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The moduli space of smooth curves

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Introduction

The purpose of this thesis is to study the moduli problem for smooth projective curves. We will outline the construction of M_g: the (coarse) moduli space for smooth projective curves of genus g. We will follow the proof by Mumford [GIT] and Gieseker [Gie82] that uses Geometric Invariant Theory (GIT).

These notes will outline the path followed to construct M_g, introducing the main tools used for the central construction.

0.1 The general problem

In algebraic geometry classification is a key question. Given a class of varieties, we could consider them up to an equivalence relation. For example, we could consider closed subvarieties of $\mathbb{P}^n$ up to projective transformation or varieties on a fixed field of dimension 1 up to isomorphism. The main goal is to construct a moduli space which is a space whose "points" are in bijection with these varieties up to a chosen equivalence relation. Furthermore, we want to encode how these varieties vary continuously. A moduli problem is essentially this kind of classification problem.

We consider maps $C \to S$ for general S, whose fibers are varieties we are interested to classify. We define *famililes* such maps $C \to S$ with some additional properties depending on the context. We will consider families up to some equivalence relation $\sim_S$. This relation will match the relation mentioned above when $S = \text{Spec } k$. We could define a pullback of classes over maps $T \to S$ making sure that properties we used to define families are still verified. Usually the pullback is a base change. We define a **moduli problem** as a contravariant functor $\mathcal{M} : Sch \to Set$

$$\begin{aligned} \mathcal{M}(S) &:= {}^{\{\text{families over } S\}}\big/_{\sim_S} \\ \mathcal{M}(f^\star : T \to S) &:= f^\star : \mathcal{M}(S) \to \mathcal{M}(T). \end{aligned}$$

In this setting the initial varieties up to equivalence are $\mathcal{M}(\text{Spec } k)$.

In order to study this functor we ask if it's representable i.e. if there exists $M \in \text{Sch}$ and a natural isomorphism $\eta : M \to \text{Hom}(-, M)$. In this case we say M is a **fine moduli space** for $\mathcal{M}$. We obtain $\mathcal{M}(\text{Spec } k) = \text{Hom}(\text{Spec } k, M)$ and the construction of M is exactly the answer to our question. In this setting every family $\mathcal{F}$ over a scheme S corresponds to a unique morphism $f : S \to M$ and we also have $\mathcal{F} = f^\star \mathcal{U}$ where $\mathcal{U} = \eta_M^{-1}(Id_M)$ is called the Universal Family.

Unfortunately, it turns out that such a fine moduli space does not exist in almost every situation, neither in the case of smooth projective curves of fixed genus. We could try to weaken the condition of representability and this leads to the introduction of coarse moduli spaces. This solution was adopted by Mumford and then Gieseker more than 40 years ago.

Definition (Coarse Moduli Space)**.** Let $\eta : \mathcal{M} \to \text{Hom}(-, M)$ be a natural transformation such that:

- $\eta_{\text{Spec } k} : \mathcal{M}(\text{Spec } k) \to \text{Hom}(\text{Spec } k, M)$ is bijective,
- for each scheme N and $\nu : \mathcal{M} \to \text{Hom}(-, N)$, there exist a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$ where $h_f : \text{Hom}(-, M) \to \text{Hom}(-, N)$ induced by f.

The first assumption maintains the requirement on k-points as before but we replace representability with a weaker request: now the existence of a universal family is not assured. The focus of the thesis is now on constructing coarse moduli spaces, in particular those of smooth projective curves of genus g.

We look for a discrete invariant and we reduce to study objects with fixed invariant's value: for instance the degree or the rank of vector bundle (clearly invariant under isomorphism), the Hilbert polynomials for closed subschemes of $\mathbb{P}^n$ and genus of curves. In this regard, a central role is played by the flatness of families (a classical requirement for moduli problems): often fibres of a flat map $f : C \to S$ (S connected) have the same invariant's value.

