x)$ for almost every x in $\mathbb{E}$.*

Proof. For all ε in $(0; +\infty)$ for all n in $\mathbb{N}$ we define

$$A_n^\varepsilon := \{x \in \mathbb{E} \mid |f_n(x) - f(x)| \geq \varepsilon\}.$$

Under our hypothesis, it holds that

$$\lim_{n\to+\infty} \mu(A_n^\varepsilon) = 0.$$

If we apply a diagonal procedure, we can find a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that for all k in $\mathbb{N}$ it holds that $\mu\left(A_{n_k}^{\frac{1}{k}}\right) \leq 2^{-k}$. Thanks to lemma Borel-Cantelli' lemma (see 3.1.16), if we define

$$A := \left\{x \in \mathbb{E} \mid x \in A_{n_k}^{\frac{1}{k}} \text{ for infinite indices } k\right\},$$

we obtain that $\mu(A) = 0$. In particular, for all x in $\mathbb{E} \setminus A$, there exists k_0 in $\mathbb{N}$ such that for all k greater than k_0 it holds that $|f_{n_k}(x) - f(x)| \leq \frac{1}{k}$. □

Example 3.1.26. For all n in $\mathbb{N}$ we define

$$\delta_n := \sum_{i=1}^{n} \frac{1}{i}.$$

In $[0;1)$ with the Lebesgue measure, for all n in $\mathbb{N}$ we define the measurable set

$$A_n := \{x \in [0;1) \mid \exists k \in \mathbb{Z} : x + k \in (\delta_n; \delta_{n+1}]\}.$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of measurable functions such that

$$f_n := \mathbb{1}_{A_n}.$$

It's easy to see that $\{f_n\}_{n\in\mathbb{N}}$ converges toward the zero function in measure. Since the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is not bounded and the sequence $\{\delta_{n+1} - \delta_n\}_{n\in\mathbb{N}}$ is infinitesimal, for all x in $[0;1)$ for all n in $\mathbb{N}$ there exists an integer m greater than n such that x is in A_m. This is enough to conclude that the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not converge pointwise toward zero function in any point.

Theorem 3.1.27 (Severini-Egorov's theorem)**.**
Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let us assume that $\mu(\mathbb{E})$ is finite. Let $\{f_n\}_{n\in N}$ be a sequence of measurable functions between $\mathbb{E}$ and $\mathbb{R}$; let f be a measurable function between $\mathbb{E}$ and $\mathbb{R}$. Let us suppose that $\{f_n(x)\}_{n\in\mathbb{N}}$ converges toward $f(x)$ for almost every x in $\mathbb{E}$. For all positive real number ε, there exists a measurable set E_ε such that $\mu(E_\varepsilon) \leq \varepsilon$ and $\{f_n\}_{n\in\mathbb{N}}$ converge toward f uniformly in $\mathbb{E} \setminus E_\varepsilon$.

Proof. Let n, k be positive integers. We define

$$B_{n;k} := \left\{x \in \mathbb{E} \mid \exists m \geq n : |f_m(x) - f(x)| \geq \frac{1}{k}\right\}.$$

Since $\mathbb{E}$ is a finite measure space, if we define

$$B^k := \left\{x \in \mathbb{E} \mid \ |f_n(x) - f(x)| \geq \frac{1}{k} \text{ for infinite indices } n\right\} = \bigcap_{n\in\mathbb{N}} B_{n;k},$$

it holds that

$$\lim_{n\to+\infty} \mu(B_{n;k}) = \mu(B^k).$$

Since $\{f_n(x)\}_{n\in\mathbb{N}}$ converges toward $f(x)$ for almost every x in $\mathbb{E}$, we obtain that $\mu(B^k) = 0$.

Let us fix a positive real number ε. We can state that there exists a subsequence $\{B_{n_k;k}\}_{k\in\mathbb{N}}$ such that for k in $\mathbb{N}$ it holds that

$$\mu(B_{n_k;k}) \leq \frac{\varepsilon}{2^{k+1}}.$$

We define

$$E_\varepsilon := \bigcup_{k\in\mathbb{N}} B_{n_k;k}$$

and we obtain that

$$\mu(E_\varepsilon) \leq \sum_{k\in\mathbb{N}} \mu(B_{n_k;k}) \leq \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^{k+1}} = \varepsilon.$$

If x is in E_ε^c, for all k in $\mathbb{N}$ for all integer m greater than n_k it holds that

$$|f_m(x) - f(x)| \leq \frac{1}{k}.$$

Hence, $\{f_n\}_{n\in\mathbb{N}}$ converges toward f uniformly in E_e^c. □

Example 3.1.28. In theorem 3.1.27, it is necessary to assume that $\mu(\mathbb{E})$ is finite. Otherwise, we can consider $\mathbb{E} = \mathbb{R}$ with the Lebesgue measure and $\left\{\mathbb{1}_{[n;+\infty)}\right\}_{n\in N}$. It's easy to see that the sequence converge toward zero function pointwise; unfortunately, if B is any finite measure subset, the convergence is not uniform in B^c.

3.1.4 Density in L^p

The aim of this subsection is to show the density of some collections of functions with respect to L^p norm.

Proposition 3.1.29. *Let $(\mathbb{E};\mathscr{E};\mu)$ be a measurable space with a measure μ. Let p be in $[1;+\infty]$. Let us define*

$$\mathcal{B}^p(\mathbb{E}) := \{f \in L^p(\mathbb{E}) \mid \exists M \in \mathbb{R} : \; |f(x)| \leq M \text{ for almost every } x \in \mathbb{E}\}.$$

Then $\mathcal{B}^p(\mathbb{E})$ is dense in $L^p(\mathbb{E})$ with respect to the L^p norm.

Proof. If p equals $+\infty$ the conclusion is trivial. Let us assume that p is a real number in $[1;+\infty)$. Let f be any function in $L^p(\mathbb{E})$. Let N be any positive integer. We define the function $T_N f : \mathbb{E} \to \mathbb{R}$ such that

$$T_N f(x) := (f(x) \wedge n) \vee (-n).$$

We say that $T_N f$ is the truncation of f between $-N$ and N. We notice that $\{T_N f\}_{N\in\mathbb{N}}$ is a sequence in $\mathcal{B}^p(\mathbb{E})$ that converges pointwise toward f almost everywhere and $|T_N f(x) - f(x)|^p \leq 2\,|f(x)|^p$ for almost every x in $\mathbb{E}$. Since $|f|^p$ is a suitable domination in $L^1(\mathbb{E})$, the dominated convergence theorem implies that

$$\lim_{N\to+\infty} \int_{\mathbb{E}} (T_N f(x) - f(x))^p \, d\mu(x) = 0.$$

□

Proposition 3.1.30. *Let $\mathscr{E}$ be a σ-algebra in $\mathbb{R}^d$ that contains the open balls; let μ be any measure over $(\mathbb{R}^d;\mathscr{E})$. Let p be a real number in $[1;+\infty)$. We define*

$$\mathcal{A}^p(\mathbb{R}^d) := \left\{f \in L^p(\mathbb{R}^d) \mid \exists M \in \mathbb{R} : \; f(x) = 0 \text{ for almost every } x \in \mathcal{B}(0;M)^c\right\}.$$

Then $\mathcal{A}^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ with respect to the L^p norm.

Proof. Let f be any function in $L^p(\mathbb{E})$. For all positive integer n, we define $f_n : \mathbb{R}^d \to \mathbb{R}$ such that

$$f_n(x) := f(x)\mathbb{1}_{\mathcal{B}(0;n)}.$$

It's immediate to see that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is in $\mathcal{A}^p(\mathbb{R}^d)$, it converges pointwise toward f for almost every x in $\mathbb{R}^d$ and $|f_n(x) - f(x)|^p \leq 2\,|f(x)|^p$ for almost every x in $\mathbb{R}^d$ for all n in $\mathbb{N}$. Since $|f|^p$ is in $L^1(\mathbb{R}^d)$, the dominated convergence theorem implies that

$$\lim_{N\to+\infty} \int_{\mathbb{E}} (f_n(x) - f(x))^p \, d\mu(x) = 0.$$

□

Example 3.1.31. In proposition 3.1.30 it is necessary to assume that p is a real number. If p equals $+\infty$, in $(\mathbb{R};\mathscr{M};\mathscr{L})$ we notice that the function $\mathbb{1}_{\mathbb{R}}$ is in $L^\infty(\mathbb{R})$, but it cannot be approximated by a sequence of function supported in bounded subsets with respect to L^∞ norm.

Proposition 3.1.32. *Let $\mathscr{E}$ be a σ-algebra in $\mathbb{R}^d$ that contains the open balls; let μ be any measure over $(\mathbb{R}^d; \mathscr{E})$. Let p be a real number in $[1; +\infty)$. We define*

$$\mathcal{C}^p(\mathbb{R}^d) \coloneqq \mathcal{A}^p(\mathbb{R}^d) \cap \mathcal{B}^p(\mathbb{R}^d).$$

Then $\mathcal{C}^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ with respect to the L^p norm.

Proof. Let f be a function in $L^p(\mathbb{R}^d)$. Let n be any positive integer. We denote as $T_n f$ the truncation of f between $-n$ and n (see 3.1.29). We define $f_n : \mathbb{R}^d \to \mathbb{R}$ such that

$$f_n(x) = T_n f(x) \mathbb{1}_{\mathcal{B}(0;n)}.$$

If we slightly modify the procedure described in 3.1.30, we obtain that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable function in $\mathcal{C}^p(\mathbb{R}^d)$ that converges toward f with respect to L^p norm. □

Remark 3.1.33. Let $(\mathbb{X}; \tau)$ be a topological space. Let A, B subsets in $\mathbb{X}$. Let us assume that B is dense in $\mathbb{X}$ and $\overline{A}$ contains B. It's immediate to see that $\overline{A} = \mathbb{X}$; in other words, A is dense in $\mathbb{X}$.

Proposition 3.1.34. *Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. Let p be in $[1; +\infty]$. We define $\mathscr{S}(\mathbb{E})$ as the set of the step functions (see 2.2.6). Then $\mathscr{S}(\mathbb{E})$ is dense in $L^p(\mathbb{E})$ with respect to L^p norm.*

Proof. Let f be any function in $L^p(\mathbb{E})$. We claim that there exists a sequence of step functions that converges toward f with respect to L^p norm. If we join 3.1.29 and 3.1.33, we can assume that f is a bounded-valued function. Let n be a positive integer. If k is a positive integer, we define

$$A_{k;n} \coloneqq f^{-1}\left(\left[\frac{k}{n}; \frac{k+1}{n}\right)\right);$$

if k is a negative integer, we define

$$A_{k;n} \coloneqq f^{-1}\left(\left[\frac{k-1}{n}; \frac{k}{n}\right)\right).$$

Then, we define $f_n : \mathbb{E} \to \mathbb{R}$ such that

$$f_n(x) \coloneqq \sum_{k\in\mathbb{Z}^*} \frac{k}{n} \mathbb{1}_{A_{k;n}}(x).$$

Since f is a bounded-valued function, f_n is defined by a finite sum; hence, $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{S}(\mathbb{E})$. It's easy to see that for all n in $\mathbb{N}$ for all x in $\mathbb{E}$ it holds that

$$|f_n(x) - f(x)| \le \frac{1}{n}.$$

In other words, we obtain that

$$\lim_{n\to+\infty} \|f_n - f\|_{L^\infty(\mathbb{E})} = 0;$$

so, if p is $+\infty$, the thesis is proved. Let us assume that p is a real number in $[1; +\infty)$. We have already shown that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise toward f for almost every x in $\mathbb{E}$.

It's easy to see that for all n in $\mathbb{N}$ for all x in $\mathbb{E}$ it holds that $|f_n(x) - f(x)|^p \leq 2\,|f(x)|^p$; since $|f|^p$ is a suitable domination in $L^1(\mathbb{E})$, the dominated convergence theorem implies that

$$\lim_{N\to+\infty} \int_{\mathbb{E}} (f_n(x) - f(x))^p \, d\mu(x) = 0.$$

□

Proposition 3.1.35. *Let p be a real number in $[1; +\infty)$. Then, the set $C_c(\mathbb{R}^d)$ of the continuous functions supported in a bounded subsets is dense in $L^p(\mathbb{R}^d)$ with respect to the L^p norm.*

Proof. Let f be a function in $L^p(\mathbb{R}^d)$. If we join 3.1.32, 3.1.34 and 3.1.33, we can assume that f is the indicator function of a bounded measurable subset E. Let us fix a positive real number ε. Thanks to 2.1.22, there exist an open set A and a close set C such that

$$C \subseteq E \subseteq A$$

and $\mathscr{L}^d(A \setminus C) \leq \varepsilon$. We recall that if X is any subset in $\mathbb{R}^d$, the function $dist(\cdot; X) : \mathbb{R}^d \to \mathbb{R}$ such that

$$dist(y; X) := \inf\{|y - z| \mid z \in \mathbb{X}\}$$

has the following properties:

- it is well defined and continuous;
- $dist(y; X) = 0$ if and only if y is in $\overline{X}$.

If we define $g_\varepsilon : \mathbb{R}^d \to [0; 1]$ such that

$$g_\varepsilon(x) := \frac{dist(x; A^c)}{dist(x; A^c) + dist(x; C)},$$

we notice that g_ε is well defined and continuous and it holds that

$$\int_{\mathbb{R}^d} |g_\varepsilon(x) - \mathbb{1}_E(x)|^p \, dx = \int_{A\setminus C} |g_\varepsilon(x) - \mathbb{1}_E(x)|^p \, dx \leq \mathscr{L}^d(A \setminus C) \leq \varepsilon.$$

□

Proposition 3.1.36. *Let p be a real number in $[1; +\infty)$. Then, $L^p(\mathbb{R}^d)$ is a separable metric space.*

Proof. We will complete the proof under the further assumption that d equals 1. Let us define $\mathcal{B}$ as the collection of indicator functions of intervals whose boundary value is rational. $\mathcal{B}$ is countable. We define

$$\mathcal{D} := \mathrm{Span}_{\mathbb{Q}}(\mathcal{B}).$$

It's immediate to see that $\mathcal{D}$ is countable. We claim that it is dense in $L^p(\mathbb{R}^d)$ with respect to L^p norm. Let f be any function in $L^p(\mathbb{R}^d)$. We have to show that if ε is any real number, there exists a function f_ε in $\mathcal{D}$ such that $\|f - f_\varepsilon\|_{L^p(\mathbb{R}^d)} \leq \varepsilon$. If we join 3.1.32 and 3.1.34, we can assume that $f = c\mathbb{1}_E$, where c is a real number and E is a measurable bounded set in $\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can also assume that c is a equal to 1. Let ε be any positive real number. Thanks to 2.1.22, there exists an open set A such that E is contained in A and $\mathscr{L}^1(A \setminus E) \le \frac{\varepsilon}{2}$. In other words, we can assume that A is an open bounded set. We know that there exists a finite of countable collection of pairwise disjoint intervals that cover A; without loss of generality, we assume that

$$A \coloneqq \bigcup_{n \in \mathbb{N}} (a_n; b_n)$$

and

$$\sum_{n \in \mathbb{N}} (b_n - a_n) < +\infty.$$

In other words, we can assume that the sum is finite, namely

$$A = \bigcup_{n=1}^{N} (a_n; b_n).$$

For all n in $\mathbb{N}$ there exist c_n and d_n in $\mathbb{Q}$ such that $(c_n; d_n)$ is contained in $(a_n; b_n)$ and

$$\mathscr{L}^1((a_n; b_n) \setminus (c_n; d_n)) \le \frac{\varepsilon}{4^n}.$$

It's immediate to see that

$$f_\varepsilon \coloneqq \sum_{i=1}^{N} \mathbb{1}_{(c_n; d_n)}$$

is in $\mathcal{D}$ and

$$\|f_\varepsilon - f\|_{L^p(\mathbb{R})} \le \varepsilon.$$

□

Remark 3.1.37. The statement of proposition 3.1.36 is false if p equals $+\infty$. In fact, $L^\infty(\mathbb{R})$ is not separable. It's enough to consider the collection of functions $\{\mathbb{1}_{[x;+\infty)}\}_{x \in \mathbb{R}}$: it is more than countable and if $x_1 \ne x_2$ it holds that

$$\left\|\mathbb{1}_{[x_1;+\infty} - \mathbb{1}_{[x_2;+\infty)}\right\|_{L^\infty(\mathbb{R})} = 1.$$

Hence, $L^\infty(\mathbb{R})$ is a metric space that admits a more than countable subset $\{f_x \mid x \in \mathbb{R}\}$ such that the open balls

$$\mathscr{B} \coloneqq \left\{ \mathcal{B}\left(f_x; \frac{1}{2}\right) \;\middle|\; x \in \mathbb{R} \right\}$$

are pairwise disjoint. This is enough to conclude that $L^\infty(\mathbb{R})$ is not separable. By contradiction, let us assume that $L^\infty(\mathbb{R})$ is a separable metric space, namely there exists a countable dense subset $\mathcal{D}$. Thanks to the choice axiom, there exists a function $\psi : \mathscr{B} \to \mathbb{D}$ such that for all $\mathcal{B}\left(f_x; \frac{1}{2}\right)$ in $\mathscr{B}$ it holds that $\psi\left(\mathcal{B}\left(f_x; \frac{1}{2}\right)\right)$ is in $\mathcal{B}\left(f_x; \frac{1}{2}\right) \cap \mathcal{D}$. Since the open balls are pairwise disjoint, the function ψ is injective. As $\mathscr{B}$ is more than countable and $\mathcal{D}$ is countable, the absurd follows immediately.

Theorem 3.1.38 (Lusin's theorem)**.**
Let E be a measurable subset in $\mathbb{R}^d$. Let $f : E \to \mathbb{R}$ be a measurable function. Let us assume that $\mathscr{L}^d(E)$ is finite. For all positive real number ε there exists a closed set E_ε with the following properties:

- *E_ε is completely contained in E;*

- $\mathscr{L}^d(E \setminus E_\varepsilon) \leq \varepsilon$;
- $f|_{E_\varepsilon}$ *is continuous.*

Proof. **Step 1**: Let us assume that f is a bounded-valued function. Then f is in $L^1(E)$; thanks to 3.1.35, there exists a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ that converges toward f with respect to L^1 norm. Up to further subsequences, not relabelled, we can assume that the convergence is pointwise for almost every x in E. Let us fix a positive real number ε. Thanks to Severini-Egorov' theorem (see 3.1.27), there exists a measurable subset E'_ε such that

- E'_ε is completely contained in E;
- $\mathscr{L}^d(E \setminus E'_\varepsilon) \leq \frac{\varepsilon}{2}$;
- $\{f_n\}_{n\in\mathbb{N}}$ converges toward f uniformly in E'_ε.

If E'_ε is closed, the theorem is proved; otherwise, thanks to 2.1.22, there exists a closed set E_ε contained in E'_ε such that $\mathscr{L}^d(E'_\varepsilon \setminus E_\varepsilon) \leq \frac{\varepsilon}{2}$. Hence, we obtain that $\mathscr{L}^d(E \setminus E_\varepsilon) \leq \varepsilon$.

Step 2: Let f be any real-valued measurable function. It is enough to show that for all positive real number ε there exists a measurable set A_ε in E such that $\mathscr{L}(E \setminus A_\varepsilon) \leq \varepsilon$ and f is bounded in A_ε. For all positive integer n we define

$$A_{\frac{1}{n}} \coloneqq f^{-1}([-n; n]).$$

Since f is a real-valued function, $\{A_{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is a decreasing sequence of measurable sets and it holds that

$$\bigcap_{n\in\mathbb{N}} A_{\frac{1}{n}} = \emptyset.$$

Since E is a finite measure set, this is enough to conclude that

$$\lim_{n\to+\infty} \mathscr{L}(A_{\frac{1}{n}}) = 0.$$

□

3.2 Convolution

Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions, the function $f * g$ is called convolution between f and g. We will give reasonable hypothesis to make sure that the function is well defined and we will study the its main properties. However, the convolution plays a fundamental role to show the density of smooth functions in L^p, for p in $[0; +\infty)$. Qualitatively, $f * g(x)$ is a kind of weighted average of the value of f in the neighborhood of x with respect to the value of g. Hence, it is not surprising that the convolution makes functions more regular.

3.2.1 Definition and main properties

Definition 3.2.1 (Convolution for nonnegative functions)**.**
Let $f, g : \mathbb{R}^d \to [0; +\infty]$ be measurable functions; let x be any point in $\mathbb{R}^d$. We define

$$(f * g)(x) \coloneqq \int_{R^d} f(x - y) g(y) dy.$$

Remark 3.2.2. Since the integral make sense for nonnegative functions, we notice that definition 3.2.1 is always well posed, i. e. $f * g(x)$ is in $[0; +\infty]$.

Remark 3.2.3. Let $f, g, h : \mathbb{R}^d \to [0; +\infty]$ be measurable functions. The convolution has the following properties:

- (commutative) if we define $z := x - y$ and we change variables, the following identities hold true:

$$(f * g)(x) = \int_{R^d} f(x-y)g(y)dy = \int_{\mathbb{R}^d} f(z)g(x-z)dz = (g * f)(x).$$

- (associative) Since the functions are nonnegative, we can use Fubini's theorem and we can switch the order of integration; if we define $t := y - z$ we obtain that

$$\begin{aligned}
[(f * g) * h](x) &= \int_{\mathbb{R}^d} (f * g)(y)h(x-y)dy \\
&= \int_{\mathbb{R}^d} \left(\int_{R^d} f(z)g(y-z)dz \right) h(x-y)dy \\
&= \int_{R^d} f(z) \left(\int_{\mathbb{R}^d} g(y-z)h(x-y)dy \right) dz \\
&= \int_{R^d} f(z) \left(\int_{\mathbb{R}^d} g(t)h(x-z-t)dt \right) dz \\
&= \int_{R^d} f(z)(g * h)(x-z)dz \\
&= [f * (g * h)](x).
\end{aligned}$$

- (linearity in both factors) If λ is any positive real number, it's easy to see that

$$(\lambda f + g) * h \equiv \lambda(f * h) + (g * h);$$

$$f * (\lambda g + h) \equiv \lambda(f * g) + (f * h).$$

- (measurability) The function $f * g : \mathbb{R}^d \to [0; +\infty]$ is measurable, as follows from Fubini's theorem.

Definition 3.2.4 (Convolution for variable sign functions)**.**
Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions; let x be any point in $\mathbb{R}^d$. We define

$$(f * g)(x) := \int_{R^d} f(x-y)g(y)dy.$$

Remark 3.2.5. Unlike the case of definition 3.2.1, the integral the defines convolution may not have sense. The purpose of next lemmas is to find reasonable hypothesis to make sure that definition 3.2.4 is well posed.

Proposition 3.2.6. *Let $f, g : \mathbb{R}^d \to [0; +\infty]$ be measurable functions. If f and g are in $L^1(\mathbb{R}^d)$, then $\|f * g\|_{L^1(\mathbb{R}^d)} = \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$.*

Proof. Thanks to Fubini's theorem, we can switch the order of integration; so, the following identities hold true:

$$\begin{aligned}\|f * g\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} f * g(x)dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)g(y)dy\right) dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)dx\right) g(y)dy \\ &= \|g\|_{L^1(\mathbb{R}^d)} \|\tau_y(f)\|_{L^1(\mathbb{R}^d)} \\ &= \|g\|_{L^1(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)} .\end{aligned}$$

□

Corollary 3.2.7. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ measurable functions in $L^1(\mathbb{R}^d)$; then definition 3.2.4 is well posed; in other words, the integral make sense and $f * g(x)$ is finite for almost every x in $\mathbb{R}^d$. In particular, the function $f * g$ is well defined for almost every x in $\mathbb{R}^d$ and it is measurable. Moreover, the following inequality holds true:*

$$\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} .$$

Proof. Thanks to proposition 3.2.6, $|f| * |g|$ is in $L^1(\mathbb{R}^d)$. In particular $|f| * |g|(x)$ is finite for almost every x in $\mathbb{R}^d$. Let x be any point in $\mathbb{R}^d$. If we define

$$[\zeta_x(f, g)](y) \coloneqq f(x-y)g(y),$$

we have already shown that $\zeta_x(f, g)$ is in $L^1(\mathbb{R}^d)$ for almost every x in $\mathbb{R}^d$; hence $f * g$ make sense and it is finite for almost every x in $\mathbb{R}^d$. Thanks to Fubini's theorem, $f * g$ is measurable. Since we can switch the order of integration, we can slightly modify the proof of proposition 3.2.6 to show that the following inequalities hold true:

$$\begin{aligned}\|f * g\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |f * g(x)|\, dx \\ &= \int_{\mathbb{R}^d} \left|\int f(x-y)g(y)dy\right| dx \\ &\leq \int_{R^d} \left(\int_{\mathbb{R}^d} |f(x-y)|\, |g(y)|\, dy\right) dx \\ &= \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} .\end{aligned}$$

□

Proposition 3.2.8. *Let $f, g : \mathbb{R}^d \to [0; +\infty]$ be measurable functions; let p be in $[1; +\infty]$. If f is in $L^p(\mathbb{R}^d)$ and g is in $L^1(\mathbb{R}^d)$, then $\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$.*

Proof. If p equals $+\infty$ the conclusion is immediate. If p is equal to 1, the thesis has already been proved in proposition 3.2.6. Let us assume that p is in $(1; +\infty)$. Let x be any point in $\mathbb{R}^d$; thanks to Hölder's inequality, we obtain that

$$\begin{aligned}f * g(x) &= \int_{\mathbb{R}^d} f(x-y)g(y)dy \\ &= \int_{\mathbb{R}^d} f(x-y)g(y)^{\frac{1}{p}} g(y)^{1-\frac{1}{p}} dy \\ &\leq \left(\int_{\mathbb{R}^d} f(x-y)^p g(y)dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} g(y)dy\right)^{1-\frac{1}{p}} .\end{aligned}$$

Integrating in x and switching the order of integrals, we find that

$$\begin{aligned}\|f * g\|_{L^p(\mathbb{R}^d)}^p &= \int_{\mathbb{R}^d} [f * g(x)]^p dx \\ &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)^p g(y) dy \right) \|g\|_{L^1(\mathbb{R}^d)}^{p-1} dx \\ &= \|g\|_{L^1(\mathbb{R}^d)}^{p-1} \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} f(x-y)^p dx \right) dy \\ &= \|g\|_{L^1(\mathbb{R}^d)}^{p-1} \|g\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}^p \\ &= \|g\|_{L^1(\mathbb{R}^d)}^p \|f\|_{L^p(\mathbb{R}^d)}^p .\end{aligned}$$

□

Corollary 3.2.9. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ measurable functions; let p be in $[1; +\infty]$. If we assume that f is in $L^p(\mathbb{R}^d)$ and g is in $L^1(\mathbb{R}^d)$, then definition 3.2.4 is well posed, namely the integral make sense and $f * g(x)$ is finite for almost every x in $\mathbb{R}^d$. In particular, the function $f * g$ is well defined and measurable. Moreover, the following inequality holds true:*

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} .$$

Proof. Thanks to proposition 3.2.8, we can slightly modify the proof of corollary 3.2.7; then thesis follows immediately. □

Proposition 3.2.10. *Let $f, g : \mathbb{R}^d \to [0; +\infty]$ measurable functions; let p be in $[1; +\infty]$; let p^* be the conjugate index of p as in 3.1.7. If f is in $L^p(\mathbb{R}^d)$ and g is in $L^{p^*}(\mathbb{R}^d)$, then $\|f * g\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p^*}(\mathbb{R}^d)}$.*

Proof. If p equals $+\infty$, thesis follows from 3.2.8. Let us assume that p is in $[1; +\infty)$. Let x be any point in $\mathbb{R}^d$; thanks to Hölder's inequality, we obtain that

$$\begin{aligned}f * g(x) &= \int_{\mathbb{R}^d} f(x-y) g(y) dx \\ &\leq \left(\int_{\mathbb{R}^d} f(x-y)^p dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} g(y)^{p^*} dy \right)^{\frac{1}{p^*}} \\ &= \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p^*}(\mathbb{R}^d)} .\end{aligned}$$

□

Corollary 3.2.11. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions; let p be in $[1; +\infty]$; let p^* be the conjugate index of p as in 3.1.7. If we assume that f is in $L^p(\mathbb{R}^d)$ and g is in $L^{p^*}(\mathbb{R}^d)$, then definition 3.2.4 is well posed, namely the integral make sense and $f * g(x)$ is finite for almost every x in $\mathbb{R}^d$. In particular, the functions $f * g$ is measurable. Moreover, the following inequality holds true:*

$$\|f * g\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p^*}(\mathbb{R}^d)} .$$

Proof. Thanks to proposition 3.2.10, we can slightly modify the proof of corollary 3.2.7; then thesis follows immediately. □

Lemma 3.2.12. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Let p be in $[1; +\infty)$. Let us assume that f is in $L^p(\mathbb{R}^d)$. Let h be any vector in $\mathbb{R}^d$; let $\tau_h f$ be as in 1.0.1 Then $\{\tau_h f\}_{h \in \mathbb{R}}$ converges toward f with respect to L^p norm, when h approaches to 0.*

Proof. **Step 1**: Let us assume that f is in $C_c^0(\mathbb{R}^d)$, namely there exists a positive real number R such that f is supported in $\mathcal{B}(0;R)$. Hence, if $|h|$ is lower than 1, the following inequalities hold true:

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathcal{B}(0;R+1)} |f(x-h) - f(x)|^p \, dx.$$

We claim that the right hand side converges toward 0 as h approaches to 0. We have that:

- since f is continuous, for all x in $\mathbb{R}^d$ it holds that

$$\lim_{h \to 0} f(x-h) - f(x) = 0;$$

- for all x in $\mathbb{R}^d$ for all h in $\mathcal{B}(0;1)$ it holds that

$$|f(x-h) - f(x)| \leq 2\,\|f\|_{L^\infty(\mathbb{R}^d)}\, \mathbb{1}_{\mathcal{B}(0;R+1)}(x).$$

  Since f is supported in a compact subset, the left hand side is function in $L^p(\mathbb{R}^d)$.

Then the conclusion follows from dominated convergence theorem.

Step 2: Let f be any function in $L^p(\mathbb{R}^d)$. Let ε be any positive real number; since p is real, there exists f in $C_c^0(\mathbb{R}^d)$ such that

$$\|f - g\|_{L^p(\mathbb{R}^d)} \leq \frac{\varepsilon}{3}.$$

Let h be any vector in $\mathbb{R}^d$. We notice that

$$\|\tau_h f - \tau_h g\|_{L^p(\mathbb{R}^d)} = \|f - g\|_{L^p(\mathbb{R}^d)} \leq \frac{\varepsilon}{3}.$$

As shown in the first step, there exists a positive real number h_0 with the following property: if h is in $\mathcal{B}(0;h_0)$, it holds that

$$\|g - \tau_h g\|_{L^P(\mathbb{R}^d)} \leq \frac{\varepsilon}{3}.$$

Hence, if h is in $\mathcal{B}(0;1)$, the following inequalities hold true:

$$\begin{aligned} \|f - \tau_h f\|_{L^p(\mathbb{R}^d)} &\leq \|f - g\|_{L^p(\mathbb{R}^d)} + \|\tau_h f - \tau_h g\|_{L^p(\mathbb{R}^d)} + \|g - \tau_h g\|_{L^p(\mathbb{R}^d)} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then, thesis follows immediately. $\square$

Proposition 3.2.13. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions; let p be in $[1;+\infty]$; let p^* be the conjugate index of p as in 3.1.7. If we assume that f is in $L^p(\mathbb{R}^d)$ and g is in $L^{p^*}(\mathbb{R}^d)$, then $f * g$ is a uniformly continuous function.*

Proof. Let x and h be any vectors in $\mathbb{R}^d$. It is not restrictive to assume that p is in $[1;+\infty)$. Thanks to Hölder's inequality, we obtain that

$$\begin{aligned}
|f * g(x-h) - f * g(x)| &\leq \int_{\mathbb{R}^d} |f(x-h-y) - f(x-y)|\, |g(y)|\, dy \\
&\leq \left(\int_{\mathbb{R}^d} |f(x-h-y) - f(x-y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |g(y)|^{p^*} \, dy \right)^{\frac{1}{p^*}} \\
&\leq \left(\int_{\mathbb{R}^d} |f(t-h) - f(t)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |g(y)|^{p^*} \, dy \right)^{\frac{1}{p^*}} \\
&= \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p^*}(\mathbb{R}^d)} .
\end{aligned}$$

Thanks to lemma 3.2.12, $\|\tau_h f - f\|_{L^p(\mathbb{R}^d)}$ is a continuity module that does not depend of x. Hence, thesis follows immediately. □

3.2.2 Regularization and approximation

Proposition 3.2.14. *Let p be in $[1;+\infty]$. Let us define p^* the conjugate index of p as in 3.1.7. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions. Let g be in $L^p(\mathbb{R}^d)$. Let i be an integer in $\{1;\ldots;d\}$. Let us assume that there exists $\frac{\partial f}{\partial x_i}$ and it is continuous; let us suppose that f and $\frac{\partial f}{\partial x_i}$ are in $L^{p^*}(\mathbb{R}^d)$. Then, there exists $\frac{\partial (f*g)}{\partial x_i}$ and it equals $\frac{\partial f}{\partial x_i} * g$.*

Proof. Thanks to proposition 3.2.11 and 3.2.13, $f * g$ and $\frac{\partial f}{\partial x_i} * g$ are well defined and uniformly continuous. Without loss of generality, we can assume that i equals 1. Let $(x_1;\ldots;x_d)$ be any vector in $\mathbb{R}^d$; if we denote $y \coloneqq (x_2;\ldots;x_d)$, we have to show that for all $(x;y)$ in $\mathbb{R}\times\mathbb{R}^{d-1}$, the following identity holds true (assuming that x is nonnegative):

$$f * g(x;y) - f * g(0;y) = \int_0^x \frac{\partial f}{\partial x_1} * g(t,y) dt.$$

Then, thesis follows immediately from the fundamental theorem of calculus. Let x be any positive real number; let $(t;w;y)$ be any vector in $[0;x]\times\mathbb{R}\times\mathbb{R}^{d-1}$. We claim that the function $\zeta : [0;x]\times\mathbb{R}\times\mathbb{R}^{d-1}$ defined as

$$\zeta(t,w,y) \coloneqq \frac{\partial f}{\partial x_1}(t-w;y) g(w;y)$$

is in $L^1([0;x]\times\mathbb{R}^d)$. In deed, we can use Hölder's inequality and Fubini's theorem for nonnegative function and we obtain that

$$\begin{aligned}
\int_{[0;x]\times\mathbb{R}^d} |\zeta(t,w,y)|\, dt dw dy &= \int_0^x \left(\int_{\mathbb{R}^d} \left| \frac{\partial f}{\partial x_1} \right| (t-w;y) g(w;y) dw dy \right) dt \\
&\leq \int_0^x \left\| \frac{\partial f}{\partial x_1} \right\|_{L^{p^*}(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)} dt \\
&= |x| \left\| \frac{\partial f}{\partial x_1} \right\|_{L^{p^*}(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)} .
\end{aligned}$$

Hence, for all $(x;y)$ in $[0;+\infty)\times\mathbb{R}^{d-1}$ the following identities hold true:

$$\begin{aligned}\int_0^x \left(\frac{\partial f}{\partial x_1} * g\right)(t;y)dt &= \int_0^x \left(\int_{\mathbb{R}^d} \frac{\partial f}{\partial x_1}(t-w;y)g(w;y)dwdy\right)dt \\ &= \int_{\mathbb{R}^d}\left(\int_0^x g(w;y)\frac{\partial f}{\partial x_1}(t-w;y)dt\right)dwdy \\ &= \int_{\mathbb{R}^d} g(w;y)\left(\int_0^x \frac{\partial f}{\partial x_1}(t-w;y)dt\right)dydw \\ &= \int_{R^d} g(w;y)[f(x-w;y)-f(0-w;y)]dwdy \\ &= f*g(x;y) - f*g(0;y).\end{aligned}$$

In the last identity we used the fact that $f*g$ is finite for every $(x;y)$ in $\mathbb{R}\times\mathbb{R}^{d-1}$, then we can split the integral. □

Corollary 3.2.15. *Let p be in $[1;+\infty]$. Let p^* be the conjugate index of p as in 3.1.7. Let $f,g:\mathbb{R}^d\to\mathbb{R}$ be measurable functions. Let g be in $L^p(\mathbb{R}^d)$. Let $(i_1;\ldots;i_n)$ be in $\{1;\ldots;d\}^n$. Let us assume that there exists $\frac{\partial^n f}{\partial x_{i_1}\ldots\partial x_{i_n}}$ and it is continuous; let us suppose that f and $\frac{\partial^k f}{\partial x_{i_1}\ldots\partial x_{i_k}}$ are in $L^{p^*}(\mathbb{R}^d)$ for all integer k in $\{1;\ldots;n\}$. Then, there exists $\frac{\partial^n (f*g)}{\partial x_{i_1}\ldots\partial x_{i_n}}$ and it equals $\frac{\partial^n f}{\partial x_{i_1}\ldots\partial x_{i_n}}*g$. In particular, if f is in $C^\infty(\mathbb{R}^d)$ then $f*g$ is in $C^\infty(\mathbb{R}^d)$.*

Proof. If is an immediate consequence of proposition 3.2.14. □

Theorem 3.2.16. *Let p be a real number in $[1;+\infty)$. Let $f,g:\mathbb{R}^d\to\mathbb{R}$ be measurable functions. Let g be in $L^1(\mathbb{R}^d)$; let f be in $L^p(\mathbb{R}^d)$. For all positive real number δ we define $\sigma_\delta g$ as in 1.0.2. Then $\sigma_\delta g * f$ is in $L^p(\mathbb{R}^d)$ for all δ greater that 0 and $\{\sigma_\delta g * f\}_{\delta>0}$ converges toward $f\,\|g\|_{L^1(\mathbb{R}^d)}$ with respect to L^p norm as δ approaches 0.*

Proof. First of all, we notice that if $\|g\|_{L^1(\mathbb{R}^d)}=0$, the conclusion is trivial. Hence, we can suppose that $\|g\|_{L^1(\mathbb{R}^d)}\neq 0$. We notice that it is not restrictive to assume that $\|g\|_{L^1(\mathbb{R}^d)}$ is equal to 1. In fact, it's easy to see that $\|g\|_{L^1(\mathbb{R}^d)} = \|\sigma_\delta g\|_{L^1(\mathbb{R}^d)}$ for all positive real number δ. Let us consider the family of functions

$$\left\{\frac{\sigma_\delta g}{\|\sigma_\delta g\|_{L^1(\mathbb{R}^d)}} * f\right\}_{\delta>0}.$$

If we show that it converges toward f with respect to L^p norm, then thesis in the most general case follows immediately.

Let x be any vectors in $\mathbb{R}^d$ and δ any positive real number. For all y in $\mathbb{R}^d$, we denote $t:=\frac{y}{\delta}$; hence, dt equals $\frac{dy}{y^d}$. Having said that, we obtain that

$$f*\sigma_\delta g(x) = \int_{\mathbb{R}^d} f(x-y)\frac{1}{\delta^d}g\left(\frac{y}{\delta}\right)dy = \int_{\mathbb{R}^d} f(x-\delta t)g(t)dt,$$

$$f(x) = \int_{\mathbb{R}^d} f(x)g(t)dt.$$

Thanks to Hölder's inequality, we have that

$$\begin{aligned}
|f * \sigma_\delta g(x) - f(x)| &= \left| \int_{\mathbb{R}^d} [f(x-\delta t) - f(x)] g(t) dt \right| \\
&\leq \int_{\mathbb{R}^d} |f(x-\delta t) - f(x)| \, |g(t)|^{\frac{1}{p}} \, |g(t)|^{1-\frac{1}{p}} \, dt \\
&\leq \left(\int_{\mathbb{R}^d} |f(x-\delta t) - f(x)|^p \, |g(t)| \, dt \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |g(t)| \, dt \right)^{1-\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} |f(x-\delta t) - f(x)|^p \, |g(t)| \, dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Since we can use Fubini's theorem with nonnegative functions, the following inequalities hold true:

$$\begin{aligned}
\|f * \sigma_\delta g - f\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |f * \sigma_\delta g(x) - f(x)|^p \, dx \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-\delta t) - f(x)|^p \, |g(t)| \, dt \right) dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-\delta t) - f(x)|^p \, |g(t)| \, dx \right) dt \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-\delta t) - f(x)|^p \, dx \right) |g(t)| \, dt \\
&= \int_{\mathbb{R}^d} |g(t)| \, \|\tau_{\delta t} f - f\|^p_{L^p(\mathbb{R}^d)} \, dt.
\end{aligned}$$

We claim that the last integral converges toward 0 as δ approaches 0. Since g is in $L^1(\mathbb{R}^d)$, $g(t)$ is finite for almost every t in $\mathbb{R}^d$; then, for almost every t in $\mathbb{R}^d$ lemma 3.2.12 implies that

$$\lim_{\delta \to 0} |g(t)| \, \|\tau_{\delta t} f - f\|^p_{L^p(\mathbb{R}^d)} = 0.$$

We notice that for almost every t in $\mathbb{R}^d$ for all δ greater than 0 it holds that

$$|g(t)| \, \|\tau_{\delta t} f - f\|^p_{L^p(\mathbb{R}^d)} \leq |g(t)| \, (2 \, \|f\|_{L^p(\mathbb{R}^d)})^p,$$

that is a suitable domination in $L^1(\mathbb{R}^d)$. Hence, the thesis is an immediate consequence of dominated convergence theorem. □

Corollary 3.2.17. *Let p be any real number in $[1;+\infty)$. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable functions. Let g be in $C_c^\infty(\mathbb{R}^d)$; let f be in $L^p(\mathbb{R}^d)$. For all positive real number δ we define $\sigma_\delta g$ as in 1.0.2. Then $\sigma_\delta g * f$ is in $L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ for all δ greater that 0 and $\{\sigma_\delta g * f\}_{\delta > 0}$ converges toward $f \, \|g\|_{L^1(\mathbb{R}^d)}$ with respect to L^p norm as δ approaches to 0. In particular, if p is any real number in $[1;+\infty)$, then $C^\infty(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d)$ are dense in $L^p(\mathbb{R}^d)$.*

Proof. Let p^* be the conjugate index of p. Since g is in $C_c^\infty(\mathbb{R}^d)$, g and all its partial derivatives are in $L^{p^*}(\mathbb{R}^d)$; as for the density of $C^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$, it is an immediate consequence of proposition 3.2.15 and theorem 3.2.16.

As for the density of $C_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$, we notice that if g is supported in $\mathcal{B}(0;R)$ and f is supported in $\mathcal{B}(0;M)$, then $f * g$ is supported in $\mathcal{B}(0;M+R)$. Hence, the thesis is a consequence of proposition 3.1.30, remark 3.1.33 and theorem 3.2.16. □

Remark 3.2.18. We notice that corollary 3.2.17 is false if p equals $+\infty$. In fact, if u is any function in $L^\infty(\mathbb{R}^d)$ and $\{g_n\}_{n\in\mathbb{N}}$ is any sequence of functions in $C(\mathbb{R}^d)$ that converges toward u with respect to L^∞ norm, there exists another function $\tilde{u}$ such that:

- $\tilde{u}$ and u coincide almost everywhere;
- $\{g_n\}_{n\in\mathbb{N}}$ converges toward $\tilde{u}$ uniformly in $\mathbb{R}^d$.

Hence, $\tilde{u}$ is a continuous function; in particular u coincides almost everywhere with a continuous function. The absurd follows taking $u := \mathbb{1}_{[0;+\infty)\times\mathbb{R}^{d-1}}$.

On the pointwise convergence of the convolution

Proposition 3.2.19. *Let $\{g_n\}_{n\in\mathbb{N}}$ be any sequence of functions in $L^1(\mathbb{R}^d)$ such that*

- *if n is any natural number, then $\|g_n\|_{L^1(\mathbb{R}^d)} = 1$;*
- *if δ is any positive real number, then it holds that*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d\setminus\mathcal{B}(0;\delta)} |g_n(x)|\, dx = 0.$$

If $f : \mathbb{R}^d \to \mathbb{R}$ is any function in $L^\infty(\mathbb{R}^d)$ and x_0 is any point in $\mathbb{R}^d$ such that f is continuous in x_0, then the following conclusion holds true:

$$\lim_{(h;n)\to(x_0;+\infty)} f * g_n(x_0+h) = f(x_0).$$

Proof. Let ε be any positive real number. Let δ be a positive real number such that if x, y are in $\mathcal{B}(0;\delta)$, then it holds that

$$|f(x_0+h-y) - f(x_0)| \le \frac{\varepsilon}{2}.$$

Let n_0 be a natural number such that if n is any integer greater that or equal to n_0, then

$$\int_{\mathbb{R}^d\setminus\mathcal{B}(0;\delta)} |g_n(y)|\, dy \le \frac{\varepsilon}{4\,\|f\|_{L^\infty(\mathbb{R}^d)}}.$$

Then, if n is any integer greater that or equal to n_0 and h is any point in $\mathcal{B}(0;\delta)$, the following inequalities hold true:

$$\begin{aligned}
|f * g_n(x_0+h) - f(x_0)| &= \left| \int_{\mathbb{R}^d} f(x_0+h-y) g_n(y) dy - \int_{\mathbb{R}^d} f(x_0) g_n(y) dy \right| \\
&\le \int_{\mathbb{R}^d} |g_n(y)|\, |f(x_0+h-y) - f(x_0)|\, dy \\
&= \int_{\mathcal{B}(0;\delta)} |g_n(y)|\, |f(x_0+h-y) - f(x_0)|\, dy \\
&\qquad + \int_{\mathbb{R}^d\setminus\mathcal{B}(0;\delta)} |g_n(y)|\, |f(x_0+h-y) - f(x_0)|\, dy \\
&\le \frac{\varepsilon}{2} \int_{\mathcal{B}(0;\delta)} |g_n(y)|\, dy + 2\,\|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d\setminus\mathcal{B}(0;\delta)} |g_n(y)|\, dy \\
&\le \frac{\varepsilon}{2} \|g_n\|_{L^1(\mathbb{R}^d)} + \frac{\varepsilon}{4\,\|f\|_{L^\infty(\mathbb{R}^d)}} 2\,\|f\|_{L^\infty(\mathbb{R}^d)} \\
&= \varepsilon.
\end{aligned}$$

So, the thesis follows immediately. ☐

Corollary 3.2.20. *Let $g : \mathbb{R}^d \to \mathbb{R}$ be any function in $L^1(\mathbb{R}^d)$ such that $\|g\|_{L^1(\mathbb{R}^d)} = 1$. Let δ be any positive real number; let us consider $\sigma_\delta g$ as in 1.0.2. If f is any function in $L^\infty(\mathbb{R}^d)$ and x_0 is any point in $\mathbb{R}^d$ such that f is continuous in x_0, then the following conclusion holds true:*

$$\lim_{(h;\delta)\to(0;0)} f * \sigma_\delta g(x_0 + h) = f(x_0).$$

Proof. Thanks to proposition 3.2.19, it is enough to show that if θ is any positive real number, then

$$\lim_{\delta\to 0} \int_{\mathbb{R}^d \setminus \mathcal{B}(0;\theta)} \sigma_\delta g(y) dy = 0.$$

Hence, let us fix θ in $(0; +\infty)$. If we denote $t := \frac{x}{\delta}$, then dt equals $\frac{dx}{\delta^d}$. So, it holds that

$$\begin{aligned} \lim_{\delta\to 0} \int_{\mathbb{R}^d \setminus \mathcal{B}(0;\theta)} \sigma_\delta g(y) dy &= \lim_{\delta\to 0} \int_{\mathbb{R}^d \setminus \mathcal{B}(0;\theta)} \frac{1}{\delta^d} g\left(\frac{y}{\delta}\right) dy \\ &= \lim_{\delta\to 0} \int_{\mathbb{R}^d \setminus \mathcal{B}\left(0;\frac{\theta}{\delta}\right)} g(t) dt \\ &= 0, \end{aligned}$$

as follows immediately from the dominated convergence theorem, because $|g|$ is a suitable domination in $L^1(\mathbb{R}^d)$. □

Chapter 4

Hilbert space

In this chapter, we will assume that any vector space is over the fields $\mathbb{F}$ that denotes either $\mathbb{C}$ of $\mathbb{R}$.

4.1 Inner product space

4.1.1 Definition and main properties

Definition 4.1.1 (Inner product space)**.**
Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$; let $<\cdot,\cdot>: \mathbb{V}\times\mathbb{V}\to\mathbb{F}$ be a map with the following properties:

- conjugate symmetry, i. e. for all x, y in $\mathbb{V}$ it holds that
$$< x, y >= \overline{< x, y >};$$
- linearity in the first argument, i.e. for all x, y, z in $\mathbb{V}$ for all α in $\mathbb{F}$ it holds that
$$< x + \alpha y, z >=< x, z > +\alpha < y, z >;$$
- positive-definiteness, i. e. for all x in $\mathbb{V}$ it holds that $< x, x >$ is real and nonnegative; moreover, $< x, x >$ equals 0 if and only if x is 0.

We say that $(\mathbb{V}; <\cdot,\cdot>)$ is an inner space product.

Definition 4.1.2 (Norm associated to the inner product)**.**
Let $(\mathbb{V}; <\cdot,\cdot>)$ be an inner space product. For all x in $\mathbb{V}$ we define
$$\|x\| := \sqrt{< x, x >}.$$
We say that $\|\cdot\| : \mathbb{V}\to[0;+\infty)$ is the norm associated to the inner product.

Lemma 4.1.3 (Cauchy-Schwarz inequality)**.**
Let $(\mathbb{V}; <\cdot,\cdot>)$ be an inner space product. For all x, y in $\mathbb{V}$ the following inequality holds true:
$$|< x, y >| \leq \sqrt{< x, x >}\sqrt{< y, y >}.$$

Proof. Let x, y be vectors in $\mathbb{V}$. If y is 0, the thesis follows immediately. Hence, we can assume that $\|y\| \neq 0$. Let us denote

$$\lambda := \frac{< x, y >}{\|y\|^2}.$$

Since $< \cdot, \cdot >$ is an inner product, the following inequality hold true:

$$\begin{aligned} 0 &\leq \|x - \lambda y\|^2 \\ &=< x, x > -\lambda < y, x > -\overline{\lambda < y, x >} + \lambda\overline{\lambda} < y, y > \\ &= \|x\|^2 - \frac{|< x, y >|^2}{\|y\|^2}. \end{aligned}$$

So, the thesis follows immediately. □

Lemma 4.1.4. *The function $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ defined in 4.1.2 is actually a norm.*

Proof. The only non-obvious property is the triangular inequality: as a matter of fact, it follows immediately from Cauchy-Schwarz inequality. □

Remark 4.1.5. Since any inner product space is a normed vector space, it has the structure of metric space and topological space.

Remark 4.1.6. Let $(\mathbb{V}; < \cdot, \cdot >)$ be an inner product space. The following identities holds for all x, y in $\mathbb{V}$:

- parallelogram identity:

$$\|x + y\|^2 + \|x - y\|^2 = \|x\|^2 + \|y\|^2;$$

- restitution formula for complex spaces:

$$< x, y >= \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right);$$

- restitution formula for real spaces:

$$< x, y >= \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right).$$

Lemma 4.1.7. *Let $(\mathbb{V}; < \cdot, \cdot >)$ be any inner product space; if we consider the normed space $\mathbb{V} \times \mathbb{V}$ with the product topology, the function $< \cdot, \cdot >: \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ is continuous.*

Proof. We claim that the functions $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, $- : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ are continuous. As a matter of facts, these statements follow immediately from the triangular inequality.

Having said that, we notice that the thesis is a consequence of restitution formula and the fact that the composition of continuous function is a continuous function. □

4.1.2 Orthonormal set and Hamel's basis

Definition 4.1.8 (Orthogonal).
Let $(\mathbb{V}; < \cdot, \cdot >)$ be an inner product space. Let X be a subset in $\mathbb{V}$. We define the orthogonal of X as the set

$$X^{\perp} := \{v \in \mathbb{V} \mid \forall x \in X \;\; < x, v >= 0\}.$$

Definition 4.1.9 (Orthonormal set).
Let $(\mathbb{V}; < \cdot, \cdot >)$ be an inner product space. Let $\mathscr{F}$ be a set in $\mathbb{V}$ such that for all e_1, e_2 in $\mathscr{F}$, with $e_1 \neq e_2$ it holds that $\|e_1\| = 1$ and $< e_1, e_2 >= 0$. We say that $\mathscr{F}$ is an orthonormal set.

If $\mathscr{F}$ is an orthonormal set, we say that it is maximal if for all orthonormal set $\mathscr{F}'$ such that $\mathscr{F} \subseteq \mathscr{F}'$ it holds that $\mathscr{F} = \mathscr{F}'$.

If $\mathscr{F}$ is an orthonormal set, we say that it is complete if $\mathrm{Span}(\mathscr{F})$ is dense in $\mathbb{V}$.

Remark 4.1.10. If $(\mathbb{V}; < \cdot, \cdot >)$ is an inner product space and $\{v_1; \ldots; v_n\}$ is any finite set in $\mathbb{V}$ on linearly independent vectors, we can use the Gram–Schmidt process to orthonormalise them.

If we assume Zorn's lemma, we can easily show the existence of maximal orthonormal set. If $\mathscr{F}$ is any orthonormal set, we define

$$\mathscr{G} := \{\mathscr{F}' \mid \mathscr{F} \subseteq \mathscr{F}', \mathscr{F}' \text{ is an orthonormal set in } \mathbb{V}\}$$

partially ordered with the relation of inclusion. We notice that any totally ordered set $\mathscr{H}$ has an upper bound, i.e.

$$\overline{\mathscr{H}} := \bigcup_{\mathscr{K} \in \mathscr{H}} \mathscr{K}.$$

Thanks to Zorn's lemma, we can state that there exists at least a maximal element $\mathscr{J}$. It's easy to see that $\mathscr{J}$ is a maximal orthonormal set of $\mathbb{V}$.

More precisely, we have shown that any orthonormal set in $\mathbb{V}$ can be extended to a maximal orthonormal set.

Remark 4.1.11. Let $(\mathbb{V}; < \cdot, \cdot >)$ be an inner product space. If $\mathscr{F}$ is a complete orthonormal set, then it is maximal. Let us assume that $\mathscr{F}$ is not maximal; as shown in 4.1.10, there exists a vector x in $\mathbb{V} \setminus \mathscr{F}$ with the following properties:

- $\|x\| = 1$;
- for all e in $\mathscr{F}$ it holds that $< x, e >= 0$.

Hence, we have that x is in $\mathrm{Span}(\mathscr{F})^{\perp}$. If $\{v_n\}_{n \in \mathbb{N}}$ is any sequence in $\mathrm{Span}(\mathscr{F})$, then for all n in $\mathbb{N}$ it holds that $< x, v_n >= 0$. If we join 4.1.7 and the fact that $x \neq 0$, the sequence $\{v_n\}_{n \in \mathbb{N}}$ cannot converge toward x with respect to the norm of $\mathbb{V}$.

Definition 4.1.12 (Hamel's basis).
Let $\mathbb{V}$ be any vector space; let $\mathscr{F}$ be a subset of $\mathbb{V}$. We say that $\mathscr{F}$ is an algebraic basis (or Hamel's basis) if it holds that

$$\mathbb{V} = \mathrm{Span}(\mathscr{F}).$$

Remark 4.1.13. If we slightly modify the procedure shown in 4.1.10, we can easily prove that any set of linearly independent vectors can be completed to an algebraic basis of $\mathbb{V}$.

4.2 Hilbert space

4.2.1 Definition and main properties

Definition 4.2.1 (Hilbert space)**.**
A Hilbert space $\mathbb{H}$ is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

Example 4.2.2. Let $\mathbb{H}$ be any n-dimensional real (complex) Hilbert space; there exists an isometry ψ between $\mathbb{V}$ and $\mathbb{R}^n$ (or $\mathbb{C}^n$). Let us denote $\{v_1; \dots; v_n\}$ an orthonormal basis of $\mathbb{H}$ and $\{e_1; \dots; e_n\}$ the canonical basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$); if i is any integer in $\{1; \dots; n\}$, we can define $\psi(v_i) = e_i$.

Example 4.2.3. Let $(\mathbb{E}; \mathscr{E}; \mu)$ be a measurable space with a measure μ. We claim that $(L^2(\mathbb{E}); \|\cdot\|_{L^2(\mathbb{E})})$ is an Hilbert space. We have already shown that it is complete. We notice that the function $< \cdot, \cdot >: L^2(\mathbb{E}) \times L^2(\mathbb{E}) \to \mathbb{R}$ such that

$$< f, g > := \int_{\mathbb{E}} f(x) g(x) d\mu(x)$$

is well defined and it is the inner product that induces the L^2 norm. As for $L^2_{\mathbb{C}}(\mathbb{E})$, we notice that the function $< \cdot, \cdot >: L^2_{\mathbb{C}}(\mathbb{E}) \times L^2_{\mathbb{C}}(\mathbb{E}) \to \mathbb{C}$ such that

$$< f, g > := \int_{\mathbb{E}} f(x) \overline{g(x)} d\mu(x)$$

is well defined and it is the inner product that induces the L^2 norm.

In particular, ℓ^2 is an Hilbert space with the inner product $< \cdot, \cdot >: \ell^2 \times \ell^2 \to \mathbb{R}$ such that for all $x := \{x_n\}_{n \in \mathbb{N}}, y := \{y_n\}_{n \in \mathbb{N}}$ in ℓ^2 it holds that

$$< x, y > := \sum_{n \in \mathbb{N}} x_n y_n.$$

As for $\ell^2_{\mathbb{C}}$ the function $< \cdot, \cdot >: \ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}} \to \mathbb{C}$ such that for all $x := \{x_n\}_{n \in \mathbb{N}}, y := \{y_n\}_{n \in \mathbb{N}}$ in ℓ^2 it holds that

$$< x, y > := \sum_{n \in \mathbb{N}} x_n \overline{y_n}$$

is the inner product that induces the $\ell^2_{\mathbb{C}}$ norm.

4.2.2 Hilbert's basis theorem

Lemma 4.2.4. *Let $\mathbb{H}$ be an Hilbert space. Let $\mathscr{F}$ be any countable orthonormal set in $\mathbb{H}$, namely*

$$\mathscr{F} := \{e_n \mid n \in \mathbb{N}\}.$$

Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be any sequence in $\mathbb{C}$. For all n in $\mathbb{N}$, we define

$$S_n := \sum_{i=1}^{n} \alpha_i e_i,$$

$$\tilde{S}_n := \sum_{i=1}^{n} |\alpha_i|^2 .$$

Then $\{S_n\}_{n\in\mathbb{N}}$ converges toward x_0 with respect to the norm of $\mathbb{H}$ if and only if $\{\tilde{S}_n\}_{n\in\mathbb{N}}$ converges in $\mathbb{R}$. If we denote

$$x_0 \coloneqq \sum_{n\in\mathbb{N}} \alpha_n e_n,$$

the following conclusions hold true:

- *if i is any natural number, then $< x_0, e_i >= \alpha_i$;*
- $\|x_0\|^2 = \sum_{n\in\mathbb{N}} |\alpha_i|^2$.

Proof. Since $\mathscr{F}$ is an orthonormal set, we notice that for all n, m in $\mathbb{N}$ ($n > m$) it holds that

$$\|S_n - S_m\|^2 = \left\| \sum_{i=m+1}^{n} \alpha_i e_i \right\|^2 = \sum_{i=m+1}^{n} |\alpha_1|^2 = \left| \tilde{S}_n - \tilde{S}_m \right|.$$

We have that there exists x_0 in $\mathbb{H}$ such that $\{S_n\}_{n\in\mathbb{N}}$ converges toward x_0 with respect to the norm of $\mathbb{H}$ if and only if $\{S_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the norm of $\mathbb{H}$. Hence, $\{S_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $\{\tilde{S}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, that is equivalent to assume that it is convergent.

As for the second part of the statement, since the inner product is a continuous function (see 4.1.7), if i is any integer, we can state that

$$< x_0, e_i >= \lim_{n\to+\infty} < \sum_{j=1}^{n} \alpha_j e_j, e_i >= \alpha_i.$$

Thanks to the continuity of the norm, we can state that

$$\|x_0\|^2 = \lim_{n\to+\infty} \|S_n\|^2 = \lim_{n\to+\infty} \sum_{i=1}^{n} |\alpha_i|^2 = \sum_{n\in\mathbb{N}} |\alpha_n|^2.$$

□

Theorem 4.2.5 (Hilbert's basis theorem).
Let $\mathbb{H}$ be an Hilbert space; let $\mathscr{F}$ be countable orthonormal set in $\mathbb{H}$, namely

$$\mathscr{F} \coloneqq \{e_n \mid n \in \mathbb{N}\}.$$

Let x be any vector in $\mathbb{H}$; for all n in $\mathbb{N}$ we define

$$x_n \coloneqq < x, e_n >,$$

$$S_n(x) \coloneqq \sum_{i=1}^{n} x_i e_i.$$

Then, the following conclusions hold true:

- $\sum_{n\in\mathbb{N}} |x_n|^2 \le \|x\|^2$, *also known as Bessel's inequality;*
- *there exists x_0 in $\mathbb{H}$ such the sequence $\{S_n(x)\}_{n\in\mathbb{N}}$ converges toward x_0 with respect to the norm of $\mathbb{H}$;*

- $\|x_0\|^2 = \sum_{n\in\mathbb{N}} |x_n|^2 \leq \|x\|^2$;
- $x - x_0$ *is in* $\mathrm{Span}(\mathscr{F})^{\perp}$;
- *if* $\mathscr{F}$ *is a maximal orthonormal set, then* x *equals* x_0*. In particular, it holds that*

$$x = \sum_{n\in\mathbb{N}} < x, e_n > e_n,$$

$$\|x\|^2 = \sum_{n\in\mathbb{N}} |< x, e_n >|^2 .$$

Proof. **Step 1**: Let x be any vector in $\mathbb{H}$. For all n in $\mathbb{N}$ there exists y_n in $\mathbb{H}$ such that

$$x = y_n + \sum_{i=1}^{n} x_i e_i.$$

We notice that if j is any natural number lower than or equal to n it holds that

$$< y_n, e_j >=< x - \sum_{i=1}^{n} x_i e_i, e_j >= 0.$$

Hence, we can state that

$$\|x\|^2 = \sum_{i=1}^{n} |x_i|^2 + \|y_n\|^2 \geq \sum_{i=1}^{n} |x_i|^2 .$$

Therefore, the Bessel's inequality follows taking the supremum in n.

Step 2: As for the second, the third and the fourth statement, they follow immediately from lemma 4.2.4. Hence, let us denote

$$x_0 \coloneqq \sum_{n\in\mathbb{N}} x_n e_n.$$

Step 3: If $\mathscr{F}$ is maximal, we claim that $\mathrm{Span}(\mathscr{F}) = \{0\}$. If there exists v in $\mathrm{Span}(\mathscr{F}) \setminus \{0\}$, then we can extend $\mathbb{F}$ to an orthonormal set

$$\mathscr{F}' \coloneqq \mathbb{F} \cup \left\{ \frac{v}{\|v\|} \right\};$$

this is against the fact that $\mathscr{F}$ is maximal. Therefore, we can state that $x - x_0$ equals 0. □

Remark 4.2.6. Let $\mathbb{H}$ be an Hilbert space with a countable orthonormal maximal set $\mathscr{F}$. The following statements are immediate consequences of the theorem 4.2.5:

- $\mathscr{F}$ is complete if and only if $\mathscr{F}$ is maximal;
- $\mathbb{H}$ is separable. If $\mathbb{H}$ is a real space, it's easy to see that

$$\mathrm{Span}_{\mathbb{Q}}(\mathscr{F}) \coloneqq \{v \in \mathrm{Span}(\mathbb{F}) \mid \forall i \in \mathbb{N} \; < v, e_i >\in \mathbb{Q}\}$$

  is a dense countable set. If $\mathbb{H}$ is a complex space, it is enough to define

$$\mathrm{Span}_{\mathbb{Q}}(\mathscr{F}) \coloneqq \{v \in \mathrm{Span}(\mathbb{F}) \mid \forall i \in \mathbb{N} \; \Re < v, e_i >\in \mathbb{Q}, \; \Im < v, e_i >\in \mathbb{Q}\}.$$

- For all x in $\mathbb{H}$, we define $T(x) = \{< x, e_n >\}_{n\in\mathbb{N}}$ and we notice that $T(x)$ is in ℓ^2. In other words, the map $T : \mathbb{H} \to \ell^2$ is linear and well defined; we also know that T is injective and surjective. It is an isometry between $\mathbb{H}$ and ℓ^2. Thanks to restitution formula, it preserves the inner product; more precisely, if $\mathbb{H}$ is a real space, for all x, y in $\mathbb{H}$ the following identity (also know as Parseval's identity) holds true:

$$\begin{aligned} < x, y >_{\mathbb{H}} &= \frac{1}{4}\left(\|x+y\|^2_{\mathbb{H}} - \|x-y\|^2_{\mathbb{H}}\right) \\ &= \frac{1}{4}\left(\|T(x+y)\|^2_{\ell^2} - \|T(x-y)\|^2_{\ell^2}\right) \\ &=< T(x), T(y) >_{\ell^2} \\ &= \sum_{n\in\mathbb{N}} < x, e_n >_{\mathbb{H}} \cdot < y, e_n >_{\mathbb{H}} . \end{aligned}$$

  If $\mathbb{H}$ is complex space, the proof can be easily adapted;

- Span($\mathscr{F}$) is an Hilbert's basis that is not an Hamel's basis. Thanks to lemma 4.2.4, we can state that

$$\sum_{n\in\mathbb{N}} \frac{1}{2^n} e_n$$

  is a well defined vector in $\mathbb{H} \setminus$ Span($\mathscr{F}$).

Remark 4.2.7. As a matter of facts, if $(\mathbb{V}; < \cdot, \cdot >)$ is an inner product space and $\mathscr{F}$ is a non-countable infinite complete set, then $\mathbb{H}$ cannot be a separable space. We notice that if x, y are different point in $\mathscr{F}$, then it holds that $\|x - y\| = \sqrt{2}$. If $\mathscr{D}$ is any dense subset, thanks to choice axiom, there exists a function $\psi : \mathscr{F} \to \mathscr{D}$ such that for all x in $\mathscr{F}$ it holds that $\|\psi(x) - x\| \leq \frac{1}{2}$; this is enough to state that the function ψ is injective. So, $\mathcal{D}$ is a non-countable infinite set.

In deed, $\mathscr{F}$ (countable or non-countable) is an example of closed and bounded set that is non-compact. In fact, we have shown that it is not totally bounded.

4.2.3 A step toward duality

Lemma 4.2.8. *Let $(\mathbb{X}; d)$ be a separable metric space; let Y a subspace in $\mathbb{X}$. Then, Y is separable.*

Proof. Let $\mathscr{D}$ be a countable dense subset in $\mathbb{X}$. If we denote

$$\mathscr{D} := \{q_n \mid n \in \mathbb{N}\},$$

$$\mathscr{F} := \left\{\mathcal{B}\left(q_n; \frac{1}{m}\right) \;\middle|\; n, m \in \mathbb{N}\right\},$$

we notice that $\mathscr{F}$ is countable basis for the topology. We say that (n, m) in $\mathbb{N}^2$ are in $\mathscr{I}$ if $\mathcal{B}\left(q_n; \frac{1}{m}\right) \cap Y \neq \emptyset$. For all (n, m) in $\mathscr{I}$, we choose $p_{n;m}$ in $\mathcal{B}\left(q_n; \frac{1}{m}\right) \cap Y \neq \emptyset$. If we define

$$\mathscr{G} := \{p_{n;m} \mid (n; m) \in \mathscr{I}\},$$

we claim that $\mathscr{G}$ is a countable dense subset in Y with respect to the subspace topology. Let A be any non-empty open set in Y. By definition of subspace topology, there exists an open set V in $\mathbb{X}$ such that $A = V \cap Y$. Since there exists a subset $\mathscr{J}$ in $\mathbb{N}^2$ such that

$$V = \bigcup_{(n;m)\in\mathscr{J}} \mathcal{B}\left(q_n; \frac{1}{m}\right),$$

we can state that there exists $(n_0; m_0)$ in $\mathscr{J}$ such that

$$\mathcal{B}\left(q_{n_0}; \frac{1}{m_0}\right) \cap Y \neq \emptyset;$$

hence, $p_{n_0;m_0}$ is in $\mathscr{D} \cap A$. □

Theorem 4.2.9 (Projection on a closed vector subspace)**.**
Let $\mathbb{H}$ be a separable Hilbert space; let Y be a closed vector subspace in $\mathbb{H}$. If x is any vector in $\mathbb{H}$, there exist $\pi(x)$ in Y and $\xi(x)$ in $Y^\perp$ such that

$$x = \pi(x) + \xi(x).$$

$\pi(x)$ and $\xi(x)$ are uniquely determined. We will write

$$\mathbb{H} = Y \oplus Y^\perp.$$

Moreover, $\pi(x)$ is the unique point in Y that minimizes the function $d_x : Y \to \mathbb{R}$ such that for all y in Y it holds that $d_x(y) = \|x - y\|^2$.

Proof. **Step 1**: Thanks to lemma 4.2.8, Y is a separable metric space. Since Y is closed, it inherits from $\mathbb{H}$ the structure of Hilbert space. If we join 4.2.7 and theorem 4.2.5, we can state that there exists $\mathscr{F}$ countable Hilbert's basis of Y. If we denote

$$\mathscr{F} \coloneqq \{e_n \mid n \in \mathbb{N}\},$$

theorem 4.2.5 implies that there exists $\pi(x)$ in Y such that

$$\pi(x) \coloneqq \sum_{n\in\mathbb{N}} < x, e_n > e_n.$$

Moreover, $x - \pi(x)$ is in $\mathrm{Span}(\mathscr{F})^\perp$. Thanks to 4.1.7, we immediately notice that $x - \pi(x)$ is in $\overline{\mathrm{Span}(\mathscr{F})}^\perp$ that equals $Y^\perp$. So, we can define $\xi(x) \coloneqq x - \pi(x)$.

Step 2: If there exist y in Y and z in $Y^\perp$ such that $x = y + z$, we immediately notice that $y - \pi(x) = \xi(x) - z$ is a vector in $Y \cap Y^\perp$; hence, it equals 0. In other words, the decomposition of x is unique.