Introducing coarse moduli spaces we sacrifice the existence of a universal family, but we can still define a local version. For a moduli problem $\mathcal{M}$ we say that a family $\mathcal{F}$ over a scheme S has the **local universal property** if for any family $\mathcal{G}$ over T and any k-point $t \in T$, there exist a neighbourhood $t \in U \subset T$ and a morphism $f : U \to S$ such that $\mathcal{G}|_U \sim_U f^\star \mathcal{F}$. In that case, by taking $T = \text{Spec } k$, we have a surjection of k-points of S onto $\mathcal{M}(\text{Spec } k)$. Assume moreover that we have a group acting on S in such a way that k-points corresponding to the same element in $\mathcal{M}(\text{Spec } k)$ are identified. We would like then to "quotient" S to obtain a moduli space. We have the following:

Proposition. *Let $\mathcal{F}$ be a family with the local universal property over a scheme S. Furthermore, suppose that there is an algebraic group G acting on S such that two k-points s, t lie in the same G-orbit if and only if $\mathcal{F}_t \sim \mathcal{F}_s$. Then:*

- *any coarse moduli space is a "categorical quotient" of the G-action on S,*
- *a "categorical quotient" of the G-action on S is a coarse moduli space if and only if it is an orbit space.*

This proposition deals with the action of groups on schemes and its thesis will be clearer after the reading of next section.

0.2 Geometric invariant theory

Given an algebraic group G acting on a scheme X, we are now interested in defining a quotient scheme. To construct M_g we need to study the action of linearly reductive groups, which are groups for which every finite representation is completely reducible.

A first notion is given by a **categorical quotient** via a universal property. Unfortunately, this first notion gives us a scheme that, in general, is far from the topological notion of quotient. Specifically, we do not obtain a *geometric quotient* i.e. a quotient where the preimage of each point is a single orbit.

Looking at the sheaf of regular function we have an action on $\mathcal{O}_X(U)$ (where $U \subset X$ open and invariant). We would like regular function on quotient to be exactly G-invariant regular functions on X. With that idea, we introduce the notion of **good quotient**. It turns out that any good quotient is indeed a categorical one.

In order to obtain that the quotient is a variety, we have to answer the following question: when are the G invariants of a finite generated k algebra finitely generated?

That's exactly Hilbert's 14th problem and we will use the theorem of Nagata in the case of a rational action of a linearly reductive group. We can now define a GIT quotient for affine schemes.

Given G reductive acting on an affine scheme we define $X//G := \text{Spec } \mathcal{O}(X)^G$. This turns out to be a finite type scheme over k. $X//G$ is a good quotient and an affine scheme. As mentioned before, when a group acts on affine schemes in general a geometric quotient does not exist. The main obstacle is that the action is not closed, i.e. not every orbit is closed. With this in mind, we define $X^s \subset X$ **stable** points as an open subset where the restriction of GIT quotient is geometric.

We are interested in defining a GIT quotient for quasi-projective varieties. If X is projective, we can write $X = \text{Proj } (R(X))$ where $R(X) = {}^{k[x_i]}\!/_{I(X)}$. Let's suppose this action is linear i.e. it's a restriction of the classical action of GL_{n+1} on $\mathbb{P}^n$. We have:

$$X \dashrightarrow X//G := \text{Proj } (R(X)^G).$$

We introduce X^{ss} (semistable points) as the open set where the map is defined.

Consider now a line bundle $L \to X$. We can define a **linearization** of a given action on X. Roughly, it is an action on L such that: $L \to X$ is G-equivariant and it is linear on fibres of L. In such a way we generalize our action:

$$G \curvearrowright R(X, L) := \bigoplus_{r \geq 0} H^0(X, L^{\otimes^r})$$

and we set $X//_L G := \text{Proj } (R(X, L)^G)$. In the same way as before, we define $X^{ss}(L)$ and $X^s(L)$ as respectively semistable and stable point relative to a given linearization. We have the following:

Theorem (Mumford)**.** *Let G be a reductive group acting on a quasi-projective variety X with a linearization for L. Then $X//_L G$ is quasi-projective and there is a good quotient $\phi : X^{ss}(L) \to X//_L G$ of the G-action on $X^{ss}(L)$. Furthermore there is $Y^s \subset X//_L G$ open such that $\phi^{-1}(Y^s) = X^s(L)$ and $\phi_{|X^s(L)}$ is a geometric quotient.*

To conclude this exposition of GIT, we will state a numerical criterion which is the "typical" result used to determine (semi)stability of points. First of all, we define Hilbert-Mumford weight $\mu(x, \lambda)$. Consider a one-parameter subgroup (1-PS) $\lambda : \mathbb{G}_m \to X$ that we can lift to an action on $\mathbb{A}^{n+1}$. This is a representation of torus and we define $\mu(x, \lambda)$ as one specific weight of this representation.