Step 3: If y is any point in Y we notice that $x - \pi(x)$ and $\pi(x) - y$ are orthogonal. Hence, the following inequalities hold true:

$$\begin{aligned} d_x(y) &= \|x - y\|^2 \\ &= \|x - \pi(x) + \pi(x) - y\|^2 \\ &= \|x - \pi(x)\|^2 + \|\pi(x) - y\|^2 \\ &\geq \|x - \pi(x)\|^2 \\ &= d_x(\pi(x)). \end{aligned}$$

More precisely, the the identity $d_x(y) = d_x(\pi(x))$ holds if and only if $y = \pi(x)$. □

Remark 4.2.10. In theorem 4.2.9 it is necessary to assume that Y is a closed subspace. If $\mathscr{F}$ is an Hilbert's basis in $\mathbb{H}$, we have shown in 4.2.6 that $\text{Span}(\mathscr{F}) \neq \mathbb{H}$; thanks to 4.1.7, we can state that

$$\text{Span}(\mathscr{F})^{\perp} = \overline{\text{Span}(\mathscr{F})}^{\perp} = \mathbb{H}^{\perp} = \{0\}.$$

Theorem 4.2.11 (Riesz's representation theorem)**.**
Let $\mathbb{H}$ be a separable Hilbert space. Let $\lambda : \mathbb{H} \to \mathbb{F}$ a continuous linear functional. There exists a unique y_λ in $\mathbb{H}$ such that for all x in $\mathbb{H}$ it holds that

$$\lambda(x) =< x, y_\lambda > .$$

Proof. Since λ is continuous, we notice that $\text{Ker}(\lambda)$ is a closed vector subspace of $\mathbb{H}$. Thanks to theorem 4.2.9, we have the following decomposition:

$$\mathbb{H} = \text{Ker}(\lambda) \oplus \text{Ker}(\lambda)^{\perp}.$$

If $\lambda(x) = 0$ for all x in $\mathbb{H}$, we can define $y_\lambda = 0$. Otherwise, it is true that $\text{Ker}(\lambda) \neq \mathbb{H}$. We claim that $\text{Ker}(\lambda)^{\perp}$ is a 1-dimensional vector space; otherwise there exists a 2-dimensional subspace X such that for all x in X it holds that $\lambda(x) \neq 0$. Let x_0 be any point in $\text{Ker}(\lambda)^{\perp}$ such that $x_0 \neq 0$. We define

$$y_\lambda := \frac{\overline{\lambda(x_0)}}{\|x_0\|^2} x_0.$$

If x is any vector in $\text{Ker}(\lambda)$, we have that

$$< x, y_\lambda >= 0 = \lambda(x).$$

If x is a vector in $\text{Ker}(\lambda)$, there exists α in $\mathbb{F}$ such that $x = \alpha x_0$. So, the following identities hold true:

$$< x, y_\lambda >=< \alpha x_0, \frac{\overline{\lambda(x_0)}}{\|x_0\|^2} x_0 >= \alpha \frac{\lambda(x_0)}{\|x_0\|^2} < x_0, x_0 >= \alpha\lambda(x_0) = \lambda(\alpha x_0) = \lambda(x).$$

Since λ is linear, we have that for all x in $\mathbb{H}$ it holds that

$$\lambda(x) =< x, y_\lambda > .$$

As for the uniqueness, if there exists y_1, y_2 in $\mathbb{H}$ such that for all x in $\mathbb{H}$ it holds that

$$< x, y_1 >= \lambda(x) =< x, y_2 >,$$

then $y_1 - y_2$ is in $\mathbb{H}^{\perp}$, that is equivalent to $y_1 - y_2 = 0$. $\square$

Remark 4.2.12. If $(\mathbb{V}; < \cdot, \cdot >)$ is a infinite dimensional inner product space, there exists a linear, non continuous functional. Let $\mathscr{F}$ be a countable orthonormal set, namely

$$\mathscr{F} := \{e_n \mid n \in \mathbb{N}\}.$$

As shown in 4.1.13, we can extend $\mathscr{F}$ to an Hamel's basis $\mathscr{G}$. For all n in $\mathbb{N}$, we define

$$\lambda(e_n) := 2^n;$$

for all e in $\mathscr{G} \setminus \mathscr{F}$, we define

$$\lambda(e) \coloneqq 0.$$

Since $\mathscr{G}$ is an Hamel's basis, we can extend λ to a linear functional over $\mathbb{H}$. We notice that it is non-continuous: we notice that $\left\{\frac{e_n}{2^n}\right\}_{n\in\mathbb{N}}$ converges toward 0 with respect to the distance in $\mathbb{H}$, but for all n in $\mathbb{N}$ it holds that

$$\lambda\left(\frac{e_n}{2^n}\right) = 1.$$

Remark 4.2.13. Thanks to 4.1.7, we notice that if λ is a non-continuous functional, it cannot be represented by the inner product as in theorem 4.2.11.

Remark 4.2.14. In theorem 4.2.11, it is necessary that $\mathbb{H}$ is an Hilbert space. Otherwise, we can consider in ℓ^2 the vector subspace

$$X \coloneqq \left\{ \{x_n\}_{n\in\mathbb{N}} \in \ell^2 \mid x_n = 0 \text{ definitively } \right\}$$

and the functional $\lambda : \ell^2 \to \mathbb{F}$ such that if $x = \{x_n\}_{n\in\mathbb{N}}$ il in ℓ^2, then

$$\lambda(x) \coloneqq \sum_{n\in\mathbb{N}} \frac{x_n}{2^n}.$$

If we denote $y \coloneqq \{2^{-n}\}$, we notice that for all x in ℓ^2 it holds that $\lambda(x) =< x, y >$; so, λ is continuous. By restriction, λ defines a continuous functional $\lambda|_X$ over X; obviously, $\lambda|_X$ cannot by represented by a vector in X.

Chapter 5

Fourier series

5.1 Complex Fourier series

5.1.1 Definition and main properties

Definition 5.1.1 (Fourier coefficient)**.**
Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let n be any integer. We define the n-Fourier coefficient as follows:

$$c_n(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Remark 5.1.2. Since f is in $L^2_{\mathbb{C}}((-\pi;\pi))$, it's immediate to see that definition 5.1.1 is well posed.

Definition 5.1.3 (Fourier partial sum)**.**
Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$; for any integer i we define $c_i(f)$ as in 5.1.1. For all n in $\mathbb{N}$ we define the n-Fourier partial sum $S_n(f) : [-\pi;\pi] \to \mathbb{C}$ such that

$$S_n f(x) := \sum_{j=-n}^{n} c_j(f)e^{ijx}.$$

Theorem 5.1.4. *Let n be any integer: we define $e_n : [-\pi;\pi] \to \mathbb{C}$ such that*

$$e_n(x) := \frac{e^{inx}}{\sqrt{2\pi}}.$$

If we denote

$$\mathscr{F} := \{e_n \mid n \in \mathbb{Z}\},$$

then $\mathscr{F}$ is an Hilbert's basis of $L^2_{\mathbb{C}}((-\pi;\pi))$.

Proof. **Step 1**: Let n be any integer. We notice that

$$< e_n, e_n >= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-inx}dx = 1.$$

Let n, m be different integers; since the function e_{n-m} is $\frac{2\pi}{|n-m|}$-periodic, it holds that

$$< e_n, e_m >= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix(n-m)}dx = 0.$$

This is enough to state that $\mathscr{F}$ is an orthonormal set in $L^2_{\mathbb{C}}((-\pi;\pi))$.

Step 2: We claim that $\mathscr{F}$ is complete. Let x, y be any points in $[-\pi;\pi]$. We say that x, y are equivalent if and only if $x = y$ or x, y are in $\{-\pi;\pi\}$. We will write $x \sim y$. Let us denote as $\mathcal{K} \coloneqq [-\pi;\pi]/_\sim$ the quotient space where $-\pi$ and π have been identified. We also denote $i : [-\pi;\pi] \to \mathcal{K}$ the identification. It's easy to see that $\mathcal{K}$ is a compact Hausdorff space.

Since $e_n(-\pi) = e_n(\pi)$ for all integer n, we notice that the identification induces a set $\tilde{\mathscr{F}}$ of continuous functions between $\mathcal{K}$ and $\mathbb{C}$, namely

$$\tilde{\mathscr{F}} \coloneqq \{\tilde{e}_n \mid n \in \mathbb{Z}\}.$$

We claim that $\mathrm{Span}(\tilde{\mathscr{F}})$ is a set of complex-valued continuous functions over $\mathcal{K}$ with the following properties:

- it is an algebra: let n, m be any integers; it's easy to see that $\tilde{e}_n \cdot \tilde{e}_m = \tilde{e}_{n+m}$;
- it is closed under complex conjugation: if n is any integer, it's easy to see that $\overline{\tilde{e}_n} = \tilde{e}_{-n}$;
- it separates point: if x, y are different points in $\mathcal{K}$, it holds that $\tilde{e}_1(x) \neq \tilde{e}_1(y)$ (we notice that this is the reason why we introduce the quotient space $\mathcal{K}$);
- since $\tilde{e}_0(x) = 1$ for all x in $\mathcal{K}$, the constant functions are in $\mathrm{Span}(\tilde{\mathscr{F}})$.

Step 3: We can apply Stone-Weierstrass theorem (see 5.3.7) and we can state that $\mathrm{Span}(\tilde{\mathscr{F}})$ is dense in the set of the continuous function between $\mathcal{K}$ and $\mathbb{C}$ with respect to the norm of the uniform convergence. We define $\mathbb{X}$ as the set of the continuous functions between $[-\pi;\pi]$ and $\mathbb{C}$ that coincide in $-\pi$ and π. We notice that a function f belongs to $\mathbb{X}$ if and only if there exists a continuous function $\tilde{f}$ between $\mathcal{K}$ and $\mathbb{C}$ such that $f = \tilde{f} \circ i$. Then, it's easy to see that $\mathrm{Span}(\mathscr{F})$ is dense $\mathbb{X}$ with respect to the norm of the uniform convergence.

Since $[-\pi;\pi]$ is a finite measure space, we have that $\mathrm{Span}(\mathscr{F})$ is dense in $\mathbb{X}$ with respect to L^2 norm. Since $\mathbb{X}$ is dense in $C(\mathcal{K};\mathbb{C})$ with respect to L^2 norm and $C(\mathcal{K};\mathbb{C})$ is dense in $L^2_{\mathbb{C}}((-\pi;\pi))$ with respect to L^2 norm, then $\mathrm{Span}(\mathscr{F})$ is dense in $L^2_{\mathbb{C}}((-\pi;\pi))$ with respect to L^2 norm. □

Corollary 5.1.5. *Let f be in $L^2_{\mathbb{C}}((-\pi;\pi))$. For all integer n we define $c_n(f)$ as in 5.1.1; for all n in $\mathbb{N}$ we define $S_n f$ as in 5.1.3. Then, the following conclusions hold true:*

- *$\{S_n f\}_{n\in\mathbb{N}}$ converges toward f with respect to L^2 norm;*
- $2\pi \sum_{n\in\mathbb{Z}} |c_n(f)|^2 = \|f\|^2_{L^2((-\pi;\pi))}$;
- *if g is in $L^2_{\mathbb{C}}((-\pi;\pi))$, then it holds that*

$$< f, g >= \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx = 2\pi \sum_{n\in\mathbb{Z}} c_n(f)c_n(g).$$

Proof. Let $\mathscr{F}$ be as in theorem 5.1.4; we notice that if n is any integer, then it holds that

$$c_n(f) = \frac{1}{\sqrt{2\pi}} < f, e_n > .$$

Then, the thesis in an immediate consequence of theorems 4.2.5 and 5.1.4. □

Remark 5.1.6. We remark that if f in any function in $L^2_{\mathbb{C}}((-\pi;\pi))$, then $\{S_n f\}_{n\in\mathbb{N}}$ converges toward f with respect to the L^2 norm; so, there exists a subsequence that converges pointwise toward f for almost every x in $(-\pi;\pi)$. This is not enough to state that the whole sequence $\{S_n f\}_{n\in\mathbb{N}}$ converges pointwise toward f for almost every x in $(-\pi;\pi)$. As a matter of fact, this is a consequence of a theorem proved by the Swedish mathematician L. Carleson in 1966. However, the pointwise convergence of a specific subsequence is enough to characterize some punctual properties of the functions with other relations among the Fourier coefficient.

Proposition 5.1.7. *Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let us define the Fourier coefficient $\{c_n(f)\}_{n\in\mathbb{Z}}$ as 5.1.1. Then f is a real-valued function if and only if for all integer n it holds that $c_{-n}(f) = \overline{c_n(f)}$.*

Proof. Let f be a real-valued function; let n be any integer; we have that

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\overline{\int_{-\pi}^{\pi} \overline{f(x)e^{-inx}}dx} \\ &= \frac{1}{2\pi}\overline{\int_{-\pi}^{\pi} f(x)e^{inx}dx} = \overline{c_{-n}(f)}. \end{aligned}$$

Let us assume that f is such that for all integer n it holds that $c_{-n}(f) = \overline{c_n(f)}$. We notice that $c_0(f)$ is a real number. Let N be any positive integer; if we define the Fourier partial sum as in 5.1.3, we obtain that

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} c_n(f)e^{inx} \\ &= c_0(f) + \sum_{n=1}^{N}\left[c_n(f)e^{inx} + c_{-n}(f)e^{-inx}\right] \\ &= c_0(f) + \sum_{n=1}^{N}\left[c_n(f)e^{inx} + \overline{c_n(f)e^{inx}}\right] \\ &= c_0(f) + 2\sum_{n=1}^{N}\Re\{c_n(f)e^{inx}\}. \end{aligned}$$

Let us denote $\{S_{N_k} f\}_{k\in\mathbb{N}}$ the subsequence that converges toward f for almost every x in $[-\pi;\pi]$ Since $\{S_{N_k} f\}_{k\in\mathbb{N}}$ is a real-valued sequence of functions that converges toward f for almost every x in $[-\pi;\pi]$ and $\mathbb{R}$ is a closed set in $\mathbb{C}$, then f coincides almost everywhere with a real-valued function. □

Remark 5.1.8. We notice that if f is in $L^1_{\mathbb{C}}((-\pi;\pi))$ the definition of the Fourier coefficients (see 5.1.1) make sense. So, if $\mathbb{C}^{\mathbb{Z}}$ denotes the collection of the complex-valued sequences $\{a_n\}_{n\in\mathbb{Z}}$, we can well define the function $\Theta : L^1_{\mathbb{C}}((-\pi;\pi)) \to \mathbb{R}^{\mathbb{Z}}$ such that

$$\Theta(f) \coloneqq \{c_n(f)\}_{n\in\mathbb{Z}}.$$

First of all, we notice that for all f in $L^1_{\mathbb{C}}((-\pi;\pi))$, the sequence $c_n(f)$ is infinitesimal. In fact, this is a consequence of Riemann-Lebesgue' lemma (see 6.1.3), assuming that $f(x)$ is equal to 0 if x is not in $(-\pi;\pi)$.

We claim that Θ is injective. Since Θ is obviously linear, it is enough to show that $c_n(f) = 0$ for all n in $\mathbb{Z}$ implies that $f(x) = 0$ for almost every x in $(-\pi; \pi)$. We notice that if f belongs to $L^2_{\mathbb{C}}((-\pi;\pi))$ this statement has already been proved in 5.1.5. So, let f be a function in $L^1_{\mathbb{C}}((-\pi;\pi))$ such that for all n in $\mathbb{Z}$ it holds

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0.$$

For all n in $\mathbb{Z}$, we define the function $e_n : [-\pi;\pi] \to \mathbb{C}$ as in 5.1.4, i. e.

$$e_n(x) \coloneqq \frac{1}{\sqrt{2\pi}}e^{inx};$$

we also define

$$\mathcal{F} \coloneqq \{e_n \mid n \in \mathbb{Z}\}.$$

So, if g is $\text{Span}(\mathscr{F})$, we have that

$$\int_{-\pi}^{\pi} f(x)g(x) = 0.$$

Let us define $C_{per}([-\pi;\pi])$ the collection of the continuous functions that coincides in $-\pi$ and π. We have shown in theorem 5.1.4 that $\mathscr{F}$ is dense in $C_{per}([-\pi;\pi])$ with respect to the uniform convergence. So, if g is any function in $C_{per}([-\pi;\pi])$, there exists a sequence of function $\{g_n\}_{n\in\mathbb{N}}$ that converges toward g uniformly in $[-\pi;\pi]$. So, there exists a real number M such that $|g_n(x)| \leq M$ for all n in $\mathbb{N}$ for all x in $[-\pi;\pi]$. We claim that

$$\lim_{n\to+\infty}\int_{-\pi}^{\pi} f(x)g_n(x)dx = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

In fact, the $\{fg_n\}_{n\in\mathbb{N}}$ converges pointwise toward gf for almost every x in $(-\pi;\pi)$ and $M|f|$ is a suitable domination in L^1. So, we can easily use the dominated convergence theorem. In particular, we have that for all g in $C_{per}([-\pi;\pi])$ it holds that

$$\int_{-\pi}^{\pi} f(x)g(x)dx = 0.$$

Let h be any real-valued function in $L^\infty((-\pi;\pi))$. There exists a sequence of real-valued functions $\{h_n\}_{n\in\mathbb{N}}$ in $C_{per}([-\pi;\pi])$ that converges toward h with respect to L^2 norm. So, up to subsequences, not relabelled, the convergence is pointwise for almost every x in $(-\pi;\pi)$. We define $T_{\|h\|_{L^\infty}}$ as the truncation between $-\|h\|_{L^\infty((-\pi;\pi))}$ and $\|h\|_{L^\infty((-\pi;\pi))}$, i. e.

$$T_{\|h\|_{L^\infty}}(x) \coloneqq \begin{cases} \|h\|_{L^\infty((-\pi;\pi))} & \text{if } x \geq \|h\|_{L^\infty((-\pi;\pi))}; \\ x & \text{if } x \in \left[-\|h\|_{L^\infty((-\pi;\pi))}; \|h\|_{L^\infty((-\pi;\pi))}\right]; \\ -\|h\|_{L^\infty((-\pi;\pi))} & \text{if } x \leq -\|h\|_{L^\infty((-\pi;\pi))}. \end{cases}$$

Since T is a 1-Lipschitz function, we have that $\left\{T_{\|h\|_{L^\infty}} \circ h_n\right\}_{n\in\mathbb{N}}$ is a sequence of continuous equibounded function in $C_{per}([-\pi;\pi])$ that converges pointwise for almost every x in $(-\pi;\pi)$ toward h. Thanks to the dominated convergence theorem, the sequence converges with respect to L^1 norm toward h. Hence, we have that

$$\int_{-\pi}^{\pi} f(x)h(x)dx = \lim_{n\to+\infty}\int_{-\pi}^{\pi} f(x)h_n(x)dx = 0.$$

In particular, we can take $h(x) = sgn(f(x))$. So, we can conclude that

$$0 = \int_{-\pi}^{\pi} f(x) sgn(f(x)) dx = \int_{-\pi}^{\pi} |f(x)| \, dx;$$

in other words, $f(x)$ is equal to 0 for almost every x in $(-\pi; \pi)$.

5.1.2 On the convergence of the Fourier series

Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$; we have already shown that the Fourier series converges toward f with respect to L^2 norm. The aim of this section is to tie up the regularity of f and convergence of the Fourier series. We will find reasonable hypothesis on f under which the Fourier series converges toward f punctually or uniformly. We will show that the rapid convergence of the Fourier series force f to be regular. We will also investigate the link between the decay of Fourier coefficients and the speed of convergence of the Fourier series.

C^k functions vs convergence of the Fourier series

Definition 5.1.9. Let k be any positive integer. We define $\mathbb{X}^k_{per}$ as the space of functions with the following properties:

- f is in $C^{k-1}([-\pi;\pi];\mathbb{C})$;
- there exists a partition of $[-\pi;\pi]$, namely
$$-\pi := x_0 < x_1 < \cdots < x_j < x_{j+1} := \pi$$
such that if i is any integer in $\{0; \dots ; j\}$, then $f\big|_{[x_i;x_{i+1}]}$ is in $\in C^k([x_i;x_{i+1}];\mathbb{C})$;
- for all i in $\{0;\dots;k-1\}$ it holds that $f^i(\pi) = f^i(-\pi)$.

Lemma 5.1.10. *Let k be any positive integer; if f is any function in $\mathbb{X}^k_{per}$ and φ is any 2π-periodic smooth function, it holds that*

$$\int_{-\pi}^{\pi} \left[\frac{d^k f}{dx^k}(x)\right] \varphi(x) dx = (-1)^k \int_{-\pi}^{\pi} f(x) \left[\frac{d^k \varphi}{dx^k}(x)\right] dx.$$

Proof. The statement can be easily proved by induction on k. Let us assume that k equals 1; let us denote

$$-\pi := x_0 < x_1 < \cdots < x_j < x_{j+1} := \pi$$

the partition as declared in definition 5.1.9. Let φ any 2π-periodic smooth function. Since f is a piecewise C^1 function and it is globally continuous, we can integrate by parts and delete the boundary terms. So, we obtain that

$$\begin{aligned}
\int_{-\pi}^{\pi} f'(x)\varphi(x)dx &= \sum_{i=0}^{j} \int_{x_i}^{x_{i+1}} f'(x)\varphi(x)dx \\
&= \sum_{i=0}^{j} \left[f(x_{i+1})\varphi(x_{i+1}) - f(x_i)\varphi(x_i) - \int_{x_i}^{x_{i+1}} f(x)\varphi'(x)dx \right] \\
&= f(\pi)\varphi(\pi) - f(-\pi)\varphi(-\pi) - \int_{-\pi}^{\pi} f(x)\varphi'(x)dx \\
&= - \int_{-\pi}^{\pi} f(x)\varphi'(x)dx.
\end{aligned}$$

The inductive step is completely similar to the basis. □

Theorem 5.1.11. *Let k be any positive integer; let f be a function in $\mathbb{X}^k_{per}$. For all n in $\mathbb{Z}$, let $c_n(f)$ be as in definition 5.1.1. Then, the following conclusions hold true:*

- $\sum_{n\in\mathbb{Z}} n^{2k}\left|c_n(f)\right|^2 < +\infty$*; in particular $c_n(f)$ is $o\left(\frac{1}{|n|^k}\right)$;*
- *if α is any real number such that $\alpha < k - \frac{1}{2}$, then it holds that*

$$\sum_{k\in\mathbb{Z}} |n|^\alpha \left|c_n(f)\right| < +\infty;$$

- *for all integer j in $\{0;\dots;k-1\}$ it holds that*

$$\left\{\frac{d^j S_n f}{dx^j}\right\}_{n\in\mathbb{N}}$$

converges toward $\frac{d^j f}{dx^j}$ totally in $[-\pi;\pi]$ and for all x in $[-\pi;\pi]$ we have that

$$\frac{d^j f}{dx^j}(x) = \sum_{n\in\mathbb{Z}} (in)^j c_n(f) e^{inx}.$$

Proof. **Step 1**: First of all, we notice that $\frac{d^k f}{dx^k}$ is in $L^2_{\mathbb{C}}((-\pi;\pi))$. Thanks to lemma 5.1.10, for all n in $\mathbb{N}$ it holds that

$$\begin{aligned} c_n\left(\frac{d^k f}{dx^k}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{d^k f}{dx^k}(x)\right]e^{-inx}dx \\ &= \frac{(-1)^k}{2\pi}\int_{-\pi}^{\pi} f(x)(-in)^k e^{-inx}dx \\ &= (in)^k c_n(f). \end{aligned}$$

If apply the Parseval's identity (see 5.1.5) to $\frac{d^k f}{dx^k}$, we obtain that

$$\left\|\frac{d^k f}{dx^k}\right\|_{L^2((-\pi;\pi))} = 2\pi \sum_{n\in\mathbb{Z}} n^{2k}\left|c_n(f)\right|^2.$$

Step 2: Let α be a positive real number such that $\alpha < k - \frac{1}{2}$. We can apply the Cauchy-Schwartz' inequality in ℓ^2 (see 4.2.3 and 4.1.3) and we obtain that

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |n|^\alpha \left|c_n f\right| &= \sum_{n\in\mathbb{Z}} |n|^k \left|c_n f\right| \frac{1}{|n|^{k-\alpha}} \\ &\le \left(\sum_{n\in\mathbb{Z}} |n|^{2k}\left|c_n f\right|^2\right)\left(\sum_{n\in\mathbb{Z}}\frac{1}{|n|^{2k-2\alpha}}\right) < +\infty, \end{aligned}$$

thanks to the previous step and our assumption on α.

Step 3: If we join step 1 and step 2, we have that for all integer j in $\{0;\dots;k-1\}$ it holds that

$$\sum_{n\in\mathbb{Z}} \sup_{[-\pi;\pi]}\left\{c_n\left(\frac{d^j f}{dx^j}\right)e^{inx}\right\} = \sum_{n\in\mathbb{Z}} |n|^j \left|c_n(f)\right| < +\infty.$$

In other words, we have shown the total convergence in $[-\pi;\pi]$ of the series

$$\left\{\frac{d^j S_n f}{dx^j}\right\}_{n\in\mathbb{N}}.$$

Hence, for all x in $[-\pi;\pi]$, we obtain that

$$\frac{d^j f}{dx^j}(x) = \sum_{n\in\mathbb{Z}} c_n\left(\frac{d^j f}{dx^j}\right)e^{inx} = \sum_{n\in\mathbb{Z}}(in)^j c_n(f)e^{inx}.$$

□

Remark 5.1.12. If f is any function in $C^1([-\pi;\pi])$ such that $f(\pi)\neq f(-\pi)$, it cannot be that

$$\sum_{n\in\mathbb{Z}}|c_n(f)| < +\infty.$$

Otherwise, the Fourier series would converge punctually in $-\pi$ and π; in particular, it should be $f(-\pi)=f(\pi)$.

Theorem 5.1.13. *Let f be a function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let us define the Fourier coefficient as in 5.1.1. Let k be any integer greater than or equal to 0. Let us assume that one of the following alternatives holds true:*

- *there exists $\alpha > k+1$ such that $c_n(f)$ is $O\left(\frac{1}{|n|^\alpha}\right)$;*
- *there exists $\beta > k+\frac{1}{2}$ such that*

$$\sum_{n\in\mathbb{Z}}|n|^{2\beta}|c_n(f)|^2 < +\infty.$$

Then f is in $C^k([-\pi;\pi])$ and for all j in $\{0;\ldots;k\}$ it holds that

$$\frac{d^j f}{dx^j}(-\pi) = \frac{d^j f}{dx^j}(\pi).$$

Proof. Let us assume that the first condition holds true; then, there exists a positive real number M such that for all integer n we have that

$$|c_n(f)| \leq \frac{M}{|n|^\alpha}.$$

If $\alpha > k-1$, we can state that

$$\sum_{n\in\mathbb{Z}}|n|^k|c_n(f)| \leq M\sum_{n\in\mathbb{Z}}\frac{1}{|n|^{\alpha-k}} < +\infty.$$

As a matter of facts, we have already shown in theorem 5.1.11 (see step 2) that the second condition implies that

$$\sum_{n\in\mathbb{Z}}|n|^k|c_n(f)| < +\infty.$$

However, for all integer j in $\{0; \dots; k\}$ it holds that

$$\sum_{n\in\mathbb{Z}} \sup_{[-\pi;\pi]} \left\{ \left| c_n(f) \frac{d^j}{dx^j} e^{inx} \right| \right\} = \sum_{n\in\mathbb{Z}} \sup_{[-\pi;\pi]} \left| c_n(f)(in)^j e^{inx} \right| = \sum_{n\in\mathbb{Z}} |c_n(f)|\,|n|^j < +\infty.$$

In other words, if we define e_n as in 5.1.4, for all integer j in $\{0; \dots; k\}$ we have shown the total convergence in $[-\pi; \pi]$ of the series

$$\sum_{n\in\mathbb{Z}} c_n(f) \frac{d^j e_n}{dx^j}.$$

In particular, we have that the Fourier series converges uniformly toward f. If we derive the series, for all integer j in $\{1; \dots; k\}$ we can state that f is a function in $C^j([-\pi; \pi])$ and for all x in $[-\pi; \pi]$ it holds that

$$\begin{aligned} \frac{d^j f}{dx^j}(x) &= \frac{d^j \left(\displaystyle\sum_{n\in\mathbb{Z}} c_n(f) e_n \right)}{dx^j}(x) \\ &= \sum_{n\in\mathbb{Z}} c_n(f) \frac{d^j e_n}{dx^j}(x) \\ &= \sum_{n\in\mathbb{Z}} c_n(f)(in)^j e^{inx}. \end{aligned}$$

Since the series converges totally, we obtain that

$$\frac{d^j f}{dx^j}(-\pi) = \frac{d^j f}{dx^j}(\pi).$$

□

Remark 5.1.14. If we join theorem 5.1.11 and theorem 5.1.13, we immediately obtain that f is in $C^\infty_{per}([-\pi; \pi])$ if and only if $c_n(f)$ is $o\left(\frac{1}{|n|^\alpha}\right)$ for all α greater than 0.

Example 5.1.15. Let us consider the function f in $L^2((-\pi; \pi))$ such that $f(x) := x^2$. Integrating twice by parts, we can easily compute the Fourier coefficients and we obtain that

$$c_n(f) = \begin{cases} \dfrac{2(-1)^n}{n^2} & \text{if } n \neq 0; \\ \dfrac{\pi^2}{3} & \text{if } n = 0. \end{cases}$$

Since f is in $\mathbb{X}^0_{per}$, we can apply theorem 5.1.11 and the following identities hold true:

$$\pi^2 = \sum_{n\in\mathbb{Z}} c_n(f) e^{in\pi} = \frac{\pi^2}{3} + 2 \sum_{n=1}^{+\infty} \frac{2}{n^2}.$$

If we rearrange terms, we obtain the very well know identity

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hölder's function vs convergence of the Fourier series

Proposition 5.1.16. *Let α be a real number in $(0;1)$; let f be a function in $C^{0;\alpha}([-\pi;\pi])$ such that $f(-\pi)=f(\pi)$. If β is any real number such that $\beta>\alpha-\frac{1}{2}$, then it holds that*

$$\sum_{n\in\mathbb{Z}}|n|^{\beta}|c_n(f)|<+\infty.$$

If α is greater that or equal to $\frac{1}{2}$, the Fourier series converges toward f totally in $[-\pi;\pi]$.

Proof. Since $f(-\pi)=f(\pi)$, we can extend f to a 2π-periodic function in $C^{0;\alpha}(\mathbb{R})$ with constant c_f; we will denote the extension as f.

Step 1: Let γ be any real number. We define

$$I(\gamma)\coloneqq\int_0^1\frac{1}{h^{2\gamma}}\left(\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2\,dx\right)dh.$$

We claim that if $\gamma<\alpha+\frac{1}{2}$, then $I(\gamma)$ is a real number. Since $2\gamma-2\alpha<1$ we obtain that

$$\begin{aligned}I(\gamma)&=\int_0^1\frac{1}{h^{2\gamma}}\left(\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2\,dx\right)dh\\&\le c_f\int_0^1\frac{1}{h^{2\gamma}}\left(\int_{-\pi}^{\pi}|h|^{2\alpha}\,dx\right)dh\\&=2\pi c_f\int_0^1\frac{1}{h^{2\gamma-2\alpha}}dh<+\infty.\end{aligned}$$

Step 2: Let h be any real number; let $\tau_h f$ be as in definition 1.0.1; we claim that for all n in $\mathbb{Z}$ it holds that

$$c_n(\tau_h f)=e^{inh}c_n(f).$$

If we recall that f has been extended to a 2π-periodic function, we have that

$$\begin{aligned}c_n(\tau_h f)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+h)e^{-inx}dx\\&=\frac{e^{inh}}{2\pi}\int_{-\pi}^{\pi}f(x+h)e^{-in(x+h)}dx\\&=\frac{e^{inh}}{2\pi}\int_{-\pi+h}^{\pi+h}f(x+h)e^{-in(x+h)}dx\\&=\frac{e^{inh}}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx=e^{inh}c_n(f).\end{aligned}$$

Step 3: Let h be any real number. We define $g_h:[-\pi;\pi]\to\mathbb{C}$ such that

$$g_h(x)\coloneqq f(x+h)-f(x).$$

We claim that

$$\int_{.\pi}^{\pi}|g_h(x)|^2\,dx=2\pi\sum_{n\in\mathbb{Z}}\left|e^{inh}-1\right|^2|c_n(f)|^2.$$

Since g_h is in $L^2_{\mathbb{C}}((-\pi;\pi))$, for all n in $\mathbb{Z}$ it holds that

$$c_n(g_h)=c_n(\tau_h f)-c_n(f)=(e^{inh}-1)c_n(f).$$

We can apply the Parseval's identity to g_h (see 5.1.5) and we obtain that

$$\|g_h\|^2_{L^2((-\pi;\pi))} = \sum_{n\in\mathbb{Z}} \left|e^{inh} - 1\right|^2 |c_n(f)|^2 .$$

Step 4: For all integer n we define

$$\alpha(\gamma; n) := 2\pi \int_0^1 \frac{\left|e^{inh} - 1\right|^2}{h^{2\gamma}} dh.$$

Since α in $(0;1)$, γ is lower than $\frac{3}{2}$; hence, it's immediate to see that the sequence $\{\alpha(\gamma; n)\}_{n\in\mathbb{Z}}$ is well defined. Since we have shown in step 3 the total convergence of the series

$$\sum_{n\in\mathbb{Z}} \left|e^{inh} - 1\right|^2 |c_n(f)|^2 ,$$

we can switch the series and the integral and we obtain that

$$\begin{aligned}
I(\gamma) &= \int_0^1 \frac{1}{h^{2\gamma}} \left(\int_{-\pi}^{\pi} |f(x+h) - f(x)|^2 \, dx \right) dh \\
&= \int_0^1 \left(\frac{1}{h^{2\gamma}} 2\pi \sum_{n\in\mathbb{Z}} \left|e^{inh} - 1\right|^2 |c_n(f)|^2 \right) dh \\
&= 2\pi \sum_{n\in\mathbb{Z}} |c_n(f)|^2 \left(\int_0^1 \frac{1}{h^{2\gamma}} \left|e^{inh} - 1\right|^2 dh \right) \\
&= \sum_{n\in\mathbb{Z}} \alpha(\gamma; n) \, |c_n(f)|^2 .
\end{aligned}$$

Step 5: We claim that there exists a positive real number $\alpha(\gamma)$, such that for all n in $\mathbb{Z} \setminus \{0\}$ it holds that

$$\alpha(\gamma; n) \geq \alpha(\gamma) \, |n|^{2\gamma - 1} .$$

Let n be any integer such that $n \neq 0$; if we denote $t := nh$, then $dt = ndh$; hence, the following identities hold true:

$$\begin{aligned}
\alpha(\gamma; n) &= 2\pi \int_0^1 \frac{\left|e^{inh} - 1\right|^2}{h^{2\gamma}} dh \\
&= 2\pi \int_0^n \frac{\left|e^{it} - 1\right|^2}{t^{2\gamma}} \, |n|^{2\gamma-1} \, dt \\
&\geq |n|^{2\gamma-1} \, 2\pi \int_0^1 \frac{\left|e^{it} - 1\right|^2}{t^{2\gamma}} dt.
\end{aligned}$$

So, it is enough to define

$$\alpha(\gamma) := \int_0^1 \frac{\left|e^{it} - 1\right|^2}{t^{2\gamma}} dt.$$

Step 6: If $\gamma < \alpha + \frac{1}{2}$ we join step 1, step 4 and step 5, we obtain that

$$\alpha(\gamma) \sum_{n\in\mathbb{Z}} |n|^{2\gamma-1} \, |c_n(f)|^2 \leq \sum_{n\in\mathbb{Z}} \alpha(\gamma, n) \, |c_n(f)|^2 = I(\gamma) < +\infty.$$

In particular, we have that

$$\sum_{n\in\mathbb{Z}} |n|^{2\gamma-1} |c_n(f)|^2 < +\infty. \tag{5.1}$$

Let β be any real number such that $\beta < \alpha - \frac{1}{2}$. Let ε be a positive real number such that $\beta + \frac{\varepsilon}{2} < \alpha - \frac{1}{2}$. If we apply Cauchy-Schwartz inequality in ℓ^2 (see 4.1.3 and 4.2.3), we obtain that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |n|^{\beta} |c_n(f)| &= \sum_{n\in\mathbb{Z}} |n|^{\beta-\alpha+\frac{\varepsilon}{2}} |c_n(f)| \, |n|^{\alpha-\frac{\varepsilon}{2}} \\ &\leq \left(\sum_{n\in\mathbb{Z}} |n|^{2\beta-2\alpha+\varepsilon}\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} |n|^{2\alpha-\varepsilon} |c_n(f)|^2\right)^{\frac{1}{2}}.\end{aligned}$$

We notice that $2\beta - 2\alpha + \varepsilon < -1$; moreover, if we define $\gamma_0 \coloneqq \alpha + \frac{1}{2} - \frac{\varepsilon}{2}$, we have that $\gamma_0 < \alpha + \frac{1}{2}$ and $2\alpha - \varepsilon = 2\gamma_0 - 1$. This is enough to state that the series at right hand side are both convergent. To conclude, it's easy to see that if α is grater than $\frac{1}{2}$, we can choose $\beta = 0$ and it holds that

$$\sum_{n\in\mathbb{Z}} |c_n(f)| < +\infty;$$

so, the Fourier series converges totally toward f. □

Definition 5.1.17 (Convolution for 2π-periodic functions)**.** .
Let f, φ be any functions in $L^2_{\mathbb{C}}((-\pi;\pi))$. We denote as f and φ their extensions by periodicity over $\mathbb{R}$. We define the convolution between f and φ as the function $f *_{2\pi} \varphi : [-\pi;\pi] \to \mathbb{C}$ such that

$$f *_{2\pi} \varphi(x) \coloneqq \int_{-\pi}^{\pi} f(t)\varphi(x-t)dt.$$

Definition 5.1.18 (Dirichlet kernel)**.**
Let N be any positive integer. We define the function $D_N : [-\pi;\pi] \to \mathbb{C}$ such that

$$D_N(t) \coloneqq \sum_{n=-N}^{N} e^{int}.$$

The sequence of functions $\{D_N\}_{N\in\mathbb{N}}$ is called Dirichlet kernel.

Remark 5.1.19. Since D_N is the finite sum of complex exponentials for all N in $\mathbb{N}$, we can state that the Dirichlet kernel is a sequence of analytic functions. Moreover, it's easy to see that for all positive integer N it holds that

$$\int_{-\pi}^{\pi} D_N(t)dt = 2\pi.$$

Remark 5.1.20. Let N be any positive integer; let t be any point in $(-\pi;\pi)$. The following identities hold true:

$$\begin{aligned} D_N(t) &= \sum_{n=-N}^{N} e^{int} = \sum_{n=-N}^{N} \left(e^{it}\right)^n = \left(e^{it}\right)^{-N} \sum_{n=0}^{2N} \left(e^{it}\right)^n \\ &= \left(e^{it}\right)^{-N} \frac{\left(e^{it}\right)^{2N+1} - 1}{e^{it} - 1} = \frac{e^{it\left(N+\frac{1}{2}\right)} - e^{-it\left(N+\frac{1}{2}\right)}}{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}} \\ &= \frac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}. \end{aligned}$$

Remark 5.1.21. Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let us define the sequence of Fourier coefficients $\{c_n(f)\}_{n\in\mathbb{Z}}$ as in 5.1.1 and the sequence of the Fourier partial sum $\{S_n f\}_{n\in\mathbb{N}}$ as in 5.1.3. Let N be any natural number; for all x in $[-\pi;\pi]$ it holds that

$$\begin{aligned} S_N(f) &= \sum_{n=-N}^{N} c_n(f) e^{inx} \\ &= \sum_{n=-N}^{N} \frac{e^{inx}}{2\pi} \left(\int_{-\pi}^{\pi} f(y) e^{-iny} dy\right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(y) \sum_{n=-N}^{N} e^{in(x-y)}\right) dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x-y) dy \\ &= \frac{1}{2\pi} f *_{2\pi} D_N(t). \end{aligned}$$

We notice that if there exists M in $\mathbb{R}$ such that $\|D_N\|_{L^1((-\pi;\pi))}$ is lower that M for all positive integer N, then we could use something like proposition 3.2.19; therefore, if f is any continuous function in $L^2_{\mathbb{C}}((-\pi;\pi))$, we could conclude that the Fourier series converges point-wise toward f for all x in $(-\pi;\pi)$. As a matter of facts, there exists a continuous function f in $L^2_{\mathbb{C}}((-\pi;\pi))$ and a subset $\mathcal{D}$ dense in $[-\pi;\pi]$ such that for all x in $\mathcal{D}$ the Fourier series does not converge pointwise toward f. In fact, it can be proved that

$$\liminf_{n\to+\infty} \int_{-\pi}^{\pi} |D_n(t)|\, dt = +\infty.$$

Proposition 5.1.22. *Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let x_0 be any point in $[-\pi;\pi]$. Let us assume that there exists α in $(0;1]$ such that f is α-Hölder in x_0, i. e. there exists a positive real number C such that for all x in $[\pi;\pi]$ it holds that*

$$|f(x_0) - f(x)| \leq C\, |x - x_0|^{\alpha}.$$

Let us define the sequence $\{S_n f\}_{n\in\mathbb{N}}$ as in 5.1.3. Then, it holds that

$$\lim_{n\to+\infty} S_n f(x_0) = f(x_0).$$

Proof. We will also denote as f the extension of the function by periodicity over $\mathbb{R}$. Under our hypothesis, it's easy to see that for all x in $\mathbb{R}$ it holds that

$$|f(x_0) - f(x)| \leq C\, |x - x_0|^{\alpha}.$$

If we join 5.1.19, 5.1.21 and 5.1.20, for all positive integer N the following inequalities hold true:

$$\begin{aligned}
|S_N f(x_0) - f(x_0)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x_0-t)D_N(t)dt - \int_{-\pi}^{\pi} f(x_0)D_N(t)dt\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} (f(x_0-t)-f(x_0))D_N(t)dt\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} C\frac{(f(x_0-t)-f(x_0))}{\sin\left(\frac{t}{2}\right)}\sin\left(\left(N+\frac{1}{2}\right)t\right)dt\right|
\end{aligned}$$

If we define $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(t) := \frac{f(x_0-t)-f(x_0)}{2\pi\sin\left(\frac{t}{2}\right)}\mathbb{1}_{[-\pi;\pi]}(t),$$

we have shown that for all positive integer N it holds that

$$|S_N f(x_0) - f(x_0)| \le \left|\int_{\mathbb{R}} g(t)\sin\left(\left(N+\frac{1}{2}\right)t\right)dt\right|.$$

If we show that g is in $L^1(\mathbb{R})$, then we can apply Riemann-Lebesgue's lemma (see 6.1.3) to conclude that

$$\lim_{N\to+\infty}\int_{\mathbb{R}} g(t)e^{i\left(N+\frac{1}{2}\right)t}dt = 0.$$

If we consider the imaginary part, the thesis follows immediately. So it is enough to show that g is in $L^1(\mathbb{R})$. It's easy to see that if t is any point in $[0;\pi]$ then $\sin\left(\frac{t}{2}\right) \ge \pi t$. Since α is in $(0;1]$, we obtain that

$$\begin{aligned}
\|g\|_{L^1(\mathbb{R})} &= \int_{-\pi}^{\pi} |g(t)|\,dt \\
&= \int_{-\pi}^{\pi} \frac{|f(x_0-t)-f(x_0)|}{\left|\sin\left(\frac{t}{2}\right)\right|}dt \\
&\le \frac{C}{2\pi}\int_0^{\pi}\frac{|t|^{\alpha}}{\pi\,|t|}dt \\
&= \frac{C}{2\pi^2}\int_0^{\pi}\frac{1}{|t|^{1-\alpha}}dt < +\infty.
\end{aligned}$$

□

Proposition 5.1.23. *Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let us assume that there exists a partition of $[-\pi;\pi]$ i. e.*

$$-\pi := x_0 < x_1 < \cdots < x_k < x_{k+1} := \pi$$

such that for all integer i in $\{0;\ldots;k\}$ it holds that

- *$f|_{[x_i;x_{i+1}]}$ is in $C^1((x_i;x_i+1))$;*
- *$f'|_{[x_i;x_{i+1}]}$ is in $L^2_{\mathbb{C}}((x_i;x_{i+1}))$.*

Let us assume that for all integer i in $\{0; \dots ; k\}$ there exists l_i^+ such that

$$f(x_i)^+ := \lim_{x \to x_i^+} f(x)$$

and for all integer i in $\{1; \dots ; k+1\}$ there exists l_i^- such that

$$f(x_i)^- := \lim_{x \to x_i^-} f(x).$$

Let us define the sequence of the Fourier partial sum as in 5.1.3; then, the following conclusions hold true:

- *$\{S_n f\}_{n \in \mathbb{N}}$ converges toward f uniformly in any closed interval that does not intersect $\{x_i \mid i \in \{0; \dots ; k+1\}\}$;*
- *for all i in $\{1; \dots ; k\}$ it holds that*

$$\lim_{n \to +\infty} S_n f(x_i) = \frac{f(x_i)^+ + f(x_i)^-}{2};$$

- $\lim\limits_{n \to +\infty} S_n f(\pi) = \dfrac{f(x_0)^+ + f(x_{k+1})^-}{2}.$

Proof. **Step 1**: Without loss of generality, we can assume that

$$f(x_i) = \frac{f(x_i)^+ + f(x_i)^-}{2}$$

for all integer i in $\{1; \dots ; k\}$ and

$$f(\pi) = f(-\pi) = \frac{f(x_0)^+ + f(x_{k+1})^-}{2}.$$

Let us define $g_0 : [0; 2\pi) \to \mathbb{R}$ such that $g_0(0) := 0$ and $g_0(x) := \pi - x$ for all x in $(0; 2\pi)$. So, g_0 can be extended in $\mathbb{R}$ be periodicity. We also denote as g_0 this extension. For all h in $[-\pi; \pi)$ we define $g_h : \mathbb{R} \to \mathbb{R}$ such that $g_h(x) := g_0(x-h)$. Obviously, g_h is a piecewise affine function; moreover, h is the unique discontinuity point in $[-\pi; \pi)$. Moreover, we have that

$$g_h(h)^+ := \lim_{x \to h^+} g_h(x) = \pi,$$

$$g_h(h)^- := \lim_{x \to h^-} g_h(x) = -\pi,$$

$$0 = g_h(h) = \frac{g_h(h)^+ + g_h(h)^-}{2}.$$

Step 2: For all integer i in $\{1; \dots ; k\}$ we define

$$d_i := \frac{f(x_i)^+ - f(x_i)^-}{2\pi},$$

we also define

$$d_0 := \frac{f(x_0)^+ - f(x_{k+1}^-)}{2\pi}.$$

We denote as $f_c : [-\pi;\pi) \to \mathbb{C}$ the function such that

$$f_c(x) := f(x) - \sum_{i=0}^{k} d_i g_{x_i}(x).$$

Thanks to the properties of g_h, we have that f_c is a piecewise C^1 function; it is continuous and $f_c(-\pi) = f_c(\pi)$. So, if we define the sequence $\{S_n f_c\}_{n\in\mathbb{N}}$ of the Fourier partial sum for f_c as in 5.1.3, we can use theorem 5.1.11 and we obtain that the $\{S_n f_c\}_{n\in\mathbb{N}}$ converges toward f_c uniformly in $[\pi;\pi]$.

Step 3: We show the theorem assuming that f is equal to g_0. We define the sequence $\{c_n(g_0)\}_{n\in\mathbb{N}}$ of the Fourier coefficient for g_0 as in 5.1.1. It's easy to compute that

$$c_n(g_0) = \begin{cases} 0 & \text{if } n = 0; \\ -\frac{i}{n} & \text{if } n \neq 0. \end{cases}$$

Moreover, for all n in $\mathbb{N}$ for all x in $[-\pi;\pi]$ we have the

$$\begin{aligned} S_n g_0(x) &= \sum_{j=1}^{n} \left[c_j(g_0)e^{ijx} + c_{-j}(g_0)e^{-ijx}\right] \\ &= i\sum_{j=1}^{n} \frac{1}{j}\left[e^{-ijx-e^{ijx}}\right] \\ &= 2\Im\left(\sum_{j=1}^{n} e^{ijx}\right). \end{aligned}$$

For all n in $\mathbb{N}$ we define $a_n(x) := e^{inx}$; we also define

$$B_n(x) := \begin{cases} \frac{1}{n} \text{ if } n \geq 1, \\ 1 \qquad\qquad \text{if } n = 0. \end{cases}$$

$$A_n(x) := \sum_{k=0}^{n} e^{ixk} = \frac{1 - e^{ix(n+1)}}{1 - e^{ix}}.$$

For all positive integer n we denote

$$b_n = B_n - B_{n-1} = \begin{cases} -\frac{1}{n(n-1)} & \text{if } n \geq 2, \\ 0 & \text{if } n = 1. \end{cases}$$

So, we can use the summation by parts formula and we obtain that

$$\sum_{j=1}^{m} e^{ijx} = \frac{1}{n}A_n(x) - 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)}A_{k-1}(x).$$

Let ε be any positive real number. Since the sequence of functions $\{A_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $[\varepsilon; 2\pi - \varepsilon]$, the sequence $\{S_n g_0\}_{n\in\mathbb{N}}$ converges uniformly toward g_0 (as a matter of facts, we know that g_0 is the limit with respect to L^2 norm and it is unique). Moreover, $S_n(0)$ is equal to 0 for all n in $\mathbb{N}$; hence, the theorem is completely proved assuming that f is equal to g_0.

Step 4: Let h be any point in $[-\pi;\pi)$; we notice that $\S_n g_h(x) = S_n g_0(x-h)$ for all n in $\mathbb{N}$ for all x in $[-\pi;\pi]$. So, the theorem is true if f is equal to g_h.

In conclusion, we notice that

$$S_m f(x) = S_m f_c(x) + \sum_{i=0}^{k} d_i S_n g_{x_i}(x)$$

for all m in $\mathbb{N}$ for all x in $[-\pi;\pi]$. So, if we join the second step and the third step, the conclusion follows immediately. □

5.1.3 Application of the complex Fourier series to PDE

Heat equation with periodic boundary conditions

Definition 5.1.24. Let $u_0 : [-\pi;\pi] \to \mathbb{C}$ be any function. Let us consider the following partial derivative equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t;x) = \dfrac{\partial^2 u}{\partial x^2}(t;x) & \text{if } (t;x) \in (0;T) \times [-\pi;\pi] \\ u(t;\pi) = u(t;-\pi) & \text{if } t \in (0;T) \\ \dfrac{\partial u}{\partial x}(t;\pi) = \dfrac{\partial u}{\partial x}(t;-\pi) & \text{if } t \in (0;T) \\ u(0;x) = u_0(x) & \text{if } x \in [-\pi;\pi] \end{cases} \tag{5.2}$$

We say that (5.2) is the heat equation in $[-\pi;\pi]$ with periodic boundary conditions.

Definition 5.1.25. Let $u_0 : [-\pi;\pi] \to \mathbb{C}$ be any function; let T be any positive real number. We say that $u : [0;T) \times [-\pi;\pi] \to \mathbb{C}$ is a solution of (5.2) if it has the following properties:

- u is continuous in $[0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$, there exists
$$\frac{\partial^2 u}{\partial x^2}(t;x)$$
and it is continuous in $(0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$, there exists
$$\frac{\partial u}{\partial t}(t;x)$$
and it is continuous in $(0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$ the following identity holds true:
$$\frac{\partial^2 u}{\partial x^2}(t;x) = \frac{\partial u}{\partial t}(t;x);$$
- for all t in $(0;T)$ it holds that
$$u(t;\pi) = u(t;-\pi),$$
$$\frac{\partial u}{\partial x}(t;\pi) = \frac{\partial u}{\partial x}(t;-\pi);$$

- for all x in $[-\pi;\pi]$ it holds that

$$u(0;x) = u_0(x).$$

We are looking for reasonable hypothesis on u_0 to make sure that there exist a time T in $(0;+\infty)$ and a function $u : [0;T) \times [-\pi;\pi] \to \mathbb{C}$ that is a solution of (5.2) in the sense of definition 5.1.25.

Theorem 5.1.26 (Existence and uniqueness of the solution for heat equation with periodic boundary conditions)**.**
Let $u_0 : [-\pi;\pi] \to \mathbb{C}$ be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$; let us define the Fourier coefficient $\{c^0_n\}_{n\in\mathbb{Z}}$ as in 5.1.1. Let us assume that

$$\sum_{n\in\mathbb{Z}} \left|c^0_n\right| < +\infty.$$

Let $u : [0;+\infty) \times [-\pi;\pi] \to \mathbb{C}$ such that

$$u(t;x) \coloneqq \sum_{n\in\mathbb{Z}} c^0_n e^{-n^2 t} e^{inx}.$$

Then the following conclusions hold true:

- *u is a well defined complex-valued function in $[0;+\infty) \times [-\pi;\pi]$;*
- *u is in $C^\infty((0;+\infty) \times [-\pi;\pi])$;*
- *u is a solution of (5.2) in the sense of definition 5.1.25;*
- *if u_0 is a real-valued function, then u is a real-valued function;*
- *if v is a solution of (5.2) in the sense of 5.1.25, then v is equals to u.*

Proof. **Step 1**: We claim that u is well defined and it continuous in $[0;+\infty) \times [-\pi;\pi]$. We notice that

$$\sum_{n\in\mathbb{Z}} \sup_{[0;+\infty)\times[-\pi;\pi]} \left\{\left|c^0_n e^{-n^2 t} e^{inx}\right|\right\} \leq \sum_{n\in\mathbb{Z}} \left|c^0_n\right| < +\infty.$$

If u_0 is a real-valued function, then $c_n(f) = \overline{c_{-n}(f)}$ for all integer n, as shown in 5.1.7. For all N in $\mathbb{N}$ we define $S_N u : [0;+\infty) \times [-\pi;\pi] \to \mathbb{C}$ such that

$$S_N u(t;x) \coloneqq \sum_{n=-N}^{N} c^0_n e^{-n^2 t} e^{inx}.$$

We notice that for all N in $\mathbb{N}$ for all $(t;x)$ in $[0;+\infty) \times [-\pi;\pi]$ it holds that

$$\begin{aligned} S_N u(t;x) &= c^0_0 + \sum_{n=1}^{N} \left[c^0_n e^{-n^2 t} e^{inx} + c^0_{-n} e^{-^2 t} e^{-inx}\right] \\ &= c^0_0 + 2\sum_{n=1}^{N} e^{-n^2 t} \Re\{c^0_n e^{inx}\}. \end{aligned}$$

Hence, we obtain that $\{S_n u\}_{n\in\mathbb{N}}$ is real-valued sequence of functions that converges toward u uniformly in $[0;+\infty)\times[-\pi;\pi]$. Since $\mathbb{R}$ is a closed set in $\mathbb{C}$, then u is a real-valued function.

We claim that u is in $C^\infty((0;+\infty)\times[\pi;\pi])$. Let δ be a positive real number. Let k,j be nonnegative integers. We notice that for all integer n it holds that

$$\sup_{(\delta;+\infty)\times[-\pi;\pi]}\left\{\left|c_n^0\right|\left|\frac{\partial^{k+j}}{\partial x^k\partial t^h}(e^{-n^2t}e^{inx})\right|\right\}=\left|c_n^0\right|\left|n\right|^{2h+k}e^{-n^2\delta}.$$

Hence, we obtain that

$$\sum_{n\in\mathbb{Z}}\sup_{(\delta;+\infty)\times[-\pi;\pi]}\left\{\left|c_n^0\right|\left|\frac{\partial^{k+j}}{\partial x^k\partial t^h}(e^{-n^2t}e^{inx})\right|\right\}\leq\sum_{n\in\mathbb{Z}}\left|c_n^0\right|\left|n\right|^{2h+k}e^{-n^2\delta}$$

Since δ is a positive real number and the sequence $\{c_n^0\}_{n\in\mathbb{Z}}$ is bounded, we can state that the right hand side series converges. So, we derive the series and we obtain that u is in $C^\infty((\delta;+\infty)\times[-\pi;\pi])$. This is enough to state that u is a smooth function in $(0;+\infty)\times[-\pi;\pi]$. In particular, if $(t;x)$ is in $(0;+\infty)\times[-\pi;\pi]$, it is true that

$$\frac{\partial u}{\partial t}(t;x)=\sum_{n\in\mathbb{Z}}c_n^0\frac{\partial}{\partial t}(e^{-n^2t}e^{inx})=\sum_{n\in\mathbb{Z}}-n^2c_n^0e^{-n^2t}e^{inx};$$

$$\frac{\partial^2 u}{\partial x^2}(t;x)=\sum_{n\in\mathbb{Z}}c_n^0\frac{\partial^2}{\partial x^2}(e^{-n^2t}e^{inx})=\sum_{n\in\mathbb{Z}}-n^2c_n^0e^{-n^2t}e^{inx}.$$

As for the periodic boundary conditions, if t is any positive real number, we have that

$$u(t;\pi)=\sum_{n\in\mathbb{Z}}c_n^0e^{-n^2t}e^{in\pi}=\sum_{n\in\mathbb{Z}}c_n^0e^{-n^2t}e^{-in\pi}=u(t;-\pi),$$

$$\frac{\partial u}{\partial x}(t;\pi)=\sum_{n\in\mathbb{Z}}inc_n^0e^{-n^2t}e^{in\pi}=\sum_{n\in\mathbb{Z}}inc_n^0e^{-n^2t}e^{-in\pi}=\frac{\partial u}{\partial x}(t;-\pi).$$

As for the initial datum, we know that the Fourier series of u_0 is totally convergent (see 5.1.11); hence, if x is any point in $[-\pi;\pi]$, we have that

$$u(0;x)=\sum_{n\in\mathbb{Z}}c_n^0e^{inx}=u_0(x).$$

This is enough to state that u is a solution of (5.2) in the sense of definition 5.1.25.

Step 2: Let v be any solution of (5.2) in the sense of definition 5.1.25. Let t be any real number in $[0;T)$. We define

$$c_n(v(t;\cdot)):=\frac{1}{2\pi}\int_{-\pi}^{\pi}v(t;x)e^{-inx}dx.$$

For all integer n we define

$$c_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial v}{\partial t}(t;x)e^{-inx}dx.$$

Since v is continuous in $[0;T)\times[-\pi;\pi]$, we can apply theorem 2.3.1 and we have that for all n in $\mathbb{Z}$ the function $c_n(v(-;\cdot)):[0;T)\to\mathbb{C}$ is continuous. Since $\frac{\partial v}{\partial t}$ is continuous in $(0;T) \times [-\pi;\pi]$, we can apply theorem 2.3.2 and we have that if n is any integer, then $c_n(v(t;\cdot))$ is in $C^1((0;T))$ and for all t in $(0;T)$ it holds that

$$c_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial v}{\partial t}(t;x)e^{-inx}dx = c_n(v(t;\cdot))'.$$

We recall that v is such that for all t in $(0;T)$ it holds that

$$v(t;\pi) = v(t;-\pi),$$

$$\frac{\partial v}{\partial x}(t;\pi) = \frac{\partial v}{\partial x}(t;-\pi).$$

Let n be any integer; we define

$$c_n\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial^2 x}(t;x)e^{-inx}dx.$$

Thanks to lemma 5.1.10, for all integer n it holds that

$$\begin{aligned} c_n\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial x^2}(t;x)e^{-inx}dx \\ &= \frac{-n^2}{2\pi}\int_{-\pi}^{\pi} v(t;x)e^{-inx}dx \\ &= -n^2 c_n(v(t;\cdot)). \end{aligned}$$

We recall that v is such that for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$ it holds that

$$\frac{\partial v}{\partial t}(t;x) = \frac{\partial^2 v}{\partial x^2}(t;x).$$

Hence, for all integer n for all t in $(0;T)$, we have that

$$c_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right) = c_n\left(\frac{\partial^2 v}{\partial x^2}(t;\cdot)\right).$$

In other words, for all integer n for all t in $(0;T)$ it holds that

$$c_n(v(t;\cdot))' = -n^2 c_n(v(t;\cdot)).$$

Since $v(0;x) = u_0(x)$ for all x in $[-\pi;\pi]$, we can state that

$$c_n(v(0;\cdot)) = c_n^0.$$

We have that $c_n(v(t;\cdot))$ is a solution of the following differential problem

$$\begin{cases} y'(t) = -n^2 y(t) & \text{if } t > 0, \\ y(0) = 0, \end{cases}$$

and it is continuous in 0; this is equivalent to state that $c_n(v(t;\cdot))$ is a solution of the following Cauchy's problem:

$$\begin{cases} y'(t) = -n^2 y(t) & \text{if } t \geq 0, \\ y(0) = 0. \end{cases}$$

Hence, we have that for all t in $[0;+\infty)$ it holds that

$$c_n(v(t;\cdot)) = c_n^0 e^{-n^2 t}.$$

This is enough to state that that for all t in $[0;+\infty)$ the function $v\big|_{\{t\}\times[-\pi;\pi]}$ coincides with the function $u\big|_{\{t\}\times[-\pi;\pi]}$, i.e. for all $(t;x)$ in $[0;+\infty)\times[-\pi;\pi]$ it holds that $v(t;x) = u(t;x)$. □

Remark 5.1.27. We can show that there exist an initial datum u_0 such that for all positive real number δ the problem (5.2) has no solution in the sense of definition 5.1.25 in $(-\delta;0]\times[-\pi;\pi]$. Let $u_0 : [-\pi;\pi] \to \mathbb{C}$ be the initial datum; we denote $\{c_n^0\}_{n\in\mathbb{Z}}$ the sequence of the Fourier coefficients. Let us assume that there exist a positive real number δ and a function $v : (-\delta;0]\times[-\pi;\pi] \to \mathbb{C}$ that is a solution of (5.2) in the sense of definition 5.1.25. For all integer n for all t in $(-\delta;0]$ we define

$$c_n(v(t;\cdot)) := \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t;x)e^{-inx}dx.$$

As shown in further details in the proof of theorem 5.1.26 (see the second step), the function $c_n(v(t;\cdot))$ has the following properties:

- it is well defined and it is continuous in $(-\delta;0]$;
- $c_n(v(0;\cdot))$ equals c_n^0;
- it is in $C^1((-\delta;0))$;
- it is a solution of the following Cauchy's problem

$$\begin{cases} y(t)' = -n^2 y(t) & \text{if } t \in (-\delta;0], \\ y(0) = c_n^0. \end{cases}$$

Hence, we can state that for all n in $\mathbb{N}$ for all t in $(-\delta;0]$ it holds that

$$c_n(v(t;\cdot)) = c_n^0 e^{-n^2 t}.$$

Since the function $v\big|_{\{-\frac{\delta}{2}\}\times[-\pi;\pi]}$ is in $C^1_{per}([-\pi;\pi])$, we can apply theorem 5.1.11 and we obtain that

$$\sum_{n\in\mathbb{Z}} \left| c_n\left(v\left(-\frac{\delta}{2};\cdot\right)\right)\right| < +\infty.$$

In other words, we have that

$$\sum_{n\in\mathbb{Z}} \left| c_n^0 e^{n^2\frac{\delta}{2}}\right| < +\infty.$$

If u_0 is such that $c_n^0 = e^{-|n|}$ for all integer n, it's easy to see that u_0 is in $C^\infty_{per}([-\pi;\pi])$ (see theorem 5.1.11 and 5.1.13) but it holds that

$$\lim_{n\to+\infty} \left| c_n^0 e^{n^2\frac{\delta}{2}}\right| = +\infty.$$

Wave equation with periodic boundary conditions

Definition 5.1.28. Let $u_0, u_1 : [-\pi;\pi] \to \mathbb{C}$ be any functions. Let c be any real number. Let us consider the following partial derivative equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t}(t;x) = c^2 \dfrac{\partial^2 u}{\partial x^2}(t;x) & \text{if } (t;x) \in (0;T) \times [-\pi;\pi] \\ u(t;\pi) = u(t;-\pi) & \text{if } t \in (0;T) \\ \dfrac{\partial u}{\partial x}(t;\pi) = \dfrac{\partial u}{\partial x}(t;-\pi) & \text{if } t \in (0;T) \\ u(0;x) = u_0(x) & \text{if } x \in [-\pi;\pi] \\ \dfrac{\partial u}{\partial t}(0;x) = u_1(x) & \text{if } x \in [-\pi;\pi] \end{cases} \tag{5.3}$$

We say that (5.3) is the wave equation in $[-\pi;\pi]$ with periodic boundary conditions.

Definition 5.1.29. Let $u_0, u_1 : [-\pi;\pi] \to \mathbb{C}$ be any functions; let c be any real number. Let T be any positive real number. We say that $u : [0;T) \times [-\pi;\pi] \to \mathbb{C}$ is a solution of (5.3) if it has the following properties:

- u is continuous in $[0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $[0;T) \times [-\pi;\pi]$ there exists
$$\frac{\partial u}{\partial t}(t;x)$$
and it is continuous in $[0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$, there exists
$$\frac{\partial^2 u}{\partial x^2}(t;x)$$
and it is continuous in $(0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$, there exists
$$\frac{\partial^2 u}{\partial t^2}(t;x)$$
and it is continuous in $(0;T) \times [-\pi;\pi]$;
- for all $(t;x)$ in $(0;T) \times [-\pi;\pi]$ the following identity holds true:
$$c^2 \frac{\partial^2 u}{\partial x^2}(t;x) = \frac{\partial^2 u}{\partial t^2}(t;x);$$
- for all t in $(0;T)$ it holds that
$$u(t;\pi) = u(t;-\pi),$$
$$\frac{\partial u}{\partial x}(t;\pi) = \frac{\partial u}{\partial x}(t;-\pi);$$

- for all x in $[-\pi; \pi]$ it holds that

$$u(0; x) = u_0(x);$$

- for all x in $[-\pi; \pi]$ it holds that

$$\frac{\partial u}{\partial t}(0; x) = u_1(x).$$

We are looking for reasonable hypothesis on u_0 and u_1 to make sure that there exists a time T in $(0; +\infty)$ and a function $u : [0; T) \times [-\pi; \pi] \to \mathbb{C}$ that is a solution of (5.3) in the sense of definition 5.1.29.

Theorem 5.1.30 (Existence and uniqueness of the solution for wave equation with periodic boundary conditions (1))**.**
Let $u_0, u_1 : [-\pi; \pi] \to \mathbb{C}$ be any functions in $L^2_{\mathbb{C}}((-\pi; \pi))$; let c be any positive real number. Let us define the Fourier coefficient $\{c^0_n\}_{n\in\mathbb{Z}}$ for u_0 and $\{c^1_n\}_{n\in\mathbb{Z}}$ for u_1 as in 5.1.1. Let us assume that

$$\sum_{n\in\mathbb{Z}} \left|n^2 c^0_n\right| < +\infty,$$

$$\sum_{n\in\mathbb{Z}} \left|n c^1_n\right| < +\infty.$$

For all n in $\mathbb{Z} \setminus \{0\}$, we define

$$\alpha_n \coloneqq \frac{1}{2}\left[c^0_n + \frac{c^1_n}{icn}\right],$$

$$\beta_n \coloneqq \frac{1}{2}\left[c^0_n - \frac{c^1_n}{icn}\right].$$

Let $u : \mathbb{R} \times [-\pi; \pi] \to \mathbb{C}$ such that

$$u(t; x) \coloneqq c^0_0 + c^1_0 t + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right].$$

Then the following conclusions hold true:

- *u is a well defined complex-valued function in $\mathbb{R} \times [-\pi; \pi]$;*
- *u is in $C^2(\mathbb{R} \times [-\pi; \pi])$;*
- *u is a solution of (5.3) in the sense of definition 5.1.29;*
- *if u_0 and u_1 are real-valued functions, then u is a real-valued function;*
- *if v is a solution of (5.3) in the sense of 5.1.29, then v is equals to u.*

Proof. **Step 1**: We claim that u is well defined and it is continuous in $\mathbb{R} \times [-\pi; \pi]$. We notice that

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \sup_{\mathbb{R}\times[-\pi;\pi]} \left\{\left|\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right|\right\} \le \sum_{n\in\mathbb{Z}\setminus\{0\}} |\alpha_n| + |\beta_n|$$
$$\le \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|c^0_n\right| + \left|\frac{c^1_n}{cn}\right| < +\infty.$$

We claim that if u_0 and u_1 are real-valued functions, then u is a real-valued function. As shown in 5.1.7, we have that $c^0_{-n} = \overline{c^0_n}$ and $c^1_{-n} = \overline{c^1_n}$ for all integer n. In particular, c^0_0 and c^1_0 are real numbers. For all N in $\mathbb{N}$ we define $S_N u : \mathbb{R} \times [-\pi; \pi] \to \mathbb{C}$ such that

$$S_N u(t; x) \coloneqq c^0_0 + c^1_0 + \sum_{n=-N}^{1} \left[\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right] + \sum_{n=1}^{N} \left[\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right].$$

It's easy to see that for all n in $\mathbb{Z}$ for all $(t; x)$ in $\mathbb{R} \times [-\pi; \pi]$ it holds that

$$\overline{\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}} = \alpha_{-n} e^{-in(x+ct)} + \beta_{-n} e^{-in(x-ct)}.$$

Therefore, we obtain that

$$S_N u(t; x) = c^0_0 + c^1_0 t + 2 \sum_{n=1}^{N} \Re \left\{\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right\}.$$

Since $\{S_N u\}_{n \in \mathbb{N}}$ is a real-valued sequence of functions that converges toward u uniformly in $\mathbb{R} \times [-\pi; \pi]$ and $\mathbb{R}$ is a closed set in $\mathbb{C}$, it holds that u is a real-valued function.

We claim that u is a C^2 function in $\mathbb{R} \times [-\pi; \pi]$. Let h, k be integers in $\{0; 1; 2\}$ such that $h + k \leq 2$. First of all, we state that

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \sup_{\mathbb{R} \times [-\pi; \pi]} \left\{ \left| \frac{\partial^{h+k}}{\partial t^k \partial x^h} (\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}) \right| \right\} < +\infty.$$

For all n in $\mathbb{Z} \setminus \{0\}$ for all $(t; x)$ in $\mathbb{R} \times [-\pi; \pi]$ it holds that

$$\begin{aligned} \left| \frac{\partial^{h+k}}{\partial t^k \partial x^h} (\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}) \right| &= \left|\alpha_n c^k (in)^{k+h} e^{in(x+ct)} + \beta_n (-c)^k (in)^{k+h} e^{in(x-ct)}\right| \\ &\leq c^k \left(|\alpha_n| + |\beta_n|\right) |n|^{k+h} \\ &\leq c^k \left(\left|c^0_n\right| + \frac{|c^1_n|}{|cn|} \right) |n|^{k+h} \\ &= c^k \left|c^0_n\right| |n|^{k+h} + c^{k-1} \left|c^1_n\right| |n|^{k+h-1}. \end{aligned}$$

Since we are assuming that $h + k \leq 2$, we obtain that

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \sup_{\mathbb{R} \times [-\pi; \pi]} \left\{ \left| \frac{\partial^{h+k}}{\partial t^k \partial x^h} \left(\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right) \right| \right\} \leq c^k \sum_{n \in \mathbb{Z} \setminus \{0\}} \left|c^0_n\right| |n|^2 + \left|c^1_n\right| \frac{|n|}{c};$$

under our hypothesis, the right hand side series converges. In particular, if we derive the series we obtain that for all h, k in $\{0; 1; 2\}$ such that $h + k \leq 2$ for all $(t; x)$ in

$\mathbb{R} \times [-\pi;\pi]$ it holds that

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t;x) &= \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\partial^2}{\partial t^2}\left(\alpha_n e^{in(x+ct)} + \beta_n e^{in(x-ct)}\right) \\
&= \sum_{n\in\mathbb{Z}\setminus\{0\}} -n^2c^2\alpha_n e^{in(x+ct)} - n^2c^2\beta_n e^{in(x-ct)} \\
&= c^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} -n^2 e^{inx}\left[\frac{1}{2}\left(c_n^0 + \frac{c_n^1}{icn}\right)e^{icnt} + \frac{1}{2}\left(c_n^0 - \frac{c_n^1}{icn}\right)e^{-icnt}\right] \\
&= c^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} -n^2 e^{inx}\left[c_n^0 \frac{e^{icnt}+e^{-icnt}}{2} + \left(\frac{c_n^1}{cn}\right)\frac{e^{icnt}-e^{-icnt}}{2i}\right] \\
&= c^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} -n^2 e^{inx}\left[c_n^0\cos(cnt) + \frac{c_n^1}{cn}\sin(cnt)\right].
\end{aligned}$$

Similarly, it can be proved that

$$\frac{\partial^2 u}{\partial x^2}(t;x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} -n^2 e^{inx}\left[c_n^0\cos(cnt) + \frac{c_n^1}{cn}\sin(cnt)\right].$$

As for the periodic boundary conditions, for all t in $\mathbb{R}$ we have that

$$\begin{aligned}
u(t;\pi) &= c_0^0 + c_0^1 t + \sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha_n e^{in(\pi+ct)} + \beta_n e^{in(\pi-ct)} \\
&= c_0^0 + c_0^1 t + \sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha_n e^{in(-\pi+ct)} + \beta_n e^{in(-\pi-ct)} \\
&= u(t;-\pi);
\end{aligned}$$

if we derive the series, we can state that for all t in $\mathbb{R}$ it holds that

$$\begin{aligned}
\frac{\partial u}{\partial x}(t;\pi) &= \sum_{n\in\mathbb{Z}\setminus\{0\}} in\alpha_n e^{in(\pi+ct)} + in\beta_n e^{\pi-ct} \\
&= \sum_{n\in\mathbb{Z}\setminus\{0\}} in\alpha_n e^{in(-\pi+ct)} + in\beta_n e^{-\pi-ct} \\
&= \frac{\partial u}{\partial x}(t;-\pi).
\end{aligned}$$

As for the initial datum, we can similarly show that for all x in $[-\pi;\pi]$ it holds that

$$u(0;x) = u_0(x),$$

$$\frac{\partial u}{\partial t}(0;x) = u_1(x).$$

We can finally conclude that u is a solution of (5.3) in the sense of definition 5.1.29.