Theorem (Hilbert-Mumford Critereon)**.** *Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, for $x \in X(k)$, we have:*

$$x \in X^{ss} \iff \mu(x, \lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$
$$x \in X^{s} \iff \mu(x, \lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

The $\Rightarrow$ implication is easily verified, because a (semi)stable point is such for every subgroups. The converse means that if G is reductive it has enough 1-PSs to detect points in the closure of an orbit.

0.3 Construction of moduli space of curves

We would construct a moduli space M_g for projective smooth curves of genus g. To be precise we consider the moduli problem $\mathcal{M}_g : \text{Sch}/k \to \text{Set}$:

$$\mathcal{M}_g(S) := \{\text{proper, flat families } C \to S \text{ whose geometric fibres are smooth, connected 1-dimensional schemes of genus } g\}.$$

We restrict from the beginning to the category of finite type schemes over k (that we call Sch/k), usually these functors are deifned for general schemes but we do not need this generalty. It's well known that this functor is not representable due to the existence of non-trivial automorphism of curves. We could *rigidify* the problem by adding some extra structure in such a way that no non-trivial automorphisms of an underlying curve can fix the extra structure.

Given a curve $\mathcal{C}$, we have the v-canonical embedded $\phi : \mathcal{C} \to \mathbb{P}^n$ provided by the ample line bundle $K_{\mathcal{C}}^{\otimes^v}$ (for $v \geq 3$). This allows us to see the curves as points of Hilbert scheme $\mathrm{Hilb}_n^{p(x)}$ for suitable n and p of degree 1 (they depends on v and g).

Roughly we consider the locally closed subscheme $H_v \in \mathrm{Hilb}_n^{p(x)}$ made of curves on P_n of fixed genus such that the restriction of $O_{P_n}(1)$ is $K_{\mathcal{C}}^{\otimes^v}$.

The map ϕ defined above is not unique: clearly it depends on a basis in $H^0(\mathcal{C}, K_{\mathcal{C}}^{\otimes^v})$ and all differents embeddings are obtained acting with $PGL(n+1)$ on $\mathbb{P}^n$. This action induces an action on $\mathrm{Hilb}_n^{p(x)}$ and it restricts on H_v because the conditions that define H_v are invariant under automorphisms of $\mathbb{P}^n$.

Our goal is to quotient H_v by $PGL(n+1)$ obtaining a *geometric quotient* (i.e. in some sense an orbit space) for smooth curves, according to the proposition stated in the first section.

For the final step, we will study (semi)stability of points in H_v. Using a numerical criterion by Gieseker we will prove that the quotient is geometrical.

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Chapter 1

Introduction to moduli problems

Classify geometric objects is an everyday topic in mathematics: we classify them from high school. In this chapter, we will approach the problem from an algebraic geometry point of view.

In the first part we will explain the problem. Starting from a general point of view we will give a specific setting to the moduli problem. We will try to give several examples, in particular we focus on elliptic curves. This is a well-known example and a starting point for the main construction of the thesis.