Step 2: Let $v : [0;T)\times[-\pi;\pi] \to \mathbb{C}$ be any solution of (5.3) in the sense of definition 5.1.29. Let n be any integer; let t be any point in $[0;T)$. We define

$$c_n(v(t;\cdot)) \coloneqq \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t;x)e^{-inx}dx;$$

$$c_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial v}{\partial t}(t;x)e^{-inx}dx;$$

$$c_n\left(\frac{\partial^2 v}{\partial t^2}(t;\cdot)\right) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2 v}{\partial t^2}(t;x)e^{-inx}dx.$$

Since v is continuous in $[0;T)\times[-\pi;\pi]$, we can apply theorem 2.3.1 and we can state that the function

$$c_n(v(t;\cdot)) : [0;T) \to \mathbb{C}$$

is well defined and it is continuous. Since $\frac{\partial v}{\partial t}$ is continuous in $[0;T)\times[-\pi;\pi]$, we can use theorem 2.3.2 and we obtain that the function $c_n(v(t;\cdot))$ is in $C^1([0;T))$ and for all t in $[0;T)$ it holds that

$$c_n(v(t;\cdot))' = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial v}{\partial t}e^{-inx}dx = c_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right).$$

By definition 5.1.29, we have that $\frac{\partial^2 v}{\partial t^2}$ is continuous in $(0;T)\times[-\pi;\pi]$. So, thanks to theorem 2.3.2, we have that $c_n(v(t;\cdot))$ is in $C^2((0;T))$ and for all t in $(0;T)$ it holds that

$$c_n(v(t;\cdot))'' = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2 v}{\partial t^2}(t;x)e^{-inx}dx = c_n\left(\frac{\partial^2 v}{\partial t^2}(t;\cdot)\right).$$

We recall that for all x in $[-\pi;\pi]$ it holds that

$$v(0;x) = u_0(x),$$

$$\frac{\partial v}{\partial t}(0;x) = u_1(x).$$

Hence, it's immediate to see that

$$c_n(v(0;\cdot)) = c_n^0,$$

$$c_n(v(0;\cdot))' = c_n^1.$$

Let n be any integer; let t be any point in $(0;T)$. We define

$$c_n\left(\frac{\partial^2 v}{\partial x^2}(t;\cdot)\right) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2 v}{\partial x^2}(t;x)e^{-inx}dx.$$

By definition 5.1.29, for all t in $(0;T)$ it holds that

$$v(t;\pi) = v(t;-\pi),$$

$$\frac{\partial v}{\partial x}(t;\pi) = \frac{\partial v}{\partial x}(t;-\pi).$$

Hence, we can use lemma 5.1.10 and we obtain that

$$\begin{aligned} c_n\left(\frac{\partial^2 v}{\partial t^2}(t;\cdot)\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2 v}{\partial t^2}(t;x)e^{-inx}dx \\ &= \frac{-n^2}{2\pi}\int_{-\pi}^{\pi}v(t;x)e^{-inx}dx \\ &= -n^2 c_n(v(t;\cdot)). \end{aligned}$$

We recall that v is such that for all $(t; x)$ in $(0; T) \times [-\pi; \pi]$ it holds that

$$\frac{\partial^2 v}{\partial t^2}(t; x) = c^2 \frac{\partial^2 v}{\partial x^2}(t; x).$$

Hence, for all integer n for all t in $(0; T)$, we have that

$$c_n \left(\frac{\partial^2 v}{\partial t^2}(t; \cdot) \right) = c_n \left(c^2 \frac{\partial^2 v}{\partial x^2}(t; \cdot) \right).$$

In other words, we have shown that for all integer n the function $c_n(v(t; \cdot))$ is a solution of the following differential problem

$$\begin{cases} y''(t) = -n^2 y(t) & \text{if } t > 0, \\ y(0) = c_n^0, \\ y'(0) = c_n^1 \end{cases}$$

and it is continuous in 0; this is equivalent to state that $c_n(v(t; \cdot))$ is a solution of the following Cauchy's problem:

$$\begin{cases} y''(t) = -n^2 y(t) & \text{if } t \geq 0, \\ y(0) = c_n^0, \\ y'(0) = c_n^1. \end{cases}$$

So, if n is 0, for all t in $[0; +\infty)$ it holds that

$$c_0(t) = c_0^0 + c_0^1 t;$$

otherwise, it's easy to see that

$$c_n(t) = c_n^0 \cos(cnt) + \frac{c_n^1}{cn} \sin(cnt) = \alpha_n e^{inct} + \beta_n e^{-inct}.$$

This is enough to state that that for all t in $[0; +\infty)$ the function $v\big|_{\{t\} \times [-\pi; \pi]}$ coincides with the function $u\big|_{\{t\} \times [-\pi; \pi]}$, i. e. for all $(t; x)$ in $[0; +\infty) \times [-\pi; \pi]$ it holds that $v(t; x) = u(t; x)$. □

Theorem 5.1.31 (Existence of solution for wave equation with periodic boundary conditions (2))**.**
Let u_0 be any function in $C^2_{per}([-\pi; \pi])$; let u_1 be any function in $C^1_{per}([-\pi; \pi])$; let c be any positive real number. Then, there exist c_0^0 and c_0^1 in $\mathbb{C}$ and there exist complex-valued 2π-periodic functions φ^+ and φ^- in $C^2(\mathbb{R})$ with the following property: if we define $u : [-\pi; \pi] \times \mathbb{R} \to \mathbb{C}$ such that

$$u(t; x) = c_0^0 + c_0^1 t + \varphi^+(x - ct) + \varphi^-(x + ct),$$

then u is a solution of (5.3) *in the sense of 5.1.29 in $\mathbb{R} \times [-\pi; \pi]$.*

Proof. If we show that there exist c_0^0 and c_0^1 in $\mathbb{C}$ and there exist $\varphi^+, \varphi^- : \mathbb{R} \to \mathbb{C}$ such that for all x in $[-\pi; \pi]$ it holds that

$$\begin{cases} u_0(x) = c_0^0 + \varphi^+(x) - \varphi^-(x), \\ u_1(x) = c_0^1 - c\varphi^+(x)' + c\varphi^-(x)', \end{cases}$$

then thesis follows immediately (the other requests are obviously satisfied.) We can derive the first equation and we obtain that for all x in $[-\pi;\pi]$ it holds that

$$\begin{cases} \varphi^+(x)' + \varphi^-(x)' = u_0(x)', \\ -\varphi^+(x)' + \varphi^-(x)' = \dfrac{u_1(x) - c_0^1}{c}. \end{cases}$$

Hence, the following identities hold true:

$$\begin{cases} \varphi^+(x)' = \dfrac{1}{2}\left(u_0(x)' - \dfrac{u_1(x) - c_0^1}{c}\right), \\ \varphi^-(x)' = \dfrac{1}{2}\left(u_0(x)' + \dfrac{u_1(x) - c_0^1}{c}\right). \end{cases} \tag{5.4}$$

We can choose c_0^1 such that

$$\int_{-\pi}^{\pi} (u_1(x) - c_0^1)dx = 0;$$

since, u_0 is in $C^2_{per}([-\pi;\pi])$, we have that

$$\int_{-\pi}^{\pi} u_0(x)'dx = 0.$$

We notice that the right hand side of the equations 5.4 are functions with zero mean. From now on, we will identity u_0 and u_1 with their extension over $\mathbb{R}$ by periodicity. If we choose $c_0^0 := u_0(0)$, for all x in $\mathbb{R}$ we can define

$$\begin{cases} \varphi^+(x) = \dfrac{1}{2}\displaystyle\int_0^x \left(u_0(t)' - \dfrac{u_1(t) - c_0^1}{c}\right) dt, \\ \varphi^-(x) = \dfrac{1}{2}\displaystyle\int_0^x \left(u_0(t)' + \dfrac{u_1(t) - c_0^1}{c}\right) dt. \end{cases}$$

We have that φ^+ and φ^- are 2π-periodic functions in $C^2(\mathbb{R})$ that satisfy all the requests. □

5.2 Real Fourier series

5.2.1 Decomposition in sines and cosines

Theorem 5.2.1. *Let us define*

$$\mathscr{G} := \left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos(nx) \;\middle|\; n \in \mathbb{N}^*\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\sin(nx) \;\middle|\; n \in \mathbb{N}^*\right\}.$$

Then $\mathscr{G}$ is a real-valued functions Hilbert's basis of $L^2_{\mathbb{C}}((-\pi;\pi))$.

Proof. If we show that $\mathscr{G}$ is a maximal set, then thesis follows immediately (the other requests are obviously satisfied). If we define

$$\mathscr{F} := \left\{\frac{1}{\sqrt{2\pi}}e^{inx} \;\middle|\; n \in \mathbb{Z}\right\},$$

we have shown in theorem 5.1.4 that $\mathscr{F}$ is an Hilbert's basis of $L^2_{\mathbb{C}}((-\pi;\pi))$. It's immediate to see that

$$\text{Span}(\mathscr{F}) \subseteq \text{Span}(\mathscr{G});$$

this is enough to state that

$$\overline{\text{Span}(\mathscr{G})} = L^2_{\mathbb{C}}((-\pi;\pi)).$$

□

Remark 5.2.2. We remark that the maximality can be proved as a direct consequence of Stone-Weierstrass theorem; unfortunately, the proof of the fact that $\mathscr{G}$ is an algebra is a bit technical.

Definition 5.2.3 (Real Fourier coefficient)**.** Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let n be any positive integer. We define

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx;$$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

We also define

$$a_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Definition 5.2.4 (Real Fourier partial sum)**.**
Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let us define the real Fourier coefficients as in 5.2.3. Let n be any positive integer. For all x in $[-\pi;\pi]$ we define

$$G_n f(x) \coloneqq a_0(f) + \sum_{k=1}^{n}\left[a_k(f)\cos(kx) + b_k(f)\sin(kx)\right].$$

We say that $\{G_n f\}_{n\in\mathbb{N}^*}$ is the sequence of the real Fourier partial sum of f.

Corollary 5.2.5. *Let f be any function in $L^2_{\mathbb{C}}((-\pi;\pi))$. Let us define $\{G_n f\}_{n\in\mathbb{N}^*}$ as in 5.2.4; then $\{G_n f\}_{n\in\mathbb{N}^*}$ is a sequence of functions that converges toward f with respect to L^2 norm. Moreover, if f is a real-valued function, then $\{G_n f\}_{n\in\mathbb{N}^*}$ is a real-valued sequence of functions, $c_0(f) = a_0(f)$ and for all n in $\mathbb{N}^*$ it holds that*

$$a_n(f) = c_n(f) + c_{-n}(f) = 2\Re(c_n(f)),$$

$$b_n(f) = i(c_n(f) - c_{-n}(f)) = 2\Im(c_n(f)).$$

Proof. As for the first statement, it is an immediate consequence of theorems 5.2.1 and 4.2.5.

By definitions 5.1.1 and 5.2.3, it follows that $c_0(f) = a_0(f)$. Let n be any positive integer; then, we have that

$$c_n(f) + c_{-n}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[e^{-inx} + e^{inx}\right]dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = a_n(f),$$

$$i(c_n(f) - c_{-n}(f)) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left[e^{-inx} - e^{inx}\right]dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = b_n(f).$$

Let us assume that f is a real-valued function. Thanks to proposition 5.1.7, we can state that for all n in Z it holds that $c_n(f) = \overline{c_{-n}(f)}$. Hence, we can conclude that a_0 is a real number and for all n in $\mathbb{N}^*$ it holds that

$$a_n(f) = 2\Re(c_n(f)),$$

$$b_n(f) = 2\Im(c_n(f)).$$

□

5.2.2 Decomposition in sines

Definition 5.2.6. Let f be any function in $L^2_{\mathbb{C}}((0;\pi))$. Let n be any positive integer. We define

$$\beta_n(f) := \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx.$$

Theorem 5.2.7. *Let us define*

$$\mathscr{S} := \left\{ \sqrt{\frac{2}{\pi}}\sin(nx) \;\middle|\; n \in \mathbb{N}^* \right\}.$$

Then $\mathscr{S}$ is an Hilbert's basis of $L^2_{\mathbb{C}}((0;\pi))$.

Proof. It's immediate to see that $\mathscr{S}$ is an orthonormal set. We claim that $\mathscr{S}$ is complete. Let f be any function in $L^2_{\mathbb{C}}((0;\pi))$. We define $\tilde{f} : [-\pi;\pi] \to \mathbb{C}$ the odd extension of f, i.e. $\tilde{f}(x) = f(x)$ if x in $[0;\pi]$ and $\tilde{f}(x) = -f(-x)$ if x is in $[-\pi;0)$. Let us define the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}^*}$ as in 5.2.3 and the sequence of the real Fourier partial sum $\{G_n\tilde{f}\}_{n\in\mathbb{N}^*}$ as in 5.2.4. Since $\tilde{f}$ is odd, the following conclusions hold true:

- $a_0(\tilde{f}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{f}(x)dx = 0$;
- for all n in $\mathbb{N}^*$ we have that
$$a_n(\tilde{f}) = \frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\cos(nx)dx = 0;$$
- for all n in $\mathbb{N}^*$ we have that
$$b_n(\tilde{f}) = \frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\sin(nx)dx = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx = \beta_n(f).$$

For all n in $\mathbb{N}^*$ we define $J_nf : [0;\pi] \to \mathbb{C}$ such that

$$J_nf(x) := \sum_{k=1}^{n}\beta_k(f)\sin(kx).$$

Since $\{G_n\tilde{f}\}_{n\in\mathbb{N}^*}$ converges toward $\tilde{f}$ with respect to L^2 norm in $(-\pi;\pi)$, we have that $\{J_nf\}_{n\in\mathbb{N}^*}$ is a sequence in $\mathrm{Span}(\mathscr{S})$ that converges toward f with respect to L^2 norm in $(0;\pi)$. □

Lemma 5.2.8. *Let k be any positive integer. Let f be any function in $C^{2k}([0;\pi])$ such that for all integer i in $\{0;\dots;k-1\}$ it holds that $f^{(2i)}(0) = f^{(2i)}(\pi) = 0$. Let φ be a function in $C^{\infty}([0;\pi])$ such that for all integer i in $\{0;\dots;k-1\}$ it holds that $\varphi^{(2i)}(0) = \varphi^{(2i)}(\pi) = 0$. Then, the following identity holds true:*

$$\int_0^{\pi} f^{(2k)}(x)\varphi(x)dx = \int_0^{\pi} f(x)\varphi^{(2k)}(x)dx.$$

Proof. The statement can be easily proved by induction on k. Let us assume that k equals 1. If we integrate twice by parts and we use the boundary conditions, we obtain that

$$\begin{aligned}\int_0^{\pi} f''(x)\varphi(x)dx &= f'(\pi)\varphi(\pi) - f'(0)\varphi(0) - \int_0^{\pi} f'(x)\varphi'(x)dx \\ &= -\int_0^{\pi} f'(x)\varphi'(x)dx \\ &= -f(\pi)\varphi'(\pi) + f(0)\varphi'(0) + \int_0^{\pi} f(x)\varphi''(x)dx \\ &= \int_0^{\pi} f(x)\varphi''(x)dx.\end{aligned}$$

The inductive step is completely similar to the basis. □

5.2.3 Decomposition in cosines

Definition 5.2.9. Let f be any function in $L^2_{\mathbb{C}}((0;\pi))$. Let n be any positive integer. We define

$$\alpha_n(f) \coloneqq \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx.$$

We also define

$$\alpha_0(f) \coloneqq \frac{1}{\pi}\int_0^{\pi} f(x)dx.$$

Theorem 5.2.10. *Let us define*

$$\mathscr{C} \coloneqq \left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}}\cos(nx) \;\middle|\; n \in \mathbb{N}^*\right\}.$$

Then $\mathscr{C}$ is an Hilbert's basis of $L^2_{\mathbb{C}}((0;\pi))$.

Proof. It's immediate to see that $\mathscr{C}$ is an orthonormal set. We claim that $\mathscr{C}$ is complete. Let f be any function in $L^2_{\mathbb{C}}((0;\pi))$. We define $\tilde{f} : [-\pi;\pi] \to \mathbb{C}$ the even extension of f, i.e. $\tilde{f}(x) = f(x)$ if x in $[0;\pi]$ and $\tilde{f}(x) = f(-x)$ if x is in $[-\pi;0)$. Let us define the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}^*}$ as in 5.2.3 and the sequence of the real Fourier partial sum $\{G_n\tilde{f}\}_{n\in\mathbb{N}^*}$ as in 5.2.4. Since $\tilde{f}$ is even, the following conclusions hold true:

- for all n in $\mathbb{N}^*$ we have that

$$b_n(\tilde{f}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\sin(nx)dx = 0;$$

- $a_0(\tilde{f}) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \tilde{f}(x)dx = \dfrac{1}{\pi}\int_0^{\pi} f(x)dx = \alpha_0(f)$;
- for all n in $\mathbb{N}^*$ we have that
$$a_n(\tilde{f}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = \alpha_n(f).$$

For all n in $\mathbb{N}^*$ we define $H_n f : [0;\pi] \to \mathbb{C}$ such that

$$H_n f(x) := \alpha_0(f) + \sum_{k=1}^{n} \alpha_k(f)\cos(kx).$$

Since $\{G_n\tilde{f}\}_{n\in\mathbb{N}^*}$ converges toward $\tilde{f}$ with respect to L^2 norm in $(-\pi;\pi)$, we have that $\{H_n f\}_{n\in\mathbb{N}^*}$ is a sequence in $\mathrm{Span}(\mathscr{C})$ that converges toward f with respect to L^2 norm in $(0;\pi)$. □

Lemma 5.2.11. *Let k be any positive integer. Let f be any function in $C^{2k}([0;\pi])$ such that for all integer i in $\{0;\ldots;k-1\}$ it holds that $f^{(2i+1)}(0) = f^{(2i+1)}(\pi) = 0$. Let φ be any function in $C^{\infty}([0;\pi])$ such that for all integer i in $\{0;\ldots;k-1\}$ it holds that $\varphi^{(2i+1)}(0) = \varphi^{(2i+1)}(\pi) = 0$. Then, the following identity holds true:*

$$\int_0^{\pi} f^{(2k)}(x)\varphi(x)dx = \int_0^{\pi} f(x)\varphi^{(2k)}(x)dx.$$

Proof. The statement can be easily proved by induction on k. Let us assume that k equals 1. If we integrate twice by parts and we use the boundary conditions, we obtain that

$$\begin{aligned}\int_0^{\pi} f''(x)\varphi(x)dx &= f'(\pi)\varphi(\pi) - f'(0)\varphi(0) - \int_0^{\pi} f'(x)\varphi'(x)dx \\ &= -\int_0^{\pi} f'(x)\varphi'(x)dx \\ &= -f(\pi)\varphi'(\pi) + f(0)\varphi'(0) + \int_0^{\pi} f(x)\varphi''(x)dx \\ &= \int_0^{\pi} f(x)\varphi''(x)dx.\end{aligned}$$

The inductive step is completely similar to the basis. □

5.2.4 Application of the real Fourier series to PDE

Heat equation with homogeneous Dirichlet boundary conditions

Definition 5.2.12. Let $u_0 : [0;\pi] \to \mathbb{C}$ be any function. Let us consider the following partial derivative equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t;x) = \dfrac{\partial^2 u}{\partial x^2}(t;x) & \text{if } (t;x) \in (0;T) \times [0;\pi] \\ u(t;\pi) = u(t;-\pi) = 0 & \text{if } t \in (0;T) \\ u(0;x) = u_0(x) & \text{if } x \in [0;\pi] \end{cases} \tag{5.5}$$

We say that (5.5) is the heat equation in $[0;\pi]$ with homogeneous Dirichlet boundary conditions.

Definition 5.2.13. Let $u_0 : [0;\pi] \to \mathbb{C}$ be any function; let T be any positive real number. We say that $u : [0;T) \times [0;\pi] \to \mathbb{C}$ is a solution of (5.5) if it has the following properties:

- u is continuous in $[0;T) \times [0;\pi]$;
- for all $(t;x)$ in $(0;T) \times [0;\pi]$, there exists
$$\frac{\partial^2 u}{\partial x^2}(t;x)$$
and it is continuous in $(0;T) \times [0;\pi]$;
- for all $(t;x)$ in $(0;T) \times [0;\pi]$, there exists
$$\frac{\partial u}{\partial t}(t;x)$$
and it is continuous in $(0;T) \times [0;\pi]$;
- for all $(t;x)$ in $(0;T) \times [0;\pi]$ the following identity holds true:
$$\frac{\partial^2 u}{\partial x^2}(t;x) = \frac{\partial u}{\partial t}(t;x);$$
- for all t in $(0;T)$ it holds that
$$u(t;\pi) = u(t;-\pi) = 0;$$
- for all x in $[0;\pi]$ it holds that
$$u(0;x) = u_0(x).$$

We are looking for reasonable hypothesis on u_0 to make sure that there exists a time T in $(0;+\infty)$ and a function $u : [0;T) \times [0;\pi] \to \mathbb{C}$ that is a solution of (5.5) in the sense of definition 5.2.13.

Theorem 5.2.14 (Existence and uniqueness of the solution for heat equation with homogeneous Dirichlet boundary conditions)**.**
Let $u_0 : [0;\pi] \to \mathbb{C}$ be any function in $L^2_{\mathbb{C}}((0;\pi))$; let us define the real Fourier coefficient $\{\beta^0_n\}_{n\in\mathbb{N}^*}$ as in 5.2.6. Let us assume that
$$\sum_{n\in\mathbb{N}^*} \left|\beta^0_n\right| < +\infty.$$
Let $u : [0;+\infty) \times [0;\pi] \to \mathbb{C}$ such that
$$u(t;x) := \sum_{n\in\mathbb{N}^*} \beta^0_n e^{-n^2 t} \sin(nx).$$
Then the following conclusions hold true:

- *u is a well defined complex-valued function in $[0;+\infty) \times [0;\pi]$;*

- *u is in $C^{\infty}((0;+\infty)\times[0;\pi])$;*
- *u is a solution of (5.5) in the sense of definition 5.2.13;*
- *if u_0 is a real-valued function, then u is a real-valued function;*
- *if v is a solution of (5.5) in the sense of 5.2.13, then v is equal to u.*

Proof. **Step 1**: If we slightly modify the procedure described in many details in the first step of theorem 5.1.26, we immediately obtain the following statements:

- u is a well defined complex-valued function in $[0;+\infty)\times[0;\pi]$;
- u is in $C^{\infty}((0;+\infty)\times[0;\pi])$;
- u is a solution of (5.5) in the sense of definition 5.2.13;
- if u_0 is a real-valued function, then u is a real-valued function.

Step 2: Let v be any solution of (5.5) in the sense of definition 5.2.13. Let n be any positive integer. Let t be any real number in $[0;T)$. We define

$$\beta_n(v(t;\cdot)) := \frac{2}{\pi}\int_0^{\pi} v(t;x)\sin(nx)dx.$$

Let n be any positive integer. Let t be any real number in $(0;T)$. We define

$$\beta_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right) := \frac{2}{\pi}\int_0^{\pi}\frac{\partial v}{\partial t}(t;x)\sin(nx)dx.$$

Since v is continuous in $[0;T)\times[0;\pi]$, we can apply theorem 2.3.1 and we have that for all n in $\mathbb{N}^*$ the function $\beta_n(v(-;\cdot)):[0;T)\to\mathbb{C}$ is continuous. Since $\frac{\partial v}{\partial t}$ is continuous in $(0;T)\times[0;\pi]$, we can apply theorem 2.3.2 and we have that if n is any positive integer, then $\beta_n(v(t;\cdot))$ is in $C^1((0;T))$ and for all t in $(0;T)$ it holds that

$$\beta_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right) = \frac{2}{\pi}\int_0^{\pi}\frac{\partial v}{\partial t}(t;x)\sin(nx)dx = \beta_n(v(t;\cdot))'.$$

We recall that v is such that for all t in $(0;T)$ it holds that

$$v(t;\pi) = v(t;-\pi) = 0.$$

Let n be any positive integer; we define

$$\beta_n\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) := \frac{2}{\pi}\int_0^{\pi}\frac{\partial^2 v}{\partial^2 x}(t;x)\sin(nx)dx.$$

Thanks to lemma 5.2.8, for all integer n it holds that

$$\begin{aligned}\beta_n\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) &= \frac{2}{\pi}\int_0^{\pi}\frac{\partial^2 v}{\partial x^2}(t;x)\sin(nx)dx\\ &= \frac{-2n^2}{\pi}\int_0^{\pi} v(t;x)\sin(nx)dx\\ &= -n^2\beta_n(v(t;\cdot)).\end{aligned}$$

We recall that v is such that for all $(t; x)$ in $(0; T) \times [0; \pi]$ it holds that

$$\frac{\partial v}{\partial t}(t; x) = \frac{\partial^2 v}{\partial x^2}(t; x).$$

Hence, for all positive integer n for all t in $(0; T)$, we have that

$$\beta_n\left(\frac{\partial v}{\partial t}(t; \cdot)\right) = \beta_n\left(\frac{\partial^2 v}{\partial x^2}(t; \cdot)\right).$$

In other words, for all positive integer n for all t in $(0; T)$ it holds that

$$\beta_n(v(t; \cdot))' = -n^2 \beta_n(v(t; \cdot)).$$

Since $v(0; x) = u_0(x)$ for all x in $[0; \pi]$, we can state that

$$\beta_n(v(0; \cdot)) = \beta_n^0.$$

We have that $\beta_n(v(t; \cdot))$ is a solution of the following differential problem

$$\begin{cases} y'(t) = -n^2 y(t) & \text{if } t > 0, \\ y(0) = 0, \end{cases}$$

and it is continuous in 0; this is equivalent to state that $\beta_n(v(t; \cdot))$ is a solution of the following Cauchy's problem:

$$\begin{cases} y'(t) = -n^2 y(t) & \text{if } t \geq 0, \\ y(0) = 0. \end{cases}$$

Hence, we have that for all t in $[0; +\infty)$ it holds that

$$\beta_n(v(t; \cdot)) = \beta_n^0 e^{-n^2 t}.$$

Hence, for all t in $[0; +\infty)$ the function $v\big|_{\{t\}\times[0;\pi]}$ coincides with the function $u\big|_{\{t\}\times[0;\pi]}$, i.e. for all $(t; x)$ in $[0; +\infty) \times [0; \pi]$ it holds that $v(t; x) = u(t; x)$. □

Heat equation with homogeneous Neumann boundary conditions

Definition 5.2.15. Let $u_0 : [0; \pi] \to \mathbb{C}$ be any function. Let us consider the following partial derivative equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t; x) = \dfrac{\partial^2 u}{\partial x^2}(t; x) & \text{if } (t; x) \in (0; T) \times [0; \pi] \\ \dfrac{\partial u}{\partial x}(t; \pi) = \dfrac{\partial u}{\partial x}(t; -\pi) = 0 & \text{if } t \in (0; T) \\ u(0; x) = u_0(x) & \text{if } x \in [0; \pi] \end{cases} \tag{5.6}$$

We say that (5.6) is the heat equation in $[0; \pi]$ with homogeneous Neumann boundary conditions.

Definition 5.2.16. Let $u_0 : [0; \pi] \to \mathbb{C}$ be any function; let T be any positive real number. We say that $u : [0; T) \times [0; \pi] \to \mathbb{C}$ is a solution of (5.6) if it has the following properties:

- u is continuous in $[0;T) \times [0;\pi]$;
- for all $(t;x)$ in $(0;T) \times [0;\pi]$, there exists
$$\frac{\partial^2 u}{\partial x^2}(t;x)$$
and it is continuous in $(0;T) \times [0;\pi]$;
- for all $(t;x)$ in $(0;T) \times [0;\pi]$, there exists
$$\frac{\partial u}{\partial t}(t;x)$$
and it is continuous in $(0;T) \times [0;\pi]$;
- for all $(t;x)$ in $(0;T) \times [0;\pi]$ the following identity holds true:
$$\frac{\partial^2 u}{\partial x^2}(t;x) = \frac{\partial u}{\partial t}(t;x);$$
- for all t in $(0;T)$ it holds that
$$\frac{\partial u}{\partial x}(t;\pi) = \frac{\partial u}{\partial x}(t;-\pi) = 0;$$
- for all x in $[0;\pi]$ it holds that
$$u(0;x) = u_0(x).$$

We are looking for reasonable hypothesis on u_0 to make sure that there exists a time T in $(0;+\infty)$ and a function $u : [0;T) \times [0;\pi] \to \mathbb{C}$ that is a solution of (5.6) in the sense of definition 5.2.16.

Theorem 5.2.17 (Existence and uniqueness of the solution for heat equation with homogeneous Neumann boundary conditions)**.**
Let $u_0 : [0;\pi] \to \mathbb{C}$ be any function in $L^2_{\mathbb{C}}((0;\pi))$; let us define the real Fourier coefficient $\{\alpha^0_n\}_{n\in\mathbb{N}}$ as in 5.2.9. Let us assume that
$$\sum_{n\in\mathbb{N}} \left|\alpha^0_n\right| < +\infty.$$
Let $u : [0;+\infty) \times [0;\pi] \to \mathbb{C}$ such that
$$u(t;x) \coloneqq \sum_{n\in\mathbb{N}} \alpha^0_n e^{-n^2 t} \cos(nx).$$
Then the following conclusions hold true:

- *u is a well defined complex-valued function in $[0;+\infty) \times [0;\pi]$;*
- *u is in $C^\infty((0;+\infty) \times [0;\pi])$;*
- *u is a solution of (5.6) in the sense of definition 5.2.16;*

- *if u_0 is a real-valued function, then u is a real-valued function;*
- *if v is a solution of* (5.6) *in the sense of 5.2.16, then v is equal to u.*

Proof. **Step 1**: If we slightly modify the procedure described in many details in the first step of theorem 5.1.26, we immediately obtain the following statements:

- u is a well defined complex-valued function in $[0; +\infty) \times [0; \pi]$;
- u is in $C^\infty((0; +\infty) \times [0; \pi])$;
- u is a solution of (5.6) in the sense of definition 5.2.16;
- if u_0 is a real-valued function, then u is a real-valued function.

Step 2: Let v be any function of (5.6) in the sense of definition 5.2.16. Let n be a positive integer. Let t be any real number in $[0; T)$. We define

$$\alpha_n(v(t; \cdot)) := \frac{2}{\pi} \int_0^\pi v(t; x) \cos(nx) dx.$$

Let n be a positive integer. Let t be any real number in $(0; T)$. We define

$$\alpha_n \left(\frac{\partial v}{\partial t}(t; \cdot) \right) := \frac{2}{\pi} \int_0^\pi \frac{\partial v}{\partial t}(t; x) \cos(nx) dx.$$

Similarly, for all t in $[0; T)$ we define

$$\alpha_0(v(t; \cdot)) := \frac{1}{\pi} \int_0^\pi v(t; x) dx;$$

for all t in $(0; T)$ we define

$$\alpha_0 \left(\frac{\partial v}{\partial t}(t; \cdot) \right) := \frac{1}{\pi} \int_0^\pi \frac{\partial v}{\partial t}(t; x) dx.$$

Since v is continuous in $[0; T) \times [0; \pi]$, we can apply theorem 2.3.1 and we have that for all n in $\mathbb{N}$ the function $\alpha_n(v(-; \cdot)) : [0; T) \to \mathbb{C}$ is continuous. Since $\frac{\partial v}{\partial t}$ is continuous in $(0; T) \times [0; \pi]$, we can apply theorem 2.3.2 and we have that if n is any positive integer, then $\alpha_n(v(t; \cdot))$ is in $C^1((0; T))$ and for all t in $(0; T)$ it holds that

$$\alpha_n \left(\frac{\partial v}{\partial t}(t; \cdot) \right) = \frac{2}{\pi} \int_0^\pi \frac{\partial v}{\partial t}(t; x) \cos(nx) dx = \alpha_n(v(t; \cdot))'.$$

Similarly, we can also state that $\alpha_0(v(t; \cdot))$ is in $C^1((0; T))$ and for all t in $(0; T)$ it holds that

$$\alpha_0 \left(\frac{\partial v}{\partial t}(t; \cdot) \right) = \frac{1}{\pi} \int_0^\pi \frac{\partial v}{\partial t}(t; x) dx = \alpha_n(v(t; \cdot))'.$$

We recall that v is such that for all t in $(0; T)$ it holds that

$$\frac{\partial v}{\partial x}(t; \pi) = \frac{\partial v}{\partial x}(t; -\pi) = 0.$$

Let n be any positive integer; we define

$$\alpha_n\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) := \frac{2}{\pi}\int_0^\pi \frac{\partial^2 v}{\partial^2 x}(t;x)\cos(nx)dx.$$

Thanks to lemma 5.2.11, for all positive integer n it holds that

$$\begin{aligned}\alpha_n\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) &= \frac{2}{\pi}\int_0^\pi \frac{\partial^2 v}{\partial x^2}(t;x)\cos(nx)dx\\ &= \frac{-2n^2}{\pi}\int_0^\pi v(t;x)\cos(nx)dx\\ &= -n^2\alpha_n(v(t;\cdot)).\end{aligned}$$

Similarly, we define

$$\alpha_0\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) := \frac{1}{\pi}\int_0^\pi \frac{\partial^2 v}{\partial^2 x}(t;x)dx;$$

thanks to the fundamental theorem of calculus and our assumption on v, we can state that for all t in $(0;T)$ it holds that

$$\alpha_0\left(\frac{\partial^2 v}{\partial^2 x}(t;\cdot)\right) = \frac{\partial v}{\partial x}(t;\pi) - \frac{\partial v}{\partial x}(t;0) = 0.$$

We recall that v is such that for all $(t;x)$ in $(0;T)\times[0;\pi]$ it holds that

$$\frac{\partial v}{\partial t}(t;x) = \frac{\partial^2 v}{\partial x^2}(t;x).$$

Hence, for all n in $\mathbb{N}$ for all t in $(0;T)$, we have that

$$\alpha_n\left(\frac{\partial v}{\partial t}(t;\cdot)\right) = \alpha_n\left(\frac{\partial^2 v}{\partial x^2}(t;\cdot)\right).$$

In other words, for all n in $\mathbb{N}$ for all t in $(0;T)$ it holds that

$$\alpha_n(v(t;\cdot))' = -n^2\alpha_n(v(t;\cdot)).$$

Since $v(0;x) = u_0(x)$ for all x in $[0;\pi]$, we can state that

$$\alpha_n(v(0;\cdot)) = \alpha_n^0.$$

We have that $\alpha_n(v(t;\cdot))$ is a solution of the following differential problem

$$\begin{cases} y'(t) = -n^2 y(t) & \text{if } t > 0,\\ y(0) = 0, \end{cases}$$

and it is continuous in 0; this is equivalent to state that $\alpha_n(v(t;\cdot))$ is a solution of the following Cauchy's problem:

$$\begin{cases} y'(t) = -n^2 y(t) & \text{if } t \geq 0,\\ y(0) = 0. \end{cases}$$

Hence, we have that for all n in $\mathbb{N}$ for all t in $[0;+\infty)$ it holds that

$$\beta_n(v(t;\cdot)) = \beta_n^0 e^{-n^2 t}.$$

So, for all t in $[0;+\infty)$ the function $v_{\{t\}\times[0;\pi]}$ coincides with the function $u_{\{t\}\times[0;\pi]}$, i.e. for all $(t;x)$ in $[0;+\infty)\times[0;\pi]$ it holds that $v(t;x) = u(t;x)$. □

5.3 Appendix

5.3.1 Stone-Weierstrass theorem

Definition 5.3.1. Let $\mathbb{X}$ a topological space. We denote as $C(\mathbb{X};\mathbb{R})$ the set of the continuous functions among $\mathbb{X}$ and $\mathbb{R}$; we denote as $C(\mathbb{X};\mathbb{C})$ the set of the continuous functions among $\mathbb{X}$ and $\mathbb{C}$.

Definition 5.3.2. Let $\mathbb{X}$ be a topological space. Let $\mathscr{A}$ be any subset of $C(\mathbb{X};\mathbb{R})$ or $C(\mathbb{X};\mathbb{C})$. We say that $\mathscr{A}$ is an algebra if it is vector space closed under multiplication.

Let us assume that for all x_1, x_2 in K such that $x_1 \neq x_2$, there exists a continuous function in $\mathscr{A}$ between $\mathbb{X}$ and $\mathbb{R}$ (or $\mathbb{C}$) such that $f(x_1) \neq f(x_2)$. We say that $\mathcal{A}$ separates points.

Let us assume that $\mathscr{A}$ is a subset of $C(\mathbb{X};\mathbb{C})$. We say that $\mathscr{A}$ is closed under complex conjunction if f in $\mathscr{A}$ if and only if $\overline{f}$ is in $\mathcal{A}$, where $\overline{f} : \mathbb{X} \to \mathbb{C}$ is such that $\overline{f}(x) \coloneqq \overline{f(x)}$.

Remark 5.3.3. Let $\mathbb{X}$ be a topological space. It's immediate to see that if there exists an algebra of complex-valued or real-valued continuous functions, then $\mathbb{X}$ is an Hausdorff space.

First of all, we state and prove some useful lemmas.

Lemma 5.3.4. *Let M be a positive real number. There exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ that converges toward $f(t) \coloneqq \sqrt{t}$ uniformly in $[0;M]$ and such that $p_n(0) = 0$ for all n in $\mathbb{N}$.*

Proof. We notice that if $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of polynomials that converges toward f uniformly in $[0;M]$, then $\{q_n - q_n(0)\}_{n\in\mathbb{N}}$ satisfies all the requests. Let q_n be the Taylor polynomial of degree n. We claim that $\{q_n\}_{n\in\mathbb{N}}$ converges toward f uniformly in $[\varepsilon; 2M-\varepsilon]$ for all ε greater that 0. Let $g : \mathbb{C} \setminus \mathbb{R}^{\leq 0} \to \mathbb{C}$ be the holomorphic function such that $g(z) \coloneqq \sqrt{z}$. Let ε be a positive real number. Since g is holomorphic, there exists a power series centered in M that converges toward f totally in $\overline{\mathcal{B}(M; M-\varepsilon)}$, i. e.

$$g(z) = \sum_{n\in\mathbb{N}} \frac{g^{(n)}(M)}{n!}(z-M)^n,$$

where the right hand side series converges totally toward g in $\overline{\mathcal{B}(M; M-\varepsilon)}$. However, it's easy to see that $g^{(n)}(M) = f^{(n)}(M)$ for all n in $\mathbb{N}$; in other words, $g^{(n)}(M)$ is a real number for all n in $\mathbb{N}$. So, for all n in $\mathbb{N}$ we can define $q_n : \mathbb{R} \to \mathbb{R}$ such that

$$q_n(t) \coloneqq \sum_{i=0}^{n} \frac{g^{(n)}(M)}{n!}(t-M)^n.$$

By restriction, $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued polynomials that converges uniformly toward f in $[\varepsilon; 2M-\varepsilon]$ for all ε greater than 0. For all k in $\mathbb{N}$ there exists n_k in $\mathbb{N}$ such that

$$\sup_{2^{-k};2M-2^{-k}} \left\{ \left| q_{n_k}(t) - \sqrt{t} \right| \right\} \leq 2^{-k}.$$

Obviously, we can assume that the sequence $\{n_k\}_{k\in\mathbb{N}}$ is strictly monotonically increasing. For all k in $\mathbb{N}$ we define $p_k : \mathbb{R} \to \mathbb{R}$ such that

$$p_k(t) \coloneqq q_{n_k}(t + 2^{-k}).$$

The sequence $\{p_k\}_{k\in\mathbb{N}}$ converges toward f uniformly in $[0; M]$. In fact, if k is such that $2^{-k} \le M$, for all t in $[0; M]$ we have that

$$\left|p_k(t) - \sqrt{t}\right| \le \left|q_{n_k}(t + 2^{-k}) - \sqrt{t + 2^{-k}}\right| + \left|\sqrt{t + 2^{-k}} - \sqrt{t}\right| \le 2^{-k} + \sqrt{2^{-k}}.$$

□

Lemma 5.3.5. *Let $\mathscr{A}$ be an algebra closed in $C(\mathbb{K};\mathbb{R})$. Then, f belongs to $\mathscr{A}$ implies that $|f|$ belongs to $\mathscr{A}$. In particular, if f, g are functions in $\mathscr{A}$, then the pointwise maximum* $\max\{f; g\}$ *and the pointwise minimum* $\min\{f; g\}$ *are in $\mathscr{A}$.*

Proof. Since f is continuous and $\mathbb{K}$ is a compact space, there exists a real number M such that

$$M := \max_{\mathbb{K}}\{f(x)^2\}.$$

Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials that converges toward $g(t) := \sqrt{t}$ uniformly in $[0; M]$ and such that $p_n(0) = 0$ for all n in $\mathbb{N}$ (see lemma 5.3.4). Since $\mathscr{A}$ is an algebra and p_n does not have the term of degree 0, we have that $\{p_n(f)\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ that converges uniformly in $\mathbb{K}$ toward $\sqrt{f^2} = |f|$. Since $\mathscr{A}$ is closed under uniform convergence, we can conclude that $|f|$ is in $\mathscr{A}$.

In conclusion, we notice that, if f_1, f_2 are functions in $\mathscr{A}$, then

$$\max\{f_1; f_2\} = \frac{f_1 + f_2 + |f_1 + f_2|}{2},$$

$$\min\{f_1; f_2\} = \frac{f_1 + f_2 - |f_1 - f_2|}{2}.$$

So, the pointwise maximum and the pointwise minimum belongs to $\mathscr{A}$. □

Lemma 5.3.6. ***Step 2****: Let $\mathscr{A}$ be a set in $C(\mathbb{K};\mathbb{R})$ with the following properties:*

- *if f, g are in $\mathscr{A}$, then the pointwise maximum and the pointwise minimum are in $\mathscr{A}$;*
- *for all x_1, x_2 in $\mathbb{K}$ such that $x_1 \ne x_2$, for all y_1, y_2 in $\mathbb{R}$ (they can also coincide) there exists a function g in $\mathscr{A}$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.*

So $\mathscr{A}$ is dense in $C(\mathbb{K};\mathbb{R})$.

Proof. Let f be a function in $C(\mathbb{K};\mathbb{R})$; let ε be a positive real number. We are looking for a function h in $\mathscr{A}$ such that for all x in $\mathbb{K}$ it holds that

$$f(x) - \varepsilon \le h(x) \le f(x) + \varepsilon.$$

We notice that for all x, x' in $\mathbb{K}$ there e exists a function $g_{x;x'}$ in $\mathscr{A}$ such that $g_{x;x'}(x) = f(x)$ and $g_{x;x'}(x') = f(x')$. Let us fix x in $\mathbb{K}$. For all x' in $\mathbb{K}$, let us consider $g_{x;x'}$ such that $g_{x;x'}(x') = f(x') < f(x') + \varepsilon$. Since $g_{x;x'}$ and f are continuous functions, there exists an open set $U_{x'}$ such that x' belongs to $U_{x'}$ and $g_{x;x'}(y) < f(y) + \varepsilon$ for all y in $U_{x'}$. The collection $\{U_{x'} \mid x' \in \mathbb{K}\}$ covers $\mathbb{K}$; since $\mathbb{K}$ is a compact space, there exists a positive integer n and $\{x'_1; \dots; x'_n\}$ in $\mathbb{K}$ such that

$$\mathbb{K} = \bigcup_{i=1}^{n} U_{x'_i}.$$

We denote

$$h_x \coloneqq \min\{g_{x;x'_1}; \dots; g_{x;x'_n}\}.$$

Thanks to our hypothesis, h_x is a function in $\mathscr{A}$. By definition of h_x it immediately follows that $h_x(x) = f(x)$ and $h_x(y) \leq f(y) + \varepsilon$ for all y in $\mathbb{K}$. Since h_x and f are continuous functions, there exists an open set V_x such that x belongs to V_x and $h_x(y) > f(y) - \varepsilon$ for all y in V_x. The collection $\{V_x \mid x \in \mathbb{K}\}$ covers $\mathbb{K}$; since $\mathbb{K}$ is a compact space, there exists a positive integer m and $\{x_1; \dots; x_m\}$ in $\mathbb{K}$ such that

$$\mathbb{K} = \bigcup_{i=1}^{m} V_{x_i}.$$

We define

$$h \coloneqq \max\{h_{x_1}; \dots; h_{x_m}\}.$$

Thanks to our hypothesis, h is a function in $\mathscr{A}$. In conclusion, the following inequalities are an immediate consequence of the definitions given:

$$f(x) - \varepsilon \leq h(x) \leq f(x) + \varepsilon.$$

□

Theorem 5.3.7 (Stone-Weierstrass theorem).
Let $\mathbb{K}$ be a compact Hausdorff topological space. Let $\mathscr{A}$ be an algebra of real-valued continuous function that separates points and such that the constant functions belong to $\mathscr{A}$. Then, $\mathscr{A}$ is dense in $C(\mathbb{K}; \mathbb{R})$ with respect to the norm of the uniform convergence.

Let $\mathscr{A}$ is an algebra of complex-valued continuous functions that separates point, it is closed under complex conjunction and it is such that the constant functions belong to $\mathscr{A}$, then $\mathscr{A}$ is dense in $C(\mathbb{K}; \mathbb{C})$ with respect to the norm of the uniform convergence.

Proof. **Step 1**: Let us assume that $\mathscr{A}$ is in $C(\mathbb{K}; \mathbb{R})$. Let $\mathscr{A}'$ the closure of $\mathscr{A}$ in $C(\mathcal{K}; \mathbb{R})$. Thanks to the algebraic properties of the uniform limit, $\mathscr{A}'$ is an algebra in $C(\mathcal{K}; \mathbb{R})$ that separates points and such that the constant functions are in $\mathscr{A}'$. Thanks to lemma 5.3.5, is f, g are in $\mathscr{A}'$, then the pointwise maximum and the pointwise minimum are in $\mathscr{A}$. Let x_1, x_2 be in $\mathbb{K}$ such that $x_1 \neq x_2$; let y_1, y_2 be real numbers. Under our hypothesis, there exists a function f in $\mathscr{A}$ such that $f(x_1) \neq f(x_2)$. So, there exist real numbers α, β such that $\alpha + \beta f(x_1) = y_1$ and $\alpha + \beta f(x_2) = y_2$. We notice that the function $g \coloneqq \alpha + \beta f$ belongs to $\mathscr{A}$. So, we can apply lemma 5.3.6 and we can conclude that $\mathscr{A}'$ is dense in $C(\mathbb{K}; \mathbb{R})$. In particular, $\mathscr{A}$ is dense in $C(\mathbb{K}; \mathbb{R})$.

Step 2: Let us assume that $\mathscr{A}$ is in $C(\mathbb{K}; \mathbb{C})$. Let $\mathscr{A}''$ be the set of the real-valued functions in $\mathscr{A}$. Obviously, $\mathscr{A}''$ is an algebra that contains the real-valued constant functions. We claim that $\mathscr{A}''$ separates points. Let x_1, x_2 be in $\mathbb{K}$ such that $x_1 \neq x_2$. Let f be a complex valued function in $\mathscr{A}$ such that $f(x_1) \neq f(x_2)$. In particular, we have that $\Re f(x_1) \neq \Re f(x_2)$ or $\Im f(x_1) \neq \Im f(x_2)$. We recall that $\mathscr{A}$ is closed under complex conjunction and the following identities hold true:

$$\Re f = \frac{f + \overline{f}}{2},$$

$$\Im f = \frac{f - \overline{f}}{2i}.$$

So, $\Re f$ and $\Im f$ are real-valued functions in $\mathscr{A}$; in particular, they belongs to $\mathscr{A}''$. This is enough to conclude that $\mathscr{A}''$ separates points. Thanks to the previous step, $\mathscr{A}''$ is dense in $C(\mathbb{K};\mathbb{R})$. To conclude, for all f in $C(\mathbb{K};\mathbb{C})$ it holds that $f = \Re f + i\Im f$. So, $\mathscr{A}$ is dense in $C(\mathbb{K};\mathbb{C})$. □

Corollary 5.3.8 (Weierstrass theorem)**.**
Let $\mathbb{K}$ be a compact subset in $\mathbb{R}$. Then, the collection of the real polynomials between $\mathbb{K}$ and $\mathbb{R}$ is dense in $C(\mathbb{K};\mathbb{R})$.

Let $\mathbb{K}$ be a compact subset in $\mathbb{C}$. Then, the collection of the complex polynomials between $\mathbb{K}$ and $\mathbb{C}$ is dense in $C(\mathbb{K};\mathbb{C})$.

Proof. The proof is an immediate consequence of theorem 5.3.7. □

Remark 5.3.9. In conclusion, we notice that Weierstrass proved theorem 5.3.8 in 1885 and Stone showed its most general version (see 5.3.7) in 1937.

5.3.2 Isoperimetric inequality in dimension 2

We recall the Gauss-Green formula.

Theorem 5.3.10 (Gauss-Green formula)**.**
Let A be a bounded open set in $\mathbb{R}^2$ such that there exists a closed path $\gamma : [a;b] \to \mathbb{R}^2$ with the following properties:

- *γ is in $C^1([a;b])$;*
- *$\gamma(a) = \gamma(b)$;*
- *γ is a counter-clockwise parameterization of the boundary of A.*

Let ω be a differential form in any open neighborhood D of $\overline{A}$, i. e. there exist functions $P, Q : D \to \mathbb{R}$ in $C^1(D)$ such that

$$\omega(x;y) \coloneqq P(x;y)dx + Q(x;y)dy.$$

Then, the following identity holds true:

$$\int_\gamma \omega = \int_A \left(\frac{\partial Q}{\partial x}(x;y) - \frac{\partial P}{\partial y}(x;y) \right) dxdy.$$

Remark 5.3.11. In the hypothesis of theorem 5.3.10, we recall that

$$\mathscr{L}^1(A) = \int_A 1dxdy.$$

We define the differential form ω in $\mathbb{R}^2$ such that

$$\omega(x;y) \coloneqq -\frac{1}{2}(ydx - xdy).$$

Thanks to the Gauss-Green formula (see 5.3.10), the following inequalities hold true:

$$\int_\gamma \omega = \int_A 1dxdy = \mathscr{L}^1(A).$$

Definition 5.3.12. Let A be a bounded open set in $\mathbb{R}^2$ such that there exists a closed path $\gamma : [a; b] \to \mathbb{R}^2$ with the following properties:

- γ is in $C^1([a; b])$;
- $\gamma(a) = \gamma(b)$;
- γ is a counter-clockwise parameterization of the boundary of A.

If we define the perimeter of A as

$$\text{per}\,(A) = \int_a^b |\gamma'(t)|\, dt.$$

Remark 5.3.13. It can be proved that the perimeter of A does not depend on the specific parameterization.

Proposition 5.3.14. *Let A be a bounded open set in $\mathbb{R}^2$ such that there exists a closed path $\gamma : [\pi; \pi] \to \mathbb{R}^2$ with the following properties:*

- *γ is in $C^1([-\pi; \pi])$;*
- *$\gamma(-\pi) = \gamma(\pi)$;*
- *γ is a counter-clockwise parameterization of the boundary of A.*

Let us define the perimeter of A as in 5.3.12. Then, the following inequality holds true:

$$4\pi\, \mathscr{L}^1(A) \leq per\,(A)^2\,.$$

Moreover, it holds that

$$4\pi\, \mathscr{L}^1(A) = per\,(A)^2$$

if and only if A is the circle.

Proof. We recall that per (A) does not depend on the specific parameterization. We will prove the statement under the further assumption that $|\gamma'|$ is a constant function. In other words, we are assuming that for all t in $[-\pi; \pi]$ it holds that

$$|\gamma'(t)| = \frac{\text{per}\,(A)}{2\pi}.$$

Moreover, we can identify $\mathbb{R}^2$ with the complex plane; hence γ is a complex-valued function in $C^1([-\pi; \pi])$. We define the sequence of the Fourier coefficients $\{c_n(\gamma)\}_{n\in\mathbb{Z}}$ as in 5.1.1; thanks to 5.1.10, for all integer n it holds that $c_n(\gamma') = inc_n(\gamma)$. Be definition of perimeter, we have that

$$\text{per}\,(A)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{per}\,(A)^2\, dt = 2\pi \int_{-\pi}^{\pi} |\gamma'(t)|^2\, dt = 2\pi\, \|\gamma'\|^2_{L^2((-\pi;\pi))}\,.$$

If we use the Parseval's identity, we obtain that

$$\text{per}\,(A)^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} n^2\, |c_n(\gamma)|^2\,.$$

Since the $xdx + ydy$ is an exact differential form (the potential is $d\frac{x^2}{2} + d\frac{y^2}{2}$), we have that

$$\int_\gamma xdx + ydy = 0.$$

Hence, the following identities hold true:

$$\int_{-\pi}^{\pi} \overline{\gamma(t)}\gamma'(t)dt = \int_\gamma \overline{z}dz = \int_\gamma xdx + ydy + i(xdy - ydx) = i\int_\gamma xdx - ydy.$$

We also know that

$$\begin{aligned}
\mathscr{L}^1(A) &= -\frac{1}{2}\int_\gamma ydx - xdy \\
&= -\frac{i}{2}\int_{-\pi}^{\pi} \overline{\gamma(t)}\gamma'(t)dt \\
&= -\frac{i}{2}2\pi\sum_{n\in\mathbb{Z}}(inc_n(\gamma))\overline{c_n(\gamma)} \\
&= \pi\sum_{n\in\mathbb{Z}} n\left|c_n(\gamma)\right|^2.
\end{aligned}$$

We have shown that

$$4\pi\,\mathscr{L}^1(A) = \pi\sum_{n\in\mathbb{Z}} n\left|c_n(\gamma)\right|^2 \le \pi\sum_{n\in\mathbb{Z}} n^2\left|c_n(\gamma)\right|^2 = \operatorname{per}(A)^2.$$

It's immediate to see that

$$4\pi\,\mathscr{L}^1(A) = \operatorname{per}(A)^2$$

if and only if $c_n(\gamma) = 0$ for all integer n in $\mathbb{Z}\setminus\{0;1\}$, that is equivalent to require that

$$\gamma(t) = c_0(\gamma) + c_1(\gamma)e^{it},$$

that is the counter-clockwise parameterization of a circumference. □

5.3.3 Fourier series in $L^2_{\mathbb{C}}((-\pi;\pi)^d)$

Let us denote as $\mathbb{F}$ the real field or the complex field.

Definition 5.3.15. For all n in $\mathbb{Z}^d$ we define $e_n : [-\pi;\pi]^d \to \mathbb{C}$ such that

$$e_n(x) := \frac{1}{(2\pi)^{\frac{d}{2}}}e^{-i<n,x>}.$$

We also denote

$$\mathscr{F}^d := \{e_n \mid n \in \mathbb{Z}^d\}.$$

Theorem 5.3.16. *Let us define $\mathscr{F}^d$ as in 5.3.15. Then $\mathscr{F}^d$ is an Hilbert's basis of $L^2_{\mathbb{C}}((-\pi;\pi)^d)$.*

Proof. Thanks to Fubini's theorem, it's immediate to see that $\mathscr{F}^d$ is an orthonormal set. As a for the maximality, it can be proved via Stone-Weierstrass theorem (see 5.3.7): the proof is completely similar to theorem 5.1.4. □

Remark 5.3.17. Thanks to theorem 5.3.16, the theory developed in this chapter can be easily generalized in any dimension.

Remark 5.3.18. Theorem 5.3.16 is not surprising. In fact, for all n in $\mathbb{Z}^d$, it's immediate to see that

$$\frac{1}{(2\pi)^{\frac{d}{2}}} e^{-i<n,x>} = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} e^{-in_j x_j}.$$

As a matter of facts, the maximality of $\mathscr{F}^d$ can be an immediate consequence of the following theorem.

Theorem 5.3.19. *Let* $(\mathbb{E}_1; \mathscr{E}_1; \mu_1), (\mathbb{E}_2; \mathscr{E}_2; \mu_2)$ *be measurable spaces with measures* μ_1, μ_2. *Let us define the tensor product* σ*-algebra and the product measure as in 2.2.20, i.e.*

$$\mathscr{E} := \mathscr{E}_1 \otimes E_2,$$

$$\mu := \mu_1 \otimes \mu_2.$$

Let $\mathscr{F}_1, \mathscr{F}_2$ *be respectively Hilbert's basis of* $L^2(\mathbb{E}_1)$ *and* $L^2(\mathbb{E}_2)$*. We denote*

$$\mathscr{F}_1 := \{f_j \mid j \in \mathscr{J}\},$$

$$\mathscr{F}_2 := \{g_i \mid i \in \}.$$

For all $(j; i)$ *in* $\mathscr{J} \times \mathscr{I}$*, we denote* $h_{j;i} : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{C}$ *as*

$$h_{j;i}(x) := f_j(x) g_i(y).$$

If we denote

$$\mathscr{F} := \{h_{i;j} \mid (j; i) \in \mathscr{J} \times \mathscr{I}\},$$

it is an Hilbert's basis of $L^2(\mathbb{E}_1 \times \mathbb{E}_2)$*.*

Proof. Thanks to Fubini's theorem, it's immediate to see that $\mathscr{F}$ is an orthonormal set. We recall that the step function are dense in $L^2(\mathbb{E}_1 \times \mathbb{E}_2)$ (see 3.1.34). Thanks to 3.1.33, it is enough to show that $\overline{\text{Span}(\mathscr{F})}$ contains all the indicator functions of measurable sets in $\mathbb{E}_1 \times \mathbb{E}_2$. We define

$$\mathscr{K} := \left\{ E \in \mathscr{E} \mid \mathbb{1}_E \in \overline{\text{Span}(\mathscr{F})} \right\}.$$

We have to show that $\mathscr{K}$ is equal to $\mathscr{E}$. Let E_1, E_2 be measurable sets respectively in $\mathbb{E}_1$ and $\mathbb{E}_2$. Let us denote $E := E_1 \times E_2$. We claim that E in in $\mathscr{K}$. Let $\{g_n\}_{n \in \mathbb{N}}$ a sequence in $\text{Span}(\mathscr{F}_1)$ that converges toward $\mathbb{1}_{E_1}$ with respect to L^2 norm. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $\text{Span}(\mathscr{F}_2)$ that converges toward $\mathbb{1}_{E_2}$ with respect to L^2 norm. For all n in $\mathbb{N}$, we define $h_n : \mathbb{E}_1 \times \mathbb{E}_2 :\to \mathbb{C}$ such that

$$h_n(x; y) := g_n(x) f_n(y).$$

We notice that $\{h_n\}_{n \in \mathbb{N}}$ is a sequence in $\text{Span}(\mathscr{F})$; we claim that $\{h_n\}_{n \in \mathbb{N}}$ converges toward $\mathbb{1}_E$ with respect to L^2 norm. For all n in $\mathbb{N}$ we define $p_n : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{C}$ such that

$$p_n(x; y) := g_n(x) \mathbb{1}_{E_2}(y).$$

We notice that

$$\mathbb{1}_{E_1 \times E_2}(x; y) = \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(y).$$

Thanks to triangular inequality, for all n in $\mathbb{N}$ it holds that

$$\|h_n - \mathbb{1}_E\|_{L^2(\mathbb{E}_1\times\mathbb{E}_2)} \leq \|h_n - p_n\|_{L^2(\mathbb{E}_1\times\mathbb{E}_2)} + \|p_n - \mathbb{1}_E\|_{L^2(\mathbb{E}_1\times\mathbb{E}_2)}.$$

Thanks to Fubini's theorem, we have that

$$\begin{aligned}\lim_{n\to+\infty} \int_{\mathbb{E}_1\times\mathbb{E}_2} & |g_n(x)f_n(y) - g_n(x)\mathbb{1}_{E_2}(y)|^2 \, d(\mu_1\otimes\mu_2)(x;y) \\ & \int_{\mathbb{E}_1} |g_n(x)|^2 \left(\int_{\mathbb{E}_2} |f_n(y) - \mathbb{1}_{E_2}(y)|^2 \, d\mu_2(y)\right) d\mu_1(x) \\ & = \lim_{n\to+\infty} \|g_n\|^2_{L^2(\mathbb{E}_1)} \|f_n - \mathbb{1}_{E_2}\|^2_{L^2(\mathbb{E}_2)} = 0.\end{aligned}$$

The last identity is an consequence of the fact that $\left\{\|g_n\|_{L^2(\mathbb{E}_1)}\right\}_{n\in\mathbb{N}}$ is a bounded sequence. Similarly, it can be shown that

$$\lim_{n\to+\infty} \|p_n - \mathbb{1}_E\|_{L^2(\mathbb{E}_1\times\mathbb{E}_2)} = 0.$$

We have shown that the measurable boxes belong to $\mathscr{K}$. Moreover, it can be easily proved that $\mathscr{K}$ is a σ-algebra. Since $\mathscr{E}$ is the σ-algebra generated by the collection of the measurable boxes, then $\mathscr{K}$ is equal to $\mathscr{E}$. $\square$

Chapter 6

Fourier transform

6.1 Fourier transform in L^1

6.1.1 Definition and main properties

Definition 6.1.1. Let f be any measurable function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$. We define the Fourier transform $\mathcal{F}f : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\mathcal{F}f(y) = \int_{\mathbb{R}^d} f(x)e^{-i<x;y>}dx.$$

Remark 6.1.2. It's easy to see that definition 6.1.1 is well posed, namely the right hand side is finite for every y in $\mathbb{R}^d$. Moreover, $\mathcal{F}f$ is continuous function: if y is any vector in $\mathbb{R}^d$ and $\{y_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathbb{R}^d$ that converges toward y, then

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} f(x)(e^{-i<x;y_n>} - e^{-i<x;y>})dx = 0.$$

Since the point-wise convergence is obvious and $2|f|$ is a suitable domination in $L^1_{\mathbb{C}}(\mathbb{R}^d)$, the statement in an immediate consequence of dominated convergence theorem.

Lemma 6.1.3 (Riemann-Lebesgue' lemma)**.**
Let f be any measurable function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$; then, it holds that

$$\lim_{|y|\to+\infty} \mathcal{F}f(y) = 0,$$

namely $\mathcal{F}f$ is in $C_0(\mathbb{R}^d)$. In particular $\mathcal{F}f$ is uniformly continuous.

Proof. First of all, we notice that any function in $C_0(\mathbb{R}^d)$ is uniformly continuous.

Step 1: Let us assume that f is in $C_c(\mathbb{R}^d)$, namely there exists a positive real number M such that f is supported in $\mathcal{B}(0; M)$. Let y be any vector in $\mathbb{R}^d$; if we denote $t := x - \pi\frac{y}{|y|^2}$, then it holds that

$$\begin{aligned}\mathcal{F}f(y) &= \int_{\mathbb{R}^d} f(x)e^{-i<x;y>}dx \\ &= \int_{\mathbb{R}^d} f\left(t + \pi\frac{y}{|y^2|}\right) e^{-i<t;y>}e^{-i\pi\frac{<y;y>}{|y|^2}}dt \\ &= -\int_{\mathbb{R}^d} f\left(t + \pi\frac{y}{|y^2|}\right) e^{-i<t;y>}dt\end{aligned}$$

If y is any vector in $\mathbb{R}^d$ such that $\frac{\pi}{|y|} \leq 1$, we notice that $\tau_{-\pi\frac{y}{|y|^2}} f$ is supported in $\mathcal{B}(0; M+1)$; hence, the following identities hold true:

$$\begin{aligned}\mathcal{F}f(y) &= \frac{1}{2}\left[\int_{\mathbb{R}^d} f(x)e^{-i<x;y>}dx - \int_{\mathbb{R}^d} f\left(x+\pi\frac{y}{|y^2|}\right)e^{-i<x;y>}dx\right] \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\left[f(x) - f\left(x+\pi\frac{y}{|y^2|}\right)\right]e^{-i<y;x>}dx \\ &= \frac{1}{2}\int_{\mathcal{B}(0;M+1)}\left[f(x) - f\left(x+\pi\frac{y}{|y^2|}\right)\right]e^{-i<y;x>}dx.\end{aligned}$$

Since f is continuous and supported in a compact subset, we notice that

- for all x in $\mathbb{R}^d$ it holds that

$$\lim_{y\to+\infty}\left[f(x) - f\left(x+\pi\frac{y}{|y^2|}\right)\right]e^{-i<y;x>} = 0;$$

- f is bounded, so $2\|f\|_{L^\infty(\mathbb{R}^d)}\mathbb{1}_{\mathcal{B}(0;M+1)}$ is a suitable domination.

Having said that, the thesis is an immediate consequence of the dominated convergence theorem.

Step 2: Let f be any function in $L^1(\mathbb{R}^d)$; let ε be any positive real number. Thanks to 3.2.17, there exists f_ε in $C_c(\mathbb{R}^d)$ such that $\|f - f_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq \frac{\varepsilon}{2}$. There exists a positive real number M such that if $|y|$ is greater than M, then $\mathcal{F}f_\varepsilon(y) \leq \frac{\varepsilon}{2}$. Hence, if y is any vector in $\mathbb{R}^d$ such that $|y| > M$, then

$$\begin{aligned}|\mathcal{F}f(y)| &= \left|\int_{\mathbb{R}^d} f(x)e^{-i<x;y>}dx\right| \\ &\leq \left|\int_{\mathbb{R}^d}[f(x) - f_\varepsilon(x)]e^{-i<x;y>}dx\right| + \left|\int_{\mathbb{R}^d} f_\varepsilon(x)e^{-i<x;y>}dx\right| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So, the thesis follows immediately. □

Remark 6.1.4. If we consider $C_0(\mathbb{R}^d)$ with the norm of the uniform convergence, then we notice that $\mathcal{F} : L^1_{\mathbb{C}}(\mathbb{R}^d) \to C_0(\mathbb{R})$ is a continuous operator between Banach spaces. If f is any function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$, it holds that

$$\sup\left\{|\mathcal{F}f(y)| \;\middle|\; y \in \mathbb{R}^d\right\} = \sup\left\{\left|\int_{\mathbb{R}^d} f(x)e^{-i<x;y>}dx\right| \;\middle|\; y \in \mathbb{R}^d\right\} \leq \|f\|_{L^1(\mathbb{R}^d)};$$

thanks to 3.1.12, $\mathcal{F}$ is a 1-Lipschitz operator.

6.1.2 Examples of Fourier transform

Example 6.1.5. Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$; let h be any vector in $\mathbb{R}^d$; let $\tau_h f$ be define as in 1.0.1. Then, for all y in $\mathbb{R}^d$ the following identity holds true:

$$[\mathcal{F}\tau_h f](y) = e^{-i<h;y>}\mathcal{F}f(y).$$

If we denote $t := x - h$, we have that

$$\begin{aligned}[\mathcal{F}\tau_h f](y) &= \int_{\mathbb{R}^d} f(x-h)e^{-i<x;y>}dx \\ &= \int_{\mathbb{R}^d} f(t)e^{-i<t+h;y>}dt \\ &= e^{-i<h;y>}\int_{\mathbb{R}^d} f(t)e^{-i<t;y>}dt \\ &= e^{-i<h;y>}\mathcal{F}f(y).\end{aligned}$$

Example 6.1.6. Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$; let δ be any positive real number; let $\sigma_\delta f$ be as in 1.0.2. Then, for all y in $\mathbb{R}^d$ the following identity holds true:

$$[\mathcal{F}\sigma_\delta f](y) = \mathcal{F}f(\delta y).$$

If we denote $t := \frac{x}{\delta}$, then $dt = \frac{1}{\delta^d}dx$; hence, we have that

$$\begin{aligned}[\mathcal{F}\sigma_\delta f](y) &= \int_{\mathbb{R}^d} \frac{1}{\delta^d} f\left(\frac{x}{\delta}\right) e^{-i<y;x>}dx \\ &= \int_{\mathbb{R}^d} f(t)e^{-i<y;\delta t>}dt \\ &= \int_{\mathbb{R}^d} f(t)e^{-i<\delta y;t>}dt \\ &= \mathcal{F}f(\delta y).\end{aligned}$$

Example 6.1.7. Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$; let A be any matrix in $\mathbb{M}(n;\mathbb{R})$ such that $\det(A) \neq 0$. Then, for all y in $\mathbb{R}^d$ the following identity holds true:

$$[\mathcal{F}f \circ A](y) = \frac{1}{|\det(A)|}\mathcal{F}f([A^{-1}]^t y).$$

If we denote $t := Ax$, then $dt = |\det(A)|\,dx$; hence, we have that

$$\begin{aligned}[\mathcal{F}f \circ A](y) &= \int_{\mathbb{R}^d} f(Ax)e^{-i<y;x>}dx \\ &= \frac{1}{|\det(A)|}\int_{\mathbb{R}^d} f(t)e^{-i<y;A^{-1}t>}dt \\ &= \frac{1}{\det(A)}\int_{\mathbb{R}^d} f(t)e^{-i<[A^{-1}]^t y;t>}dt \\ &= \frac{1}{|\det(A)|}\mathcal{F}f([A^{-1}]^t y).\end{aligned}$$

Example 6.1.8. Let us consider $f(x) := e^{-|x|}$. Since f is in $L^1(\mathbb{R})$, we can compute the Fourier transform. For all y in $\mathbb{R}$ it holds that

$$\begin{aligned}\mathcal{F}f(y) &= \int_{\mathbb{R}} e^{-|x|-ixy}dx \\ &= \int_{-\infty}^0 e^{x(1-iy)}dx + \int_0^{+\infty} e^{x(-1-iy)}dx \\ &= \frac{1}{1-iy} - \frac{1}{-1-iy} = \frac{2}{1+y^2}.\end{aligned}$$

Example 6.1.9. Let us consider $f(x) := \mathbb{1}_{[-1;1]}$. Since f is in $L^1(\mathbb{R})$, we can compute the Fourier transform. For all y in $\mathbb{R} \setminus \{0\}$ it holds that

$$\mathcal{F}f(y) = \int_{-1}^{1} e^{-ixy} dx = \left[\frac{1}{-iy} e^{-ixy} \right]_{x=-1}^{x=1} = \frac{e^{iy} - e^{-iy}}{iy} = \frac{2 \sin y}{y}.$$

Since $\mathcal{F}f$ is continuous, it holds that $\mathcal{F}f(0) = 2$.