1.1 Basic ideas

Fixed a category $\mathcal{C}$ we could consider a specific class of objects $\mathcal{A}$ we are interested in classifying them. Some examples are circles in the euclidean plane, real three-dimension manifolds equipped by a metric, holomorphic surfaces or elliptic curves. The category fixed for these examples could be respectively: the category of algebraic sets in the plane, Riemannian manifolds, complex manifolds and finite type scheme over k.
We are interested to classify them up to a fixed equivalent relation $\sim$, that came out from the nature of the problem. This often corresponds to classify them up to isomorphism in the chosen category.
In our case we can consider them up to the followings:

$\mathcal{A}$	$\sim$	category $\mathcal{C}$
circles	congruence	algebraic sets in $\mathbb{A}^2$
real three-mainfolds with metrics	isometry	Riemannian manifolds
holomorphic surfaces	biholomorphism	complex manifolds
elliptic curves	isomoprhism	finite type scheme over k

Notice that a priori $\mathcal{A}$ could not be a set, but we can avoid dealing with this set-theoretical drawback because in our cases it is always a set.
$\mathcal{A}/_{\sim}$ is just a set without any structure that contains a point for every object up to the equivalent relation we have considered. Our intuition tells us that isomorphism classes could change by "deforming" them. Consider for example the real surfaces with a given metric, we could change the metric "smoothly" and we would like not to lose this information. We would like to encode this on the set $\mathcal{A}/_{\sim}$. Roughly this is done giving to this object a structure such that it becomes an element of the category $\mathcal{C}$.
Riemann introduced in 1857 the term "moduli space" to refer to this, where moduli refer to something that moves.

Problem 1.1 (Classification problem). *Fix a class of geometric objects $\mathcal{A}$ (in a category*

$\mathcal{C}$) and an equivalence relation on these. How can we give to $^{\mathcal{A}}\!/\!_{\sim}$ a geometric structure that encodes how the objects are (in some sense) "near"?

In order to study this problem in algebraic geometry, we introduce the notion of family. We will see that it is particularly suitable.

Definition 1.2 (Family)**.** Given a class of objects in a fixed category $\mathcal{C}$. We say that a surjective map $f : X \to S$ (where f, X, S are intended in $\mathcal{C}$) is a family if the fibres of this map are objects in $\mathcal{A}$.

We should not consider this as a rigorous definition for families, we will see soon that usually there are some extra requirements. The idea under this definition is that the fibre of a flat morphism varies "continuously" and this will tell us what to be "near" means for algebraic varieties (and scheme in general).

Suppose that there exists a family $f : X \to M$ such that the fibres of the morphism are exactly the equivalence classes of $\mathcal{A}$. $^{\mathcal{A}}\!/\!_{\sim}$ is represented, in some sense, by the points of M. This is the idea of what we call "moduli space" (M) and "universal family" ($X \to M$). This example is useless but illustrates the idea:

Example 1.3. Suppose we are interested in classifying circles in the real plane ($\mathcal{A}$) up to rigid movements ($\sim$). It is clear that the radius length define uniquely the elements in the equivalence classes. So we have a bijection between $^{\mathcal{A}}\!/\!_{\sim}$ and $\mathbb{R}_+$.
Consider an equilateral cone C defined by $x^2 + y^2 = z^2$ and cut it with the plane $z = 0$. Call C^+ the semi-cone in $z > 0$.
We can define the height map (from the plane $z = 0$) as

$$\begin{aligned} C^+ &\to \mathbb{R}^+ \\ (x, y, z) &\mapsto z. \end{aligned}$$

The fibre of a point z_0 is exactly the circle of radius z_0. It is therefore clear that this map represents in some sense a "universal family" and that $\mathbb{R}^+$ is the "moduli space".

1.2 Fine moduli space and universal family

We lose now the very general setting and we start working with schemes. We restrict to Sch/k: the schemes of finite type over k. Usually these functors are defined for general schemes but we do not need this generalty.

Without pretending to be formal, we can reformulate the classification problem:

Problem 1.4 (Moduli problem)**.** *We call moduli problem a contravariant functor $\mathcal{M}$: $Sch/k \to Set$ such that:*

$$\begin{aligned} \mathcal{M}(S) = \{&\textit{families } f : X \to S \textit{ that satisfied definedححpropreties and} \\ &\textit{eventually more structure}\} \\ \mathcal{M}(g : S \to T) = f^* &: \mathcal{M}(T) \to \mathcal{M}(S), \end{aligned}$$

Where f^* is the base change.
In order to have a well-defined functor, we ask that properties and extra structures are stable under base change.