Example 6.1.10. Let us consider

$$f(x) := \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

Since f is in $L^1(\mathbb{R})$, we can compute the Fourier transform. For all y in $\mathbb{R}$ it holds that

$$\begin{aligned}\mathcal{F}f(y) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2} - ixy} dx \\ &= \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x+iy)^2} dx.\end{aligned}$$

Let n be any natural number; let us consider the following paths in the complex plane:

- $\gamma_{1;n}$ is the path that joins the the points $-n$ and n, namely for all t in $[-n; n]$
$$\gamma_{1;n}(t) := t;$$
- $\gamma_{2;n}$ is the path that joins the the points n and $n + iy$, namely for all t in $[0; 1]$
$$\gamma_{2;n}(t) := n + iyt;$$
- $\gamma_{3;n}$ is the path that joins the the points $n + iy$ and $-n + iy$, namely for all t in $[-n; n]$
$$\gamma_{3;n}(t) := -t + iy;$$
- $\gamma_{4;n}$ is the path that joins the the points $-n + iy$ and $-n$, namely for all t in $[0; 1]$
$$\gamma_{4;n}(t) := -n + iy(1 - t).$$

Let γ_n denote the junction of those paths. We define $g : \mathbb{C} \to \mathbb{C}$ such that $g(z) := e^{-\frac{z^2}{2}}$. Since γ_n is a closed loop and f is an holomorphic function, it holds that

$$\begin{aligned}0 &= \int_{\gamma_n} g(z) dz \\ &= \int_{\gamma_{1;n}} g(z) dz + \int_{\gamma_{2;n}} g(z) dz + \int_{\gamma_{3;n}} g(z) dz + \int_{\gamma_{4;n}} g(z) dz.\end{aligned}$$

We consider the limits as n approaches $+\infty$ and we have that:

- it is well known that
$$\lim_{n \to +\infty} \int_{\gamma_{1;n}} g(z) dz = \lim_{n \to +\infty} \int_{-n}^{n} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi};$$

- as for $\gamma_{3;n}$, it holds that

$$\lim_{n\to+\infty}\int_{\gamma_{3;n}} g(z)dz = -\lim_{n\to+\infty}\int_{-n}^{n} e^{-\frac{(t+iy)^2}{2}}dt = -\int_{\mathbb{R}} e^{-\frac{(t+iy)^2}{2}}dt;$$

- as for $\gamma_{2;n}$, it holds that

$$\begin{aligned}\lim_{n\to+\infty}\left|\int_{\gamma_{2;n}} g(z)dz\right| &\leq |iy| \lim_{n\to+\infty}\int_0^1 \left|e^{-\frac{(n+iyt)^2}{2}}\right| dt\\ &= |y| \lim_{n\to+\infty}\int_0^1 e^{-\frac{n^2-y^2t^2}{2}}dt\\ &= |y|\left(\int_0^1 e^{\frac{y^2t^2}{2}}dt\right)\lim_{n\to+\infty} e^{-\frac{n^2}{2}} = 0;\end{aligned}$$

- similarly, we can prove that

$$\lim_{n\to+\infty}\left|\int_{\gamma_{4;n}} g(z)dz\right| = 0.$$

Hence, we can state that

$$0 = \sqrt{2\pi} - \int_{\mathbb{R}} e^{-\frac{(t+iy)^2}{2}}dt;$$

In particular, we have shown that for all y in $\mathbb{R}$, the following identity holds true:

$$\mathcal{F}f(y) = e^{-\frac{y^2}{2}}.$$

Example 6.1.11. Let us consider

$$f(x) := \frac{1}{x^2+1}.$$

Since f is in $L^1(\mathbb{R})$, we can compute the Fourier transform. For all y in $\mathbb{R}$ it holds that

$$\mathcal{F}f(y) = \int_{\mathbb{R}} \frac{e^{-ixy}}{x^2+1}dx.$$

Let us define $\mathcal{D} := \mathbb{C}\setminus\{i;-i\}$ and $g:\mathcal{D}\to\mathbb{C}$ such that for all z in $\mathcal{D}$ if holds that

$$g(z) := \frac{e^{-izy}}{z^2+1}.$$

We notice that g is an holomorphic function in the open set $\mathcal{D}$. Let n be any integer greater than 2. Let us consider the following paths in the complex plane:

- $\gamma_{1;n}$ is the path that joins the points $-n$ and n, namely for all t in $[-n;n]$

$$\gamma_{1;n}(t) := t;$$

- $\gamma_{2;n}$ is the semicircle that joins the points n and $-n$, namely for all t in $[0;\pi]$

$$\gamma_{2;n}(t) := ne^{it}.$$

Let γ_n be the junction of those paths. We also denote as $\text{Res}(i; g)$ the residue of the complex variable function g in i. Moreover, γ_n is the counter-clockwise parameterization of a closed loop in $\mathcal{D}$ that surrounds the point i, where the function g has a polar singularity. So, we can apply the residue theorem and for all n greater than 2 it holds that

$$2\pi i \text{Res}(i; g) = \int_{\gamma_{1;n}} g(z)dz + \int_{\gamma_{2;n}} g(z)dz.$$

Let us assume that y is a negative number. It's easy to see that

$$2\pi i \text{Res}(i; g) = \pi e^y.$$

Moreover, since f is in $L^1(\mathbb{R})$, we have that

$$\lim_{n\to+\infty} \int_{\gamma_{1;n}} g(z)dz = \lim_{n\to+\infty} \int_{-n}^{n} \frac{e^{-ity}}{1+t^2}dt = \mathcal{F}f(y).$$

It is also true that

$$\left| \int_{\gamma_{2;n}} g(z)dz \right| = \left| \int_0^1 \frac{e^{-iyne^{it}} ne^{it}}{1+n^2e^{2it}} dt \right| \le \frac{n}{n^2-1} \int_0^1 e^{yn\sin t} dt.$$

Since y is a negative number, it's easy to see that the right hand side is infinitesimal. So, we have shown that for all $y < 0$

$$\mathcal{F}f(y) = \pi e^y.$$

Since f is even, it's easy to see that $\mathcal{F}f$ is even. Moreover, it is continuous; so, we can conclude that

$$\mathcal{F}f(y) = \pi e^{-|y|}.$$

6.1.3 On the regularity of the Fourier transform

The theory will be developed for one variable functions. As a matter of facts, all the statements can be adapt to the case of several variable functions.

Lemma 6.1.12. *Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R}) \cap C^1(\mathbb{R})$. If we assume that f' is in $L^1_{\mathbb{C}}(\mathbb{R})$, then it holds that*

$$[\mathcal{F}f'](y) = iy\mathcal{F}f(y).$$

In particular, we have that

$$\lim_{|y|\to+\infty} \frac{\mathcal{F}fy}{|y|} = 0.$$

Proof. **Step 1**: Under our hypothesis, we claim that f is in $C_0(\mathbb{R})$. Since f is in $L^1_{\mathbb{C}}(\mathbb{R})$, if we show that there exists a real number l such that

$$l = \lim_{x\to+\infty} f(x),$$

then l must be 0. Since f' is in $L^1_{\mathbb{C}}(\mathbb{R})$, the dominated convergence theorem implies that f admits a limit as x approaches $+\infty$ and it holds that

$$\lim_{x\to+\infty} f(x) = f(0) + \lim_{x\to+\infty} \int_0^x f'(t)dt = f(0) + \int_0^{+\infty} f'(t)dt.$$

Step 2: Since f' is in $L^1_{\mathbb{C}}(\mathbb{R})$, for all y in $\mathbb{R}$ the following identity is an immediate consequence of the dominated convergence theorem:

$$\mathcal{F}f'(y) = \lim_{n\to+\infty} \int_{-n}^{n} f'(x)e^{-iyx}dx.$$

Integrating by parts the right hand side, for all natural number n we obtain that

$$\int_{-n}^{n} f'(x)e^{-iyx}dx = f(n)e^{-iny} - f(-n)e^{iny} + iy\int_{-n}^{n} f(x)e^{-ixy}dx.$$

If we recall that f is in $L^1_{\mathbb{C}}(\mathbb{R})$, we can take the limit as n approaches to $+\infty$ and the thesis follows from the dominated convergence theorem and the first step.

As for the second part of the statement, it is an immediate consequence of the fact that $\mathcal{F}f'$ is in $C_0(\mathbb{R})$ as shown in lemma 6.1.3. □

Corollary 6.1.13. *Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R}) \cap C^k(\mathbb{R})$. If we assume that for all integer i in $\{1;\dots;k\}$ it holds that $f^{(i)}$ is in $L^1_{\mathbb{C}}(\mathbb{R})$, then the following identity holds true:*

$$[\mathcal{F}f^{(k)}](y) = (iy)^k \mathcal{F}f(y).$$

In particular, we have that

$$\lim_{|y|\to+\infty} \frac{\mathcal{F}f(y)}{|y|^k} = 0.$$

Proof. The proof is consist of the iterated application of lemma 6.1.12. □

Lemma 6.1.14. *Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R})$. We define $\tilde{f} : \mathbb{R} \to \mathbb{C}$ such that $\tilde{f}(x) := -ixf(x)$ for all x in $\mathbb{R}$. If we assume that $\tilde{f}$ is in $L^1_{\mathbb{C}}(\mathbb{R})$, then $\mathcal{F}f$ is in $C^1(\mathbb{R})$ and for all y in $\mathbb{R}$ the following identity holds:*

$$[\mathcal{F}f]'(y) = \mathcal{F}\tilde{f}(y).$$

Proof. Since $\tilde{f}$ is in $L^1_{\mathbb{C}}(\mathbb{R})$, lemma 6.1.3 implies that $\mathcal{F}\tilde{f}$ is a uniformly continuous function. If we show that for all y in $\mathbb{R}$ it holds that

$$\mathcal{F}f(y) - \mathcal{F}f(0) = \int_0^y \mathcal{F}\tilde{f}(t)dt,$$

then thesis is an immediate consequence of the fundamental theorem of calculus. Without loss of generality, we can assume that y is a positive real number. We notice that

$$\int_{\mathbb{R}\times[0;y]} \left|\tilde{f}(x)e^{-itx}\right| dxdt \le |y| \left\|\tilde{f}\right\|_{L^1(\mathbb{R})}.$$

Therefore, we can use Fubini's theorem and switch the order of integration:

$$\begin{aligned}
\int_0^y \mathcal{F}\tilde{f}(t)dt &= \int_0^y \left(\int_{\mathbb{R}} -ixf(x)e^{-itx}dx\right) dt \\
&= \int_{\mathbb{R}} \left(\int_0^y -ixf(x)e^{-itx}dt\right) dx \\
&= \int_{\mathbb{R}} -ixf(x)\left[-\frac{1}{ix}e^{-iyx} + \frac{1}{ix}\right] dx \\
&= \int_{\mathbb{R}} f(x)[e^{-iyx} - 1]dx.
\end{aligned}$$

Since f is in $L^1_{\mathbb{C}}(\mathbb{R})$, we can split the integral and the we find that

$$\int_0^y \mathcal{F}\tilde{f}(t)dt = \int_{\mathbb{R}} f(x)e^{-iyx}dx - \int_{\mathbb{R}} f(x)dx = \mathcal{F}f(y) - \mathcal{F}f(0).$$

□

Corollary 6.1.15. *Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R})$. We define $\tilde{f}_k : \mathbb{R} \to \mathbb{C}$ such that for all x in $\mathbb{R}$ it holds that $\tilde{f}_k(x) = (-ix)^k f(x)$. If we assume that $\tilde{f}_k$ is in $L^1_{\mathbb{C}}(\mathbb{R})$, then $\mathcal{F}f$ is in $C^k(\mathbb{R})$ and the following identity holds for all y in $\mathbb{R}$:*

$$[\mathcal{F}f]^{(k)}(y) = \mathcal{F}\tilde{f}_k(y).$$

In particular, if f is supported by a compact subset, then $\mathcal{F}f$ is a smooth function in $\mathbb{R}$.

Proof. We notice that if j is any integer in $\{1; \dots ; k\}$, for all x in $\mathbb{R}$ it holds that $|x|^j \leq 1 + |x|^k$. In particular, we have that

$$\int_{\mathbb{R}} |x|^j |f(x)| dx \leq \left\|\tilde{f}_k\right\|_{L^1(\mathbb{R})} + \|f\|_{L^1(\mathbb{R})} .$$

Having said that, the proof is consist of the iterated application of lemma 6.1.14. As for the second statement, it obviously follows by the fact that $\tilde{f}_k$ is in $L^1_{\mathbb{C}}(\mathbb{R})$ for all k in $\mathbb{N}$. □

Proposition 6.1.16. *Let f be a function in $L^1_{\mathbb{C}}(\mathbb{R})$ supported by a compact subset. Then, there exists a function $g : \mathbb{C} \to \mathbb{C}$ that is analytic in the complex plane and such that $\mathcal{F}f$ is the restriction to $\mathbb{R}$ of g. In particular, $\mathcal{F}f$ is an analytic function.*

Proof. Let us define $g : \mathbb{C} \to \mathbb{C}$ such that

$$g(z) := \int_{\mathbb{R}} f(x)e^{-ixz}dx.$$

We claim that the function g is well defined. Let M be a positive real number such that f is supported by $[-M; M]$. For all z in $\mathbb{C}$ the function $h_z(x) := e^{-ixz}$ is continuous; in other words, h_z is in $L^\infty_{\mathbb{C}}((-M; M))$. Since f is in $L^1_{\mathbb{C}}((-M; M))$, it is immediate to see that g is well defined. Moreover, the dominated convergence theorem implies that g is a continuous function.

We claim that $g(z)dz$ is a exact differential form. Let $\gamma : [a; b] \to \mathbb{C}$ be a close path. We have that

$$\int_\gamma g(z)dz = \int_a^b g(\gamma(t))\gamma'(t)dt = \int_a^b \left(\int_{-M}^M f(x)e^{-ix\gamma(t)}\gamma'(t)dx\right) dt.$$

Since the function $f(x)e^{-ix\gamma(t)\gamma'(t)}$ is continuous and $[-M; M] \times [a; b]$ is a compact set, we can apply Fubini's theorem to switch the order of integration. Hence, we have that

$$\int_\gamma g(z)dz = \int_{-M}^M f(x) \left(\int_a^b e^{-ix\gamma(t)}\gamma'(t)dt\right) dx = \int_{-M}^M f(x) \left(\int_\gamma e^{-ixz}dz\right) dx.$$

Let x be any point in $[-M; M]$. Since the function $j_x(z) := e^{-ixz}$ is holomorphic in the complex plane, $j_x(z)dz$ is an exact differential form, i. e.

$$\int_\gamma e^{-ixz}dz = 0.$$

So, we can conclude that

$$\int_\gamma g(z)dz = 0;$$

in other words, $g(z)dz$ is an exact differential form. Morera's theorem implies that g is an holomorphic function. So, g is an analytic function in the complex plane. □

Example 6.1.17. We show another way to compute the Fourier transform of the function

$$f(x) := \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

We notice that

$$f'(x) = -x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = -xf(x).$$

Since f and f' are in $L^1(\mathbb{R})$, we can consider the Fourier transform at left hand side and right hand side; joining 6.1.12 and 6.1.14, for all y in $\mathbb{R}$ the following identity holds true:

$$[\mathcal{F}f]'(y) = y\mathcal{F}f(y). \tag{6.1}$$

Since we know that the space of the solutions of (6.1) is a one dimensional vector space and f is a solution of (6.1), we can conclude that there exists a real number c such that

$$\mathcal{F}f \equiv cf.$$

To conclude, we notice that

$$\frac{c}{\sqrt{2\pi}} = \mathcal{F}f(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx = 1.$$

6.1.4 Inversion Fourier theorem

Lemma 6.1.18. *Let f, g be functions in $L^1_{\mathbb{C}}(\mathbb{R})$. Then $f * g$ is in $L^1_{\mathbb{C}}(\mathbb{R})$ and for all y in $\mathbb{R}$ it holds that*

$$[\mathcal{F}f * g](y) = [\mathcal{F}f(y)][\mathcal{F}g(y)].$$

Proof. We have shown in 3.2.7 that $f * g$ is in $L^1_{\mathbb{C}}(\mathbb{R})$. If y is any real number, it holds that

$$\begin{aligned}
[\mathcal{F}f(y)][\mathcal{F}g(y)] &= \left(\int_{\mathbb{R}} f(x)e^{-iyx}dx\right)\left(\int_{\mathbb{R}} g(t)e^{-iyt}dt\right) \\
&= \int_{\mathbb{R}} f(x)e^{-ixy}\left(\int_{\mathbb{R}} g(t)e^{-iyt}dt\right)dx \\
&= \int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}} g(t)e^{-iy(t+x)}dt\right)dx.
\end{aligned}$$

If we denote $s := t + x$, we have that

$$[\mathcal{F}f(y)][\mathcal{F}g(y)] = \int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}} g(s-x)e^{-iys}ds\right)dx.$$

We notice that

$$\int_{\mathbb{R}^2} \left|f(x)g(s-x)e^{-iys}\right| dxds = \||f| * |g|\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.$$

So, we can use Fubini's theorem and switch the order of integration; we obtain that

$$\begin{aligned}[\mathcal{F}f(y)][\mathcal{F}g(y)] &= \int_{\mathbb{R}} f(x) \left(\int_{\mathbb{R}} g(s-x)e^{-iys}dx \right) ds \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)g(s-x)dx \right) e^{-iys}ds \\ &= \int_{\mathbb{R}} f * g(s)e^{-iys}ds \\ &= [\mathcal{F}f * g](y).\end{aligned}$$

□

Definition 6.1.19 (Fourier anti-transform)**.**
Let g be any function in $L^1_{\mathbb{C}}(\mathbb{R}^d)$. For all x in $\mathbb{R}^d$, we define

$$\mathcal{F}^* f(x) \coloneqq \int_{\mathbb{R}} g(x)e^{i<x;y>}dx.$$

Remark 6.1.20. If g is any function in $L^1_{\mathbb{C}}(\mathbb{R})$, we notice that $\mathcal{F}^*g(y) = \mathcal{F}g(-y)$ for all y in $\mathbb{R}$. Hence, $\mathcal{F}^* : L^1_{\mathbb{C}}(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ is a linear and continuous operator with same properties of $\mathcal{F}$.

Theorem 6.1.21 (Inversion Fourier theorem)**.**
Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R})$; if $\mathcal{F}f$ is in $L^1_{\mathbb{C}}(\mathbb{R})$, then for almost every x in $\mathbb{R}$ the following identity holds true:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}f(y)e^{ixy}dy = \frac{1}{2\pi}[\mathcal{F}^*\mathcal{F}f](x).$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be any function with the following properties:

- g is in $C(\mathbb{R}) \cap L^\infty_{\mathbb{C}}(\mathbb{R}) \cap L^1_{\mathbb{C}}(\mathbb{R})$;
- $\mathcal{F}^*g$ is in $L^1_{\mathbb{C}}(\mathbb{R})$;
- $g(0) = 1$.

Step 1: We claim that

$$\lim_{\delta \to 0} \int_{\mathbb{R}} g(\delta y)\mathcal{F}f(y)e^{ixy}dy = \int_{\mathbb{R}} \mathcal{F}f(y)e^{ixy}dy. \tag{6.2}$$

We notice that the following statements hold true:

- since g is continuous and $g(0) = 1$, for all x in $\mathbb{R}$, for almost every y in $\mathbb{R}$ we have that
$$\lim_{\delta \to 0} g(\delta y)\mathcal{F}f(y)e^{ixy} = \mathcal{F}f(y)e^{ixy};$$

- for all x in $\mathbb{R}$ for all y in $\mathbb{R}$, for all positive real number δ, we have that
$$\left|g(\delta y)\mathcal{F}f(y)e^{ixy}\right| \leq \|g\|_{L^\infty(\mathbb{R})}\left|\mathcal{F}f(y)\right|,$$
that is a suitable domination in $L^1(\mathbb{R})$.

Hence, (6.2) is an immediate consequence of the dominated convergence theorem.

Step 2: Let δ be a positive real number and x any point in $\mathbb{R}$. We have that
$$\int_{\mathbb{R}} g(\delta y)\mathcal{F}f(y)e^{ixy}dy = \int_{\mathbb{R}} g(\delta y)\left(\int_{\mathbb{R}} f(t)e^{-ity}dt\right)e^{ixy}dy.$$
We notice that
$$\int_{\mathbb{R}^2}\left|g(\delta y)f(t)e^{iy(x-t)}\right|dxdt = \left(\int_{\mathbb{R}}|f(t)|\,dt\right)\left(\int_{\mathbb{R}}|g(\delta y)|\,dy\right) = \|f\|_{L^1(\mathbb{R})}\frac{\|g\|_{L^1(\mathbb{R})}}{\delta}.$$
In particular, we can use Fubini's theorem and switch the order of integration. If we recall definition 1.0.2, we obtain that
$$\begin{aligned}\int_{\mathbb{R}} g(\delta y)\mathcal{F}f(y)e^{ixy}dy &= \int_{\mathbb{R}} f(t)\left(\int_{\mathbb{R}} g(\delta y)e^{iy(x-t)}dy\right)dt\\ &= \int_{\mathbb{R}} f(t)\frac{1}{\delta}\left[\mathcal{F}^*\sigma_{\frac{1}{\delta}}g\right](x-t)dt.\end{aligned}$$
If we join 6.1.6 and 6.1.20, we have that
$$\frac{1}{\delta}\left[\mathcal{F}^*\sigma_{\frac{1}{\delta}}g\right](x-t) = \frac{1}{\delta}\mathcal{F}^*g\left(\frac{x-t}{\delta}\right) = [\sigma_\delta(\mathcal{F}^*g)](x-t);$$
therefore, we obtain that
$$\int_{\mathbb{R}} g(\delta y)\mathcal{F}f(y)e^{ixy}dy = \int_{\mathbb{R}} f(t)\sigma_\delta(\mathcal{F}^*g)(x-t)dt = [f*\sigma_\delta g](x).$$
As shown in 3.2.16, $\{f*\sigma_\delta(\mathcal{F}^*g)\}_{\delta>0}$ converges toward $\|\mathcal{F}^*g\|_{L^1(\mathbb{R})}f$ with respect to L^1 norm. In particular, there exists an infinitesimal sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that for almost every x in $\mathbb{R}$ it holds that
$$\lim_{n\to+\infty}[f*\sigma_{\delta_n}(\mathcal{F}^*g)](x) = \|\mathcal{F}^*g\|_{L^1(\mathbb{R})}f(x).$$
Then, for almost every x in $\mathbb{R}$ the following identity holds true:
$$\int_{\mathbb{R}}\mathcal{F}f(y)e^{ixy}dy = \|\mathcal{F}^*g\|_{L^1(\mathbb{R})}f(x).$$
If we consider $g(y) \coloneqq e^{-\frac{y^2}{2}}$, joining 6.1.20 and 6.1.10, we have that
$$\mathcal{F}^*g(x) = e^{-\frac{x^2}{2}}\sqrt{2\pi};$$
so, $\|\mathcal{F}^*g\|_{L^1(\mathbb{R})}$ equals 2π and the theorem is completely proved. □

Remark 6.1.22. As shown in 6.1.9, f if $L^1_{\mathbb{C}}(\mathbb{R})$ does not imply that $\mathcal{F}f$ is in $L^1_{\mathbb{C}}(\mathbb{R})$. However, in order $\mathcal{F}^*\mathcal{F}f$ make sense, it has to be assumed in theorem 6.1.21.

Remark 6.1.23. We can easily show that the Fourier transform is a linear, injective operator between $L^1_{\mathbb{C}}((-\pi;\pi))$ and $C_0(\mathbb{R})$. Since the $\mathcal{F}$ is obviously linear, it is enough to show that if f is in $L^1_{\mathbb{C}}(\mathbb{R})$ is such that $\mathcal{F}f(y) = 0$ for almost every y in $\mathbb{R}$, then $f(x)$ is equal to 0 for almost every x in $\mathbb{R}$. Actually, since we can apply the inversion Fourier theorem (see 6.1.21), the conclusion is trivial.

6.2 Fourier transform in L^2

Our purpose is to find a suitable definition of Fourier transform for all f in $L^2_{\mathbb{C}}(\mathbb{R})$.

Lemma 6.2.1. *Let f be any function in $L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R})$. Then, it holds that*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\,\|f\|_{L^2(\mathbb{R})};$$

in particular, $\mathcal{F}f$ is in $L^2_{\mathbb{C}}(\mathbb{R})$ and $\mathcal{F} : L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R})$ is a linear $\sqrt{2\pi}$-Lipschitz operator.

Proof. Let g be any function with the following properties:

- g is in $C(\mathbb{R}) \cap L^\infty_{\mathbb{C}}(\mathbb{R}) \cap L^1_{\mathbb{C}}(\mathbb{R})$;
- $\mathcal{F}^* g$ is in $L^2_{\mathbb{C}}(\mathbb{R})$;
- $g(0) = 1$;
- g is even, nonnegative and monotonously decreasing in $[0; +\infty)$.

We notice that if y is any point in $\mathbb{R}$ and δ_1, δ_2 are positive real number such that δ_2 is greater than δ_1, then we have that

$$g(y\delta_1) \geq g(y\delta_2),$$
$$\lim_{\delta \to 0^+} g(\delta y) = g(0) = 1.$$

Thanks to Beppo Levi's theorem and our assumption on g, we have that

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}} g(\delta y)[\mathcal{F}f(y)]\overline{[\mathcal{F}f(y)]}dy = \int_{\mathbb{R}} [\mathcal{F}f(y)]\overline{\mathcal{F}f(y)}dy = \|\mathcal{F}f\|^2_{L^2(\mathbb{R})}.$$

Let δ be any positive real number; it holds that

$$\begin{aligned}
\int_{\mathbb{R}} g(\delta y)\,|\mathcal{F}f(y)|^2\,dy &= \int_{\mathbb{R}} g(\delta y)\left(\int_{\mathbb{R}} f(x)e^{-ixy}dx\right)\overline{\left(\int_{\mathbb{R}} f(t)e^{-ity}dt\right)}dy \\
&= \int_{\mathbb{R}} g(\delta y)\left(\int_{\mathbb{R}} f(x)e^{-ixy}dx\right)\left(\int_{\mathbb{R}} \overline{f(t)}e^{ity}dt\right)dy \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(\delta y)f(x)\overline{f(t)}e^{i(t-x)y}dt\right)dx\right]dy
\end{aligned}$$

We notice that

$$\begin{aligned}
\int_{\mathbb{R}^3} |g(\delta y)|\,|f(x)|\,\left|\overline{f(t)}\right| dydxdt &= \left(\int_{\mathbb{R}} |f(x)|\,dx\right)^2 \left(\int_{\mathbb{R}} |g(\delta y)|\,dy\right) \\
&= \|f\|^2_{L^1(\mathbb{R})} \frac{\|g\|_{L^1(\mathbb{R})}}{\delta}.
\end{aligned}$$

Then, we can switch the order of integration and, if we recall definition 1.0.2, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}} g(\delta y)\,|\mathcal{F}f(y)|^2\,dy &= \int_{\mathbb{R}^2} f(x)\overline{f(t)}\left(\int_{\mathbb{R}} g(\delta y)e^{i(t-x)y}dy\right)dxdt \\
&= \frac{1}{\delta}\int_{\mathbb{R}^2} f(x)\overline{f(t)}[\mathcal{F}^*\sigma_{\frac{1}{\delta}}g](t-x)dxdt.
\end{aligned}$$

If we join 6.1.6 and 6.1.20, we have that

$$\begin{aligned}\int_{\mathbb{R}} g(\delta y)\left|\mathcal{F}f(y)\right|^2 dy &= \int_{\mathbb{R}^2} f(x)\overline{f(t)}[\sigma_\delta \mathcal{F}^* g](t-x)dxdt\\ &= \int_{\mathbb{R}} \overline{f(t)}(f * \sigma_\delta(\mathcal{F}^* g))(t)dt.\end{aligned}$$

As shown in 3.2.16, we have that $\{f * \sigma_\delta(\mathcal{F}^* g)\}_{\delta>0}$ converges toward $f\left\|\mathcal{F}^* g\right\|_{L^1(\mathbb{R})}$ with respect to L^2 norm. Hence, if we use Hölder's inequality, we notice that

$$\begin{aligned}&\lim_{\delta\to 0}\left|\int_{\mathbb{R}} g(\delta y)\left|\mathcal{F}f(y)\right|^2 dy - \left\|\mathcal{F}^* g\right\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}\overline{f(y)}f(y)dy\right|\\ &= \lim_{\delta\to 0}\left|\int_{\mathbb{R}}\overline{f(y)}\left[(f*(\sigma_\delta\mathcal{F}^* g))(y) - \left\|\mathcal{F}^* g\right\|_{L^1(\mathbb{R})} f(y)\right]dy\right|\\ &\leq \lim_{\delta\to 0}\left\|f\right\|_{L^2(\mathbb{R})}\left\|f*(\sigma_\delta\mathcal{F}^* g) - \left\|\mathcal{F}^* g\right\|_{L^1(\mathbb{R})} f\right\|_{L^2(\mathbb{R})}\\ &= 0.\end{aligned}$$

This is enough to state that

$$\left\|\mathcal{F}f\right\|^2_{L^2(\mathbb{R})} = \left\|\mathcal{F}^* g\right\|_{L^1(\mathbb{R})}\left\|f\right\|^2_{L^2(\mathbb{R})^2}.$$

To conclude, if we consider $g(x) := e^{-\frac{x^2}{2}}$, then we have that $\mathcal{F}^* g(y) = \sqrt{2\pi}e^{-\frac{x^2}{2}}$; in particular, it holds that $\left\|\mathcal{F}^* g\right\|_{L^1(\mathbb{R})} = 2\pi$. □

Theorem 6.2.2. *There exists a linear and continuous operator denoted also with* $\mathcal{F} : L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R})$ *that extends the Fourier transform defined in* $L^2_{\mathbb{C}}(\mathbb{R}) \cap L^1_{\mathbb{C}}(\mathbb{R})$ *and such that if* f *in any function in* $L^2_{\mathbb{C}}(\mathbb{R})$ *it holds that*

$$\left\|\mathcal{F}f\right\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\left\|f\right\|_{L^2(\mathbb{R})^2}.$$

In other words, $\mathcal{F}$ *can be extended to an isometry on* $L^2_{\mathbb{C}}(\mathbb{R})$ *(up the factor* $\sqrt{2\pi}$*).*

Proof. It's easy to see that $L^1_{\mathbb{C}}(\mathbb{R})\cap L^2_{\mathbb{C}}(\mathbb{R})$ is dense in $L^1_{\mathbb{C}}(\mathbb{R})$. In fact, if f is any function in $L^2_{\mathbb{C}}(\mathbb{R})$, for all n in $\mathbb{N}$ we define

$$f_n(x) := f(x)\mathbb{1}_{[-n;n]}(x).$$

Then $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R})$ that converges toward f with respect to L^2 norm. In lemma 6.2.1, we have shown that

$$\mathcal{F} : L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R})$$

is a linear $\sqrt{2\pi}$-Lipschitz operator. Since it is uniformly continuous and $L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R})$ is dense in $L^2_{\mathbb{C}}(\mathbb{R})$, $\mathcal{F}$ can be extended by continuity to a linear operator that we will also denote with $\mathcal{F}$ on $L^2_{\mathbb{C}}(\mathbb{R})$. Hence, if f is any function in $L^2_{\mathbb{C}}(\mathbb{R})$ it holds that

$$\left\|\mathcal{F}f\right\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\left\|f\right\|_{L^2(\mathbb{R})^2}.$$

□

Remark 6.2.3. We can slightly modify lemma 6.2.1 and corollary 6.2.2 to show that $\mathcal{F}^*$ admits a linear $\sqrt{2\pi}$-Lipschitz extension to $L^2_{\mathbb{C}}(\mathbb{R})$, denote also with $\mathcal{F}^*$, such that if f is any function in $L^2_{\mathbb{C}}(\mathbb{R})$ it holds that

$$\|\mathcal{F}^* f\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\,\|f\|_{L^2(\mathbb{R})}\,.$$

Remark 6.2.4. It's immediate to see that both $\mathcal{F}$ and $\mathcal{F}^*$ are injective operators on $L^2_{\mathbb{C}}(\mathbb{R})$.

Lemma 6.2.5. *Let f be any function in $L^2_{\mathbb{C}}(\mathbb{R})$. Let us assume that for almost every y in $\mathbb{R}$ there exists $L(y)$ in $\mathbb{C}$ such that*

$$L(y) = \lim_{n\to+\infty}\int_{-n}^{n} f(x)e^{-ixy}dx.$$

Then, it holds that $L(y) = \mathcal{F}f(y)$ for almost every y in $\mathbb{R}$.