We notice that the set $\mathcal{M}(\text{Spec } k)$ is what we called $^{\mathcal{A}}\!/\!_{\sim}$. This brings us back to the original problem.

Notation 1.5. Let $\mathcal{F} \to S$ be a family, and suppose $s \in S$ be a set-theoretic point of S, we call $\mathcal{F}_s \to \mathrm{Spec}\ k(s)$ the family under the base change $\mathrm{Spec}\ k(s) \to S$. This is simply the fibre of s.

From now on we will study classification problems simply considering the functors that define them. This allows us to give a suitable definition of moduli space.

Definition 1.6 (Fine moduli space)**.** Consider a moduli problem $\mathcal{M} : \mathrm{Sch}/k \to \mathrm{Set}$, we say that a k-scheme of finite type M is a **fine moduli space** if there is a natural isomorfism between $\mathcal{M}$ and $\mathrm{Hom}(\_, M)$.

If such a fine moduli space exists, we obviously have that $\mathcal{M}(\mathrm{Spec}\ k) = \mathrm{Hom}(\mathrm{Spec}\ k, M)$ i.e. there is a 1:1 correspondence between k-points of M and the equivalence classes $\mathcal{A}/\sim$.

In this situation, we have a natural isomorphism η:

$$\begin{array}{ccc} \mathrm{Hom}(S, M) = h_M(S) & \xrightarrow{\eta_S} & \mathcal{M}(S) \\ \big\downarrow{\scriptstyle \mathcal{M}(f)} & & \big\downarrow{\scriptstyle h_M(f)} \\ \mathrm{Hom}(T, M) = h_M(T) & \xrightarrow{\eta_T} & \mathcal{M}(T) \end{array}$$

The natural transformation determines an element $\mathcal{U} = \eta_M(\mathrm{Id}_M) \in \mathcal{M}(M)$ that define completely the transformation (thanks to Yoneda's lemma).

Definition 1.7 (Universal family)**.** Let $\mathcal{M}$ be a moduli problem. Thanks to Yoneda lemma we have that M represent $\mathcal{M}$ if and only if there exists $\mathcal{U} \in \mathcal{M}(M)$ such that the following holds: for every $\mathcal{F} \in \mathcal{M}(S)$ there exists a unique $g \in \mathrm{Hom}(S, M)$ such that $\mathcal{F} = g^*\mathcal{U}$.
In this case we call $\mathcal{U}$ the universal family.

Thanks to standard techniques of category theory, the remark below is an immediate consequence.

Remark 1.8. If such a fine moduli space (or universal family) exists, it is unique up to a canonical isomorphism.

An example is worth more than a thousand words. We recall an object we already know in a slightly different context and that we will resume it in chapter 4: the Grassmannian.

Example 1.9. It is well known that Grassmannian $\mathrm{Gr}(d, n)$ classify the subspaces of dimension d (or the quotients of dimension $n - d$) of a fixed space of dimension n, where the equivalence relation is the equality of subspaces.
We can construct it classically as a differential (or holomorphic) manifold. It is possible to construct it in an algebraic setting, emulating the classical case. The construction works also for the tautological subbundle i.e. the subbundle of $k^n \times \mathrm{Gr}(d, n)$ whose fibre over a point is the subspace the point represents. We define a moduli problem suitable for this situation. Fix $S \in \mathrm{Sch}/k$ and consider the short exact sequences

$$0 \to \mathcal{K} \to \mathcal{O}_S^n \to \mathcal{Q} \to 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are locally free sheaves on S and the rank of $\mathcal{Q}$ is d. We say that two exact sequences of this kind are equivalent if there exist isomorphisms α and β such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}_S^n & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \alpha} & & \big\| & & \big\downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{O}_S^n & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0 \end{array}$$

commute.
Hence the moduli functor of interest is:

$$\mathcal{G}r(S) := \{\text{exact sequences of locally free sheaves on } S\ 0 \to \mathcal{K} \to \mathcal{O}_S^n \to \mathcal{Q} \to 0\} / \sim$$
$$\mathcal{G}r(h : S \to T) := h^* : \mathcal{G}r(T) \to \mathcal{G}r(S).$$

It turns out that it is representable by a scheme, we call it $\mathrm{Gr}(d, n)$. Moreover, the tautological subbundle is the universal family.

There are plenty of examples of moduli spaces. The following families have more structure but it is still very easy.

Example 1.10. Suppose k algebraically closed. Consider the problem of classify the ordered set of distinct four k-points $(p_1, p_2, p_3, p_4) \in \mathbb{P}^1_k$ up to the natural action of GL_2.
It is well known that for every three distinct points on $\mathbb{P}^1$ there exists a unique Möbius transformation that send them to 0, 1, ∞. Hence $\mathcal{A}/\sim$ are the k-points of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
We encode now this problem with the formalism just defined. The following functor $\mathcal{M}$: $\mathrm{Sch}/k \to \mathrm{Set}$ is suitable for our problem:

$$\begin{aligned} \mathcal{M}(S) = \{(f : X \to S, \sigma_1, \sigma_2, \sigma_3, \sigma_4) \mid f \text{ is proper and flat}, \\ f^{-1}(s) \simeq \mathbb{P}^1 \text{ for every } s\ k\text{-pointand } \sigma_i \text{ are section of } f\big\} / \sim \\ \mathcal{M}(g : T \to S) = g^* : \mathcal{M}(S) \to \mathcal{M}(T), \end{aligned}$$

where $(f : X \to S, \sigma_1, \sigma_2, \sigma_3, \sigma_4) \sim (f' : X' \to S, \sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4)$ if there exists an isomorphism $g : X \to X'$ such that $f = f' \circ g$ and $\sigma'_i = g \circ \sigma_i$.
It is an easy verification that $\mathcal{M}(\mathrm{Spec}\ k) = \mathcal{A}/\sim$.
In this case it is clear that we have generalized our problem replacing Spec k with $S \in \mathrm{Sch}/k$, hence the requirements are consequently adjusted: the idea of introduce σ_i and the hypothesis $f^{-1}(s) = \mathbb{P}^1$ seems reasonable.
Anyway the hypothesis f proper and flat is quite obscure. These are added to obtain that $\mathcal{M}$ is representable. We will see in a while why we need flatness.
To conclude the example we exhibit the universal family, that in this situation is the trivial one. Consider the map

$$\begin{aligned} f : \mathbb{P}^1 \setminus \{0, 1, \infty\} \times_k \mathbb{P}^1 \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \\ (s, t) \mapsto s \end{aligned}$$

and define the sections such that: $\sigma_1(s) = (s, 0)$, $\sigma_2(s) = (s, 1)$, $\sigma_3(s) = (s, \infty)$, $\sigma_4(s) = (s, s)$.
These define an element of $\mathcal{M}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ that is an universal family.

In general, it turns out that the existence of a fine moduli space is rare and there are several reasons why this could not occur.

We state here a basic request for a representable functor.

Proposition 1.11 (gluing property)**.** *Suppose that $\mathcal{M}$ is a representable functor, the following map is a bijection (between sets).*

$$\{a \in \mathcal{M}(S)\} \to \{\ (f_i)_i\ /\ f_i \in \mathcal{M}(S_i) \text{ and } f_{i|S_i \cap S_j} = f_{j|S_i \cap S_j}\ \}$$

Where S_i is an open covering of S and the map is induced by the pullback via the inclusion $S_i \subset S$.

The proof is immediate thanks to the representability of $\mathcal{M}$. This proposition can be used to exclude the existence of a fine moduli space. Several problems can occur when we are trying to construct a fine moduli space. One problem that we could have is the existence of automorphisms of the objects we are classifying.

Problem 1.12 (Problem with automorphisms)**.** *Let k be algebraically closed. Consider a moduli problem $\mathcal{M}$ such that $\mathcal{M}(S)$ are map $X \to S$ that satisfied some property. Moreover, suppose that there exists an element $B \in \mathcal{M}(Spec\ k)$ that admit a non-trivial automorphism $\phi : B \to B$. We have that these settings often prevents that there exists a fine moduli space.*

We use some terms that will be clearer after chapter 2, anyway it is possible to understand the ideas simply assuming to deal with the correspondent's classical notions.

Proof. Define $G =< \phi >$, it acts on B. Suppose that there exists a categorical quotient B/G of B. Consider now an action of G on a scheme X, this define a diagonal action of G on $B \times X$, again suppose that there exists a quotient $(B \times X)/G$. For example, The first request is satisfied when ϕ has finite order, $B = \text{Proj } A$ and the action is induced by an algebra automorphism of A.
The family $B \times X \to X$ go to the quotient in two different ways:

$$(B \times X)/G \to X/G$$
$$B \times (X)/G \to X/G.$$

Every fibre of these two families is B. If the moduli problem is representable by M, both families are obtained from the universal family by a base change with the constant map $X/G \to M$. In general we have that these two families are not isomorphic, this implies that the function is not representable. □

This is an intrinsic problem that will occur in the main construction of the thesis, we need a new and more comprehensive notion of moduli space.

1.3 Coarse moduli spaces

In this section, we will weaken the idea of moduli space, as promised. This will resolve the problem 1.12. We define:

Definition 1.13 (Coarse moduli space)**.** let $\mathcal{M}$ be a moduli problem, we say that $M \in \text{Sch}/k$ is a *coarse moduli space* for $\mathcal{M}$ if there exists a natural transformation $\eta : \mathcal{M} \to \text{Hom}(\_, M)$ such that:

- $\eta_{\text{Spec } k} : \mathcal{M}(\text{Spec } k) \to \text{Hom}(\text{Spec } k, M)$ is bijective.
- for every $N \in \text{Sch}/k$ and $\nu : \mathcal{M} \to \text{Hom}(\_, N)$ there exists a unique map $f : M \to N$ such that

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\eta} & \text{Hom}(\_, M) \\ \downarrow \nu & \swarrow f_* & \\ \text{Hom}(\_, N) & & \end{array} .$$

In that case we say that $\mathcal{M}$ is *coarsely represented* by M.

Let us unravel this definition a little. The first point says that the points of M still represent the equivalence classes of our initial problem. The second one says that there is still a natural transformation $\mathcal{M} \to \text{Hom}(\_, M)$ and M gives the best approximation with Hom functors (but it is not necessary an isomorphism).

Obviously we have:

Remark 1.14. A fine moduli space is a coarse moduli space.

Moreover, it follows easily from the second property that:

Remark 1.15. Given a moduli functor $\mathcal{M}$ there exists at most a unique coarse moduli space that represent it.

The introduction of coarse moduli spaces resolves the problem with automorphisms. In general there are still several problems that can occur. We state some classical examples where a coarse moduli space does not exist.

Remark 1.16 (Jump phenomena). We explain this phenomena with an example. Fix $k = \overline{k}$ and a moduli functor $\mathcal{M}$. Suppose that there exists a family $\mathcal{F} \to S$ and $t \in S(k)$ such that

$$\mathcal{F}_s \sim \mathcal{F}_{s'} \text{ for all } s, s' \in S(k) \setminus \{t\},$$
$$\mathcal{F}_s \nsim \mathcal{F}_t \text{ for all } s \in S(k) \setminus \{t\}\,.$$

We have that does not exist a coarse moduli space for $\mathcal{M}$.
The idea is that there is a "jump" in the fibre of $\mathcal{F} \to S$.
This phenomenon tells us that the fibre is not continuous in some sense. Often it is possible to avoid this requiring that families are flat. We will see that in our case this hypothesis is required.

Proof. Suppose that there exists a coarse moduli space M for the problem, there exist a natural morphism $\eta : \mathcal{M} \to \mathrm{Hom}(\_, M)$. We have $\eta_S(\mathcal{F}) \in \mathrm{Hom}(S, M)$ and $\eta_{\mathrm{Spec}\ k(s)}(\mathcal{F}_s) \in \mathrm{Hom}(\mathrm{Spec}\ k, M)$ (through the composition $\mathrm{Spec}\ k(s) \to S$). Thanks to the hypothesis we have that $\eta_{\mathrm{Spec}\ k(s)}(\mathcal{F}_s)$ are the same k-point for all $s \in S(k) \setminus \{t\}$ and $\eta_{\mathrm{Spec}\ k(t)}(\mathcal{F}_t)$ is a different point: this implies that the map $\mathrm{Hom}(S, M)$ should be constant on $S(k) \setminus \{t\}$ and send t to another point, this is absurd. □

Another problem is the following:

Remark 1.17 (Unbounded problem). Again, we explain this with a sort of example. Consider an easy generalization of example 1.10: instead of fixing 4 points, we fix an arbitrary number n of points (hence the functor is the same, except that we can have an arbitrary number of sections). It is possible to prove that, if we fix n a moduli space there exists. If we consider the generalized problem, the scheme that represents it should be an infinite disjoint union of objects in Sch/k: that is not a finite type scheme.
In general: suppose that we are studying a moduli functor whose at priori depends on an arbitrary fixed number (as a dimension or a rank). Suppose moreover that this number is invariant on families and that it is possible to construct a fine moduli space if we fix it. We have that if we do not fix this number, the moduli space of the problem should not be a finite type scheme.

Often we avoid this phenomenon by fixing the discrete invariants of the problem.

We will see that in our case of interest such a coarse moduli space exists.

1.4 The functor $\mathcal{M}_g$

This section is devoted to the definition of the functor $\mathcal{M}_g$ and the analysis of the problem for $g = 0, 1$.

In the first subsection we deal with the classification of elliptic curves, a starting point for the main construction.

1.4.1 A classical example: elliptic curves

This example is the start point for the main construction of the thesis, we state it here and discuss the classical solution. It will lead us through the next subsections.

Definition 1.18 (Elliptic curves in classical algebraic geometry)**.** We define an elliptic curve as a Riemann surface, connected and compact, of genus 1 with a specified point.

Thanks to classical theory every compact Riemann surface has an algebraic realization. So every elliptic curve can be viewed as a submanifold of $\mathbb{P}^2$ given by the closure of an equation of the form $y^2 = p(x)$ in $\mathbb{C}^2$, where p is a polynomial of degree 3 without multiple roots.

On the other hand, also the converse is true: it is a straightforward verification that every equation of this kind defines a holomorphic manifold in $\mathbb{P}^2$ and it is possible to verify that the genus of a surface of this kind is 1.

This is not the unique description we can give to elliptic curves: these can be described as complex tori.

Remark 1.19. Let C be an elliptic curve, it is holomorphic to a complex tori i.e. a quotient $\mathbb{C}/\Lambda$ where Λ is a discrete subgroup of $\mathbb{C}$. In this description the marked point corresponds to the origin.
Rescaling the lattice Λ does not change the isomorphism class (in the holomorphic sense) of the torus. Hence we can suppose $\Lambda =< 1, \tau >_{\mathbb{Z}}$ where $\tau \in \mathcal{H}$ (i.e. in the $\Im(z) > 0$ semiplane of $\mathbb{C}$).

This analogy is well known also because we can give a group law to an elliptic curve, that corresponds to the usual sum modulo Λ on the torus. We avoid proving these classical results, a reference for elliptic curves is [Sil09].

From a classical point of view, we could ask if there exists a manifold that parametrizes all elliptic curves up to biholomorphism, the answer is positive.

Thanks to the previous remark, we have that τ determines the isomorphism class of an elliptic curve. We would now understand when two different values of τ define the same elliptic curves. Consider the following action:

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) \times \mathcal{H} &\to \mathcal{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) &\mapsto \left(\frac{a\tau + b}{c\tau + d} \right), \end{aligned}$$

This is obtained by performing, first of all, a change of basis and then normalizing one generator of the lattice to 1 (i.e. rescaling the lattice). Two points of $\mathcal{H}$ conjugated by an element of $\mathrm{SL}_2(\mathbb{Z})$ represent the same isomorphism class. It is well known that the converse holds i.e. two points of $\mathcal{H}$ that represent the same elliptic curve are conjugated by $\mathrm{SL}_2(\mathbb