Proof. Let n be any natural number. We define

$$f_n(x) \coloneqq f(x)\mathbb{1}_{[-n;n]}.$$

Since f_n is in $L^2_{\mathbb{C}}((-n;n))$, it holds that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^1_{\mathbb{C}}(\mathbb{R})$. We notice that for all natural number n, for all y in $\mathbb{R}$ it holds that

$$\mathcal{F}f_n(y) = \int_{\mathbb{R}} f_n(x)e^{-ixy}dx = \int_{-n}^{n} f(x)e^{-ixy}.$$

We know that $\{f_n\}_{n\in\mathbb{N}}$ converges toward f with respect to L^2 norm. As shown in 6.2.2, we have that $\{\mathcal{F}f_n\}_{n\in\mathbb{N}}$ converges toward $\mathcal{F}f$ with respect to L^2 norm. Up to subsequences, not relabelled, we can assume that the convergence is pointwise for almost every y in $\mathbb{R}$. This is enough to conclude that $L(y) = \mathcal{F}f(y)$ for almost every y in $\mathbb{R}$. □

Lemma 6.2.6. *Let f be any function in $C^1(\mathbb{R})\cap L^1_{\mathbb{C}}(\mathbb{R})$; if f' is in $L^2_{\mathbb{C}}(\mathbb{R})\cap L^1_{\mathbb{C}}(\mathbb{R})$, then $\mathcal{F}f$ is in $L^1_{\mathbb{C}}(\mathbb{R})$.*

Proof. Thanks to lemma 6.1.12, if y is any point in $\mathbb{R}$ it holds that

$$[\mathcal{F}f'](y) = iy\mathcal{F}f(y).$$

Since f' is in $L^1_{\mathbb{C}}(\mathbb{R})\cap L^2_{\mathbb{C}}(\mathbb{R})$, lemma 6.2.1 implies that $\mathcal{F}f'$ is in $L^2_{\mathbb{C}}(\mathbb{R})$. If we define

$$g(y) \coloneqq (1+|y|)\,|\mathcal{F}f(y)|\,,$$

we have that g is in $L^2_{\mathbb{C}}(\mathbb{R})$. If we use Hölder's inequality, we obtain that

$$\int_{\mathbb{R}} |\mathcal{F}f(y)|\,dy \le \left(\int_{\mathbb{R}} |g(y)|^2\,dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left|\frac{1}{1+|y|}\right|^2 dy\right)^{\frac{1}{2}}$$

that is finite. □

Theorem 6.2.7. *Let f be any function in $L^2_{\mathbb{C}}(\mathbb{R})$; then, for almost every x in $\mathbb{R}$ the following identity holds true:*

$$[\mathcal{F}^*\mathcal{F}f](x) = 2\pi f(x). \tag{6.3}$$

Proof. Thanks to 6.1.21 and lemma 6.2.6, we know that formula (6.3) holds for all f in $C_c^1(\mathbb{R})$. If Id_{L^2} denotes the identity of $L^2_{\mathbb{C}}(\mathbb{R})$, we have shown that $\mathcal{F}^* \circ \mathcal{F}$ is a 2π-Lipschitz operator on $L^2_{\mathbb{C}}(\mathbb{R})$ that coincides with $2\pi Id_{L^2}$ on $C_c^1(\mathbb{R})$, that is a dense subset of $L^2_{\mathbb{C}}(\mathbb{R})$; so, they coincide on $L^2_{\mathbb{C}}(\mathbb{R})$. □

Proposition 6.2.8. *Let f, g be functions in $L^2_{\mathbb{C}}(\mathbb{R})$; then, $f \cdot g$ is in $L^1_{\mathbb{C}}(\mathbb{R})$ and for almost every x in $\mathbb{R}$ it holds that*

$$[\mathcal{F}f \cdot g](x) = \frac{1}{2\pi}[\mathcal{F}f * \mathcal{F}g](x).$$

Proof. First of all we notice that, if f, g are in $L^2_{\mathbb{C}}(\mathbb{R})$, then $\mathcal{F}f$ and $\mathcal{F}g$ are both in $L^2_{\mathbb{C}}(\mathbb{R})$; so, $\mathcal{F}f * \mathcal{F}g$ is well defined and it is in $L^1_{\mathbb{C}}(\mathbb{R})$, as shown in 3.2.11.

Let f, g be any functions in $C_c(\mathbb{R})$; if we use 6.1.21 and we generalize 6.1.18 with the operator $\mathcal{F}^*$, we obtain that for almost every x in $\mathbb{R}$ it holds that

$$f(x) \cdot g(x) = \frac{1}{(2\pi)^2}[\mathcal{F}^*\mathcal{F}f](x) \cdot [\mathcal{F}^*\mathcal{F}g](x) = \frac{1}{(2\pi)^2}[\mathcal{F}^*(\mathcal{F}f * \mathcal{F}g)](x).$$

Hence, for almost every x in $\mathbb{R}$ the following identity holds true:

$$[\mathcal{F}f \cdot g](x) = \frac{1}{(2\pi)^2}[\mathcal{F}(\mathcal{F}^*(\mathcal{F}f * \mathcal{F}g))](x).$$

Since $\mathcal{F}f * \mathcal{F}g$ is in $L^1_{\mathbb{C}}(\mathbb{R})$, we can use theorem 6.1.21 and we obtain that the following inequality holds for almost every x in $\mathbb{R}$:

$$[\mathcal{F}f \cdot g](x) = \frac{1}{2\pi}[\mathcal{F}f * \mathcal{F}g](x).$$

Let us consider the operator $H_1 : L^2_{\mathbb{C}}(\mathbb{R}) \times L^2_{\mathbb{C}}(\mathbb{R}) \to C_0(\mathbb{R})$ such that

$$H_1(f; g) \coloneqq \mathcal{F}f \cdot g.$$

As shown in 6.1.2, we can state that it is well defined; if we join Hölder's inequality and 6.1.4, we obtain that H_1 is continuous. Similarly, we can consider the operator $H_2 : L^2_{\mathbb{C}}(\mathbb{R}) \times L^2_{\mathbb{C}}(\mathbb{R}) \to C(\mathbb{R})$ such that

$$H_2(f; g) \coloneqq \frac{1}{2\pi}\mathcal{F}f * \mathcal{F}g.$$

Thanks to proposition 3.2.13, we can state that $\mathcal{F}f * \mathcal{F}g$ is uniformly continuous; in particular, H_2 is well defined. As for the continuity, it follows immediately from 6.2.2 and 3.2.11. Since $C_c(\mathbb{R}) \times C_c(\mathbb{R})$ is a dense subset in $L^2_{\mathbb{C}}(\mathbb{R}) \times L^2_{\mathbb{C}}(\mathbb{R})$ where H_1 and H_2 coincide, they coincide everywhere. □

6.3 Application of Fourier transform to PDE

Definition 6.3.1. Let $u_0 : \mathbb{R} \to \mathbb{C}$ be any function. Let us consider the following partial derivative equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t; x) = \dfrac{\partial^2 u}{\partial x^2}(t; x) & \text{if } (t; x) \in (0; T) \times \mathbb{R} \\ u(0; x) = u_0(x) & \text{if } x \in \mathbb{R} \end{cases} \tag{6.4}$$

We say that (6.4) is the heat equation in $\mathbb{R}$.

Definition 6.3.2. Let $u_0 : \mathbb{R} \to \mathbb{C}$ be any function; let T be any positive real number. We say that $u : [0;T) \times \mathbb{R} \to \mathbb{C}$ is a solution of (6.4) if it has the following properties:

- u is continuous in $[0;T) \times \mathbb{R}$;
- for all $(t;x)$ in $(0;T) \times \mathbb{R}$, there exists
$$\frac{\partial^2 u}{\partial x^2}(t;x)$$
and it is continuous in $(0;T) \times \mathbb{R}$;
- for all $(t;x)$ in $(0;T) \times \mathbb{R}$, there exists
$$\frac{\partial u}{\partial t}(t;x)$$
and it is continuous in $(0;T) \times \mathbb{R}$;
- for all $(t;x)$ in $(0;T) \times \mathbb{R}$ the following identity holds true:
$$\frac{\partial^2 u}{\partial x^2}(t;x) = \frac{\partial u}{\partial t}(t;x);$$
- for all x in $\mathbb{R}$ it holds that
$$u(0;x) = u_0(x).$$

The purpose of this section is to find reasonable hypothesis on u_0 to make sure that there exist a time T and a solution u for equation (6.4) in $[0;T) \times \mathbb{R}$. Then, we will study the regularity of u of the solution.

Definition 6.3.3 (Heat kernel)**.**
Let us define the function $g_1 : \mathbb{R} \to \mathbb{R}$ such that

$$g_1(x) \coloneqq \frac{e^{-\frac{x^2}{4}}}{\sqrt{4\pi}}.$$

Let t be any positive real number; for all x in $\mathbb{R}$ we define

$$g_t(x) \coloneqq \sigma_{\sqrt{t}} g_1(x) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

The function $\psi : (0;+\infty) \times \mathbb{R} \to \mathbb{R}$ such that $\psi(t;x) \coloneqq g_t(x)$ is called heat kernel.

Remark 6.3.4. The Fourier transform can be used to solve formally the problem (6.4). Let us consider the equation

$$\frac{\partial^2 u}{\partial x^2}(t;x) = \frac{\partial u}{\partial t}(t;x).$$

If we denote as $\mathcal{F}u$ the Fourier transform in x and we apply the formulas described in 6.1.12 and 2.3.2, we find that

$$-y^2 \mathcal{F}u(t;y) = \mathcal{F}\frac{\partial^2 u}{\partial x^2}(t;y) = \mathcal{F}\frac{\partial u}{\partial t}(t;y) = \frac{\partial \mathcal{F}u}{\partial t}(t;y).$$

Since we are looking for a function u that is continuous in $[0;T) \times \mathbb{R}$, we have that for all y in $\mathbb{R}$, the function $\mathcal{F}u$ is a solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial \mathcal{F}u}{\partial t}(t;y) = -y^2 \mathcal{F}u(t;y); \\ \mathcal{F}u(0;y) = \mathcal{F}u_0(y). \end{cases} \tag{6.5}$$

Therefore, for all $(t;y)$ in $(0;+\infty) \times \mathbb{R}$ we obtain that

$$\mathcal{F}u(t;y) = \mathcal{F}u_0(y)e^{-y^2 t}.$$

Let $g_t : \mathbb{R} \to \mathbb{R}$ be as in definition 6.3.3 for all t in $(0;+\infty)$. Let us denote $\mathcal{F}g_t$ the Fourier transform in x. If we join 6.1.10 and 6.1.6, for all $(t;y)$ in $(0;+\infty) \times \mathbb{R}$ we have that

$$\mathcal{F}u(t;y) = \mathcal{F}u_0(y)e^{-y^2 t} = [\mathcal{F}u_0(y)][\mathcal{F}g_t(y)].$$

If we use the formula described in 6.1.18, for all $(t;y) \in (0;+\infty) \times \mathbb{R}$ we obtain that

$$\mathcal{F}u(t;y) = [\mathcal{F}u_0(y)][\mathcal{F}g_t(y)] = [\mathcal{F}u_0 * g_t](y).$$

Since the Fourier transform in injective, for all $(t;y) \in (0;+\infty) \times \mathbb{R}$ we have that

$$u(t;y) = u_0 * g_t(y).$$

Despite the resolution described in 6.3.4 is only formal, it suggests a formula for the solution. The aim of the next theorem is to give reasonable hypothesis that make the procedure described in 6.3.4 a rigorous proof.

Theorem 6.3.5 (Existence of a solution for heat equation in $\mathbb{R}$)**.**
Let $u_0 : \mathbb{R} \to \mathbb{R}$ be any continuous function such that u_0 is in $L^\infty(\mathbb{R})$. Let t be any positive real number; let g_t be as in 6.3.3. Let us define the function $u : (0;+\infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$u(t;x) := \begin{cases} u_0(x) & \text{if } (t;x) \in \{0\} \times \mathbb{R}; \\ [u_0 * g_t](x) & \text{if } (t;x) \in (0;+\infty) \times \mathbb{R}. \end{cases}$$

Then, u is well defined and it is a solution of (6.4) in the sense of definition 6.3.2. Moreover, u is a smooth function in $(0;+\infty) \times \mathbb{R}$.

Proof. **Step 1**: Let g_1 be as in definition 6.4. We notice that $\|g_1\|_{L^1(\mathbb{R})} = 1$; hence, it's easy to see that if t is any positive real number, then it holds that

$$\|g_t\|_{L^1(\mathbb{R})} = 1.$$

If we join the fact that u_0 is in $L^\infty(\mathbb{R})$ and proposition 3.2.11, we can state that $u_0 * g_t$ is well defined for all positive real number t.

Step 2: If $(t;x)$ is any point in $(0;+\infty) \times \mathbb{R}$, the continuity of u in $(t;x)$ is an immediate consequence of theorem 2.3.1. If we join the fact that u_0 is a continuous function and corollary 3.2.20, we obtain that for all x in $\mathbb{R}$ the function u is continuous in $(0;x)$.

Step 3: Let k be any positive integer. We notice that there exists a polynomial p_k of degree k such that for all x in $\mathbb{R}$ for all t in $(0;+\infty)$ it holds that

$$\frac{\partial^k g_t}{\partial x^k}(t;x) = p_k\left(x;\frac{1}{t}\right) g_t(x);$$

in particular, we have that $\dfrac{\partial^k g_t}{\partial x^k}$ is in $L^1(\mathbb{R})$ for all t in $(0;+\infty)$ for all k in $\mathbb{N}$. Thanks to corollary 3.2.15, we have that for all t in $(0;+\infty)$ it holds that $u_0 * g_t$ is in $C^\infty(\mathbb{R})$ and for all k in $\mathbb{N}$ for all x in $\mathbb{R}$ the following identity holds true:

$$\frac{\partial^k(u_0 * g_t)}{\partial x^k}(x) = \left[u_0 * \frac{\partial^k g_t}{\partial x^k}\right](x).$$

In particular, for all $(t;x)$ in $0;+\infty) \times \mathbb{R}$ we have that

$$\frac{\partial^2(u_0 * g_t)}{\partial x^2}(x) = \int_{\mathbb{R}} u_0(x-y)\frac{y^2-2t}{8\sqrt{\pi}t^{\frac{5}{2}}}dy. \tag{6.6}$$

Step 4: Let x be any point in $\mathbb{R}$. Let k be any nonnegative integer. We notice that there exists a polynomial q_k such that for all t in $(0;+\infty)$ for all y in $\mathbb{R}$ it holds that

$$u_0(x-y)\frac{\partial^k g_t}{\partial t^k}(t;y) = u_0(x-y)q_k\left(x;\frac{1}{\sqrt{t}}\right)\frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}.$$

It's easy to see that there for all t_0 in $(0;+\infty)$ there exist a polynomial $\alpha_{k;t_0}$ and a real number ε greater than t_0 such that for all y in $\mathbb{R}$ for all t in $(t_0-\varepsilon;t_0+\varepsilon)$ it holds that

$$\left|u_0(x-y)\frac{\partial^k g_t}{\partial t^k}(t;y)\right| \leq |u_0(x-y)|\left|\alpha_{k;t_0}(y)\frac{e^{-\frac{y^2}{2t_0}}}{\sqrt{4\pi}}\right|.$$

We define $\beta_{k;t_0} : \mathbb{R} \to \mathbb{R}$ such that

$$\beta_{k;t_0}(y) := \left|\alpha_{k;t_0}(y)\frac{e^{-\frac{y^2}{2t_0}}}{\sqrt{4\pi}}\right|.$$

We define $\gamma_{k;t_0} : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma_{k;t_0}(y) := \sum_{i=0}^{k}\beta_{i;t_0}(y).$$

We notice that if i is any integer in $\{0;\ldots;k\}$, then $\gamma_{k;t_0}$ is domination for $\frac{\partial^i g_t}{\partial t^i}$ in $L^1((t_0-\varepsilon;t_0+\varepsilon)\times\mathbb{R})$. Since u_0 is in $L^\infty(\mathbb{R})$, we can use theorem 2.3.2. Hence, we can state that for all k in $\mathbb{N}$, for all x in $\mathbb{R}$, for all t in $(0;+\infty)$ there exists $\frac{\partial^k u}{\partial t^k}(t;x)$ and it is equal to

$$\frac{\partial^k u}{\partial t^k}(t;x) = \int_{\mathbb{R}} u_0(x-y)\frac{\partial^k g_t}{\partial t^k}(t,y)dy.$$

In particular, for all $(t;x)$ in $(0;+\infty)\times\mathbb{R}$ it holds that

$$\frac{\partial u}{\partial t}(t;x) = \int_{\mathbb{R}} u_0(x-y)\frac{y^2-2t}{8\sqrt{\pi}t^{\frac{5}{2}}}dy. \tag{6.7}$$

Step 5: If we join (6.6) and (6.7), the following identity holds for all $(t;x)$ in $(0;+\infty)\times\mathbb{R}$:

$$\frac{\partial u}{\partial t}(t;x) = \int_{\mathbb{R}} u_0(x-y)\frac{y^2-2t}{8\sqrt{\pi}t^{\frac{5}{2}}}dy = \frac{\partial^2 u}{\partial x^2}(t;x).$$

Therefore, we can state that u is a solution of (6.4) in the sense of definition 6.3.2.

Step 6: As for the regularity, we can slightly modify the procedure shown in details in step 3 and in step 4 to prove that u is in $C^\infty((0;+\infty)\times\mathbb{R})$. □

Chapter 7

Harmonic functions

7.1 Definitions and main properties

Definition 7.1.1 (Laplace operator)**.**
Let A be any open set in $\mathbb{R}^d$; let $f : A \to \mathbb{R}$ be any function in $C^2(A)$. We define the Laplacian of f as the function $\Delta f : A \to \mathbb{R}$ such that

$$\Delta f(x) \coloneqq \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x).$$

Remark 7.1.2. We recall that

$$\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} = \operatorname{div}(\nabla f).$$

Definition 7.1.3 (Harmonic function)**.**
Let A be any open set in $\mathbb{R}^d$; let $f : A \to \mathbb{R}$ be any function in $C^2(A)$. We say that f is an harmonic function if $\Delta f(x) = 0$ for all x in A.

Definition 7.1.4 (Mean value properties)**.**
Let A be any open set in $\mathbb{R}^d$; let $f : A \to \mathbb{R}$ be any continuous function. We say that f has the mean value property on the spheres if the following identity holds for all closed balls $\overline{\mathcal{B}(x_0; r)}$ completely contained in A:

$$f(x_0) = \frac{1}{\operatorname{Area}\left(\partial \mathcal{B}(x_0; r)\right)} \int_{\partial \mathcal{B}(x_0;r)} f(x) d\sigma(x).$$

We say that f has the mean value property on the balls if the following identity holds for all closed balls $\overline{\mathcal{B}(x_0; r)}$ completely contained in A:

$$f(x_0) = \frac{1}{\operatorname{Vol}\left(\mathcal{B}(x_0; r)\right)} \int_{\mathcal{B}(x_0;r)} f(x) dx.$$

Lemma 7.1.5. *Let r be any positive real number; let x_0 be any point in $\mathbb{R}^d$. Let $f : \overline{\mathcal{B}(x_0; r)} \to \mathbb{R}$ be a continuous function. Then, the following identity holds true:*

$$\int_{\mathcal{B}(x_0;r)} f(x) dx = \int_0^r \left(\int_{\partial \mathcal{B}(x_0;\rho)} f(x) d\sigma(x) \right) d\rho.$$

Proof. It can be proved by considering polar coordinates in $\mathbb{R}^d$; as a matter of facts, it is a consequence of the Coarea formula. □

Proposition 7.1.6. *Let A be any open set in $\mathbb{R}^d$; let $f : A \to \mathbb{R}$ be any continuous function. f has the mean value property on the spheres if and only if it has the mean value property on the balls.*

Proof. Let us assume that f has the mean value property on the spheres. Let $\overline{\mathcal{B}(x_0; r)}$ a closed balls completely contained in A. Then, it holds that

$$\begin{aligned}\int_{\mathcal{B}(x_0;r)} f(x)dx &= \int_0^r \left(\int_{\partial\mathcal{B}(x_0;\rho)} f(x)d\sigma(x) \right) d\rho \\ &= \int_0^r f(x_0)\text{Area}\left(\partial\mathcal{B}(x_0;\rho)\right) d\rho \\ &= f(x_0) \int_0^r \rho^{d-1} d\alpha_d d\rho \\ &= f(x_0)\alpha_d r^d \\ &= f(x_0)\text{Vol}\left(B(x_0;r)\right).\end{aligned}$$

Let us assume that f has the mean value property on the balls. Let $\overline{\mathcal{B}(x_0; r)}$ be a closed balls completely contained in A. Then, it holds that

$$\alpha_d r^d f(x_0) = \int_{\mathcal{B}(x_0;r)} f(x)dx.$$

If we derive, we obtain that

$$\alpha_d d r^{d-1} f(x_0) = \int_{\partial\mathcal{B}(x_0;r)} f(x)d\sigma(x).$$

To conclude, we notice that

$$\alpha_d d r^{d-1} f(x_0) = \text{Area}\left(\partial\mathcal{B}(x_0;r)\right) f(x_0).$$

□

Remark 7.1.7. From now on, we will denote the mean value property on the spheres and the mean value property on the balls as mean value property.

Proposition 7.1.8. *Let A be any open set in $\mathbb{R}^d$; let $f : A \to \mathbb{R}$ be any harmonic function. Then f has the mean value property.*

Proof. Let $\overline{\mathcal{B}(x_0; R)}$ be any closed balls completely contained in A. We define the function $g : (0; R] \to \mathbb{R}$ such that

$$g(r) := \frac{1}{\text{Area}\left(\partial\mathcal{B}(x_0;r)\right)} \int_{\partial\mathcal{B}(x_0;r)} f(x)d\sigma(x).$$

We claim that for all r in $(0; R)$ it holds that $g'(r) = 0$. We notice that

$$\frac{1}{\text{Area}\left(\partial\mathcal{B}(x_0;r)\right)} \int_{\partial\mathcal{B}(x_0;r)} f(x)d\sigma(x) = \frac{1}{\text{Area}\left(S^{d-1}\right)} \int_{S^{d-1}} f(x_0 + ry)d\sigma(y).$$

If we apply theorem 2.3.2 and the theorem of the divergence, we obtain that

$$\begin{aligned}
\frac{dg}{dr}(r) &= \frac{1}{\text{Area}\,(S^{d-1})}\frac{d}{dr}\left(\int_{S^{d-1}} f(x_0+ry)d\sigma(y)\right) \\
&= \frac{1}{\text{Area}\,(S^{d-1})}\int_{S^{d-1}} < \nabla f(x_0+ry), y > d\sigma(y) \\
&= \frac{1}{\text{Area}\,(S^{d-1})}\int_{S^{d-1}} < \nabla f(x_0+ry), \nu(y) > d\sigma(y) \\
&= \frac{1}{r\text{Area}\,(S^{d-1})}\int_{\mathcal{B}(0;1)} \text{div}\nabla f(x_0+ry)dy \\
&= \frac{1}{r\text{Area}\,(S^{d-1})}\int_{\mathcal{B}(0;1)} \Delta f(x_0+ry)dy = 0
\end{aligned}$$

because f is harmonic. In particular, we have that g is a constant function; since f is continuous in x_0, then it that

$$\lim_{r\to 0} g(r) = f(x_0).$$

This is enough to conclude that $g(r) = f(x_0)$ for all r in $(0; R]$. □

Theorem 7.1.9. *Let A be any open set in $\mathbb{R}^d$. Let $f : A \to \mathbb{R}$ be any continuous function with the mean value property. Then f is an harmonic smooth function in A.*

Proof. We will carry out the proof assuming that A is $\mathbb{R}^d$. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be any smooth function with the following properties:

- it is supported in $\mathcal{B}(0;1)$;
- there exists a smooth function $g : [0;+\infty) \to \mathbb{R}$ such that for all y in $\mathbb{R}^d$ it holds that $\rho(y) = g(|y|)$;
- $\displaystyle\int_{\mathbb{R}^d} \rho(x)dx = 1.$

If we join 3.2.11 and 3.2.14, we obtain that $f * \rho$ is a well defined smooth function in $\mathbb{R}^d$. Let x be any point in $\mathbb{R}^d$. The following identities hold true:

$$\begin{aligned}
f*\rho(x) &= \int_{R^d} f(x-y)\rho(y)dy \\
&= \int_0^1 \left(\int_{\partial\mathcal{B}(0;r)} f(x-y)\rho(y)d\sigma(y)\right)dr \\
&= \int_0^1 g(r)\left(\int_{\partial\mathcal{B}(0;r)} f(x-y)d\sigma(y)\right)dr \\
&= \int_0^1 g(r)\left(\int_{\partial\mathcal{B}(x;r)} f(t)d\sigma(t)\right)dr \\
&= \int_0^1 g(r)d\alpha_d r^{d-1} f(x)dr \\
&= f(x)\int_{\mathcal{B}(0;1)} \rho(y)dy \\
&= f(x).
\end{aligned}$$

Hence, f is a smooth function. As shown in 7.1.8, we have that

$$\frac{d}{dr}\left(\frac{1}{\text{Area}\,(\partial\mathcal{B}(x_0;r))}\int_{\partial\mathcal{B}(x_0;r)} f(x)d\sigma(x)\right) = \frac{1}{\text{Vol}\,(\mathcal{B}(0;1))}\int_{\mathcal{B}(0;1)} \Delta f(y)dy = 0.$$

Since f has the mean value property, for all x_0 in $\mathbb{R}^d$ for all r in $(0;+\infty)$ it holds that

$$\int_{\mathcal{B}(x_0;r)} \Delta f(x)dx = 0;$$

this is enough to state that f is harmonic. □

Proposition 7.1.10 (Maximum principle).
Let A be any bounded open set in $\mathbb{R}^d$; let $f : \overline{A} \to \mathbb{R}$ be any function such that it is continuous in $\overline{A}$ and it is harmonic in A. Let x_0 be in $\overline{A}$ any maximum or minimum point for f. Then x_0 is in ∂A. Moreover, if we also assume that A is connected and there exist maximum or minimum points in A for f, then f is a constant function.

Proof. First of all, we notice that f admits maximum and minimum in $\overline{A}$. Let x_0 be a maximum or minimum point for f; let us assume that x_0 is in A. If we denote as $\tilde{A}$ the connected component of A containing x_0, we claim that f is a constant function in $\tilde{A}$. Let us define

$$B := \{x \in \tilde{A} \mid f(x) = f(x_0)\}.$$

It's immediate to see that B is a closed non-empty set. Since $f(x_0)$ is maximum or minimum for f and f has the mean value property, if x is any point in B there exists a radius r such that for all y in $\mathcal{B}(x;r) \cap \tilde{A}$ it holds that $f(y) = f(x)$; in other words, B is an open set. This is enough to state that B equals $\tilde{A}$. □

Corollary 7.1.11. *Let A be any open set in $\mathbb{R}^d$; let $u_0; u_1 : \overline{A} \to \mathbb{R}$ be continuous functions. Let us assume that u_0 and u_1 are harmonic in A. If $u_0(x) \geq u_1(x)$ for all x in ∂A, then the inequality holds for all x in A. If we also assume that A is connected and there exists x_0 in A such that $u_0(x_0) = u_1(x_0)$, then u_0 and u_1 coincide in A.*

Proof. If we apply proposition 7.1.10 with $u_0 - u_1$, then thesis follows immediately. □

7.2 Harmonic and holomorphic functions

Proposition 7.2.1. *Let A be any open set in $\mathbb{C}$. Let $f : A \to \mathbb{C}$ be any holomorphic function. Then, if we identify as A the corresponding open set in $\mathbb{R}^2$, then $\Re f$ and $\Im f$ are harmonic functions in A.*

Proof. Since f is holomorphic in A, for all $(x;y)$ in A the Cauchy-Riemann' equations hold true:

$$\begin{cases} \dfrac{\partial \Re f}{\partial x}(x;y) = \dfrac{\partial \Im f}{\partial y}(x;y); \\ \dfrac{\partial \Re f}{\partial y}(x;y) = -\dfrac{\partial \Im f}{\partial x}(x;y). \end{cases}$$

We recall that $\Re f$ and $\Im f$ are smooth functions. If we derive with respect to x, we obtain that

$$\frac{\partial^2 \Re f}{\partial x^2} = \frac{\partial^2 \Im f}{\partial y \partial x} = \frac{\partial^2 \Im f}{\partial x \partial y} = -\frac{\partial^2 \Re f}{\partial^2 y};$$

if we derive with respect to y, we obtain that

$$\frac{\partial^2 \Im f}{\partial y^2} = \frac{\partial^2 \Re f}{\partial y \partial x} = \frac{\partial^2 \Re f}{\partial x \partial y} = -\frac{\partial^2 \Im f}{\partial^2 y}.$$

□

Theorem 7.2.2. *Let A be any simply connected open set in $\mathbb{C}$. Let $u : A \to \mathbb{R}$ be an harmonic function. There exists an holomorphic $f : A \to \mathbb{C}$ such that $\Re f = u$.*

Proof. Let us define define $g : A \to \mathbb{C}$ such that

$$g(x; y) := \frac{\partial u}{\partial x}(x; y) - i\frac{\partial u}{\partial y}(x; y).$$

We claim that g is an holomorphic function. Since u is in $C^2(A)$, we can switch the order of derivation and we obtain that

$$\frac{\partial \Re g}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 \Im g}{\partial \partial x}.$$

Since u is harmonic, we have that

$$\frac{\partial \Re g}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \Im g}{\partial y}.$$

Let z_0 be any point in A. Since A is a simply connected open set, there exists an holomorphic function $f : A \to \mathbb{C}$ such that $f(z_0) = u(z_0)$ and for all z in A it holds that $f'(z) = g(z)$. In particular, we have that

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial \Re f}{\partial x} - i\frac{\partial \Re f}{\partial y}.$$

In other words, we have that $\nabla \Re f = \nabla u$ and $\Re f(z_0) = f(z_0) = u(z_0)$. Since A is connected, we obtain that $\Re f$ and u coincides in A. □

Remark 7.2.3. In theorem 7.2.2 it is necessary to assume that A is simply connected. If A is $\mathbb{C} \setminus \{0\}$ and $u : A \to \mathbb{R}$ is such that $u(z) := \log(|z|)$, it's easy to see that u is an harmonic function, but it cannot be the real part of an holomorphic function f between A and $\mathbb{C}$. In fact, u is locally the real part of a branch of the complex logarithm.

Remark 7.2.4. In theorem 7.2.2, the function $\Im f$ is called harmonic conjugate of u. Thanks to Cauchy-Riemann' equations, we notice that

$$< \nabla \Re f, \nabla \Im f >= \frac{\partial \Re f}{\partial x}\frac{\partial \Im f}{\partial x} + \frac{\partial \Re f}{\partial y}\frac{\partial \Im f}{\partial y} = 0.$$

In other words, $\nabla \Re f$ and $\nabla \Im f$ are always orthogonal.

Corollary 7.2.5. *Let A be any open set in $\mathbb{R}^2$; let $u : A \to \mathbb{R}$ be an harmonic function. Then u is analytic.*

Proof. Let z_0 be any point in A; let r be any radius such that $\mathcal{B}(z_0; r)$ is completely contained in A. Since the open balls are simply connected, we can apply theorem 7.2.2. Let $f : \mathcal{B}(z_0; r) \to A$ an holomorphic function such that $\Re f = u$ in $\mathcal{B}(z_0; r)$. Then f is analytic in $\mathcal{B}(z_0; r)$, i. e. the power series

$$\sum_{n \in \mathbb{N}} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges toward f uniformly in $\mathcal{B}(z_0; r)$. If we consider the real part of the series, it is a power series that converges uniformly toward $\Re f$ in $\mathcal{B}(z_0; r)$. □

Corollary 7.2.6. *Let A be any open set in $\mathbb{R}^2$; let $u, v : A \to \mathbb{R}$ harmonic functions. Let us assume that there exists a set U completely contained in A with the following properties:*

- *it admits a cluster point z_0 in U;*
- *for all z in U, it holds that $u(z) = v(z)$.*

Let us denote as $\tilde{A}$ the connected component of A that contains z_0. Then f and g coincide in $\tilde{A}$.

Proof. It is a consequence of the principle of analytic continuation for holomorphic functions. □

Definition 7.2.7. Let A be any open set in $\mathbb{R}^d$; let $u_0 : \partial A \to \mathbb{C}$ be any function. Let us consider the following partial derivative problem:

$$\begin{cases} \Delta u(x) = 0 & \text{if } x \in A; \\ u(x) = u_0(x) & \text{if } x \in \partial A. \end{cases} \tag{7.1}$$

We say that (7.1) is the Laplace equation in A with Dirichlet boundary conditions.

Definition 7.2.8 (Solution of Laplace equation with Dirichlet boundary conditions)**.** Let A be any open set in $\mathbb{R}^d$; let $u_0 : \partial A \to \mathbb{C}$ be any function. Let $u : \overline{A} \to \mathbb{C}$ be any function with the following properties:

- u is continuous in $\overline{A}$;
- u is in $C^2(A)$ and for all x in A it holds that

$$\Delta u(x) = 0;$$

- for all x in ∂A it holds that $u(x) = u_0(x)$.

We say that u is a solution of the Laplace equation in A with Dirichlet boundary conditions.

Theorem 7.2.9 (Existence and uniqueness of the solution for Laplace equation)**.** *Let $g : [-\pi; \pi) \to \mathbb{C}$ be any function. Let us consider the principal arguments of a complex number $Arg : \mathbb{C} \setminus \{0\} \to [-\pi; \pi)$. Let us define $u_0 : \partial\mathcal{B}(0; 1) \to \mathbb{C}$ such that*

$$u_0(z) = g(Arg(z)).$$

Let us define the sequence of the Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$ *as in 5.1.1. Let us assume that*

$$\sum_{n\in\mathbb{Z}} |c_n| < +\infty.$$

Let us define the function $u : \overline{\mathcal{B}(0;1)} \to \mathbb{C}$ *such that*

$$u(z) := c_0(f) + \sum_{n\in\mathbb{N}^*} \left[c_n(f)z^n + c_{-n}(f)\overline{z}^n\right].$$

The function u *is well defined and it is a solution of* (7.1) *in the sense of 7.2.8.*

Proof. Since the series at right hand side converge totally in $\overline{\mathcal{B}(0;1)}$, it's immediate to see that u is well defined and continuous in $\overline{\mathcal{B}(0;1)}$. We notice that the function $f^+ : \overline{\mathcal{B}(0;1)} \to \mathbb{C}$ such that

$$f^+(z) := \sum_{n\in\mathbb{N}^*} c_n(f)z^n$$

is well defined and continuous in $\overline{\mathcal{B}(0;1)}$; moreover, it is holomorphic in $\mathcal{B}(0;1)$. In particular, it is harmonic. Similarly, if we define $f^- : \overline{\mathcal{B}(0;1)} \to \mathbb{C}$ such that

$$f^-(z) := \sum_{n\in\mathbb{N}^*} c_{-n(f)}\overline{z}^n,$$

the definition is well posed, the function is continuous in $\overline{\mathcal{B}(0;1)}$ and it is anti-holomorphic in $\mathcal{B}(0;1)$. In particular, it is harmonic. This is enough to state that u is continuous in $\overline{\mathcal{B}(0;1)}$ and it is harmonic in $\mathcal{B}(0;1)$. It's immediate to see that for all z in $\partial\mathcal{B}(0;1)$ it holds that

$$\begin{aligned}
u(z) &= c_0(f) + \sum_{n\in\mathbb{N}^*} \left[c_n(f)z^n + c_{-n}(f)\overline{z}^n\right] \\
&= c_0(f) + \sum_{n\in\mathbb{N}^*} \left[c_n(f)e^{in\mathrm{Arg}(z)} + c_{-n}(f)e^{-in\mathrm{Arg}(z)}\right] \\
&= g(\mathrm{Arg}(z)) = u_0(z).
\end{aligned}$$

As for the uniqueness, it is an immediate consequence of 7.2.6. □

Remark 7.2.10. Let u be as in theorem 7.2.9. Since the series converges totally in

$\overline{\mathcal{B}(0;1)}$, for all z in $\mathcal{B}(0;1)$ the following identities hold true:

$$\begin{aligned}
u(z) &= c_0(f) + \sum_{n\in\mathbb{N}^*} \left[c_n(f)z^n + c_{-n}(f)\overline{z}^n\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)dt + \sum_{n\in\mathbb{N}^*} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt\right) z^n + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{int}dt\right)\overline{z}^n \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\left[1 + \sum_{n\in\mathbb{N}^*}\left(z^n e^{-int} + \overline{z}^n e^{int}\right)\right] dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\left[1 + 2\Re\left(\sum_{n\in\mathbb{N}^*}\left(ze^{-it}\right)^n\right)\right] dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)2\Re\left(\sum_{n\in\mathbb{N}}\left(ze^{-it}\right)^n\right) dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)2\Re\left(\frac{ze^{-it}}{1-ze^{-it}}\right) dt \\
&= \int_{-\pi}^{\pi} g(t)\frac{1-|z|^2}{2\pi\left|e^{-it}-z\right|}dt.
\end{aligned}$$

If we define the Green function in $\mathcal{B}(0;1)$ $G : [-\pi;\pi] \times \mathcal{B}(0;1) \to \mathbb{R}$ such that

$$G(t;z) \coloneqq \frac{1-|z|^2}{2\pi\left|e^{-it}-z\right|},$$

we have shown that the solution u of (7.1) is a kind of average of the boundary datum. As a matter of fact, this principle holds true under reasonable hypothesis on the regularity of the open set A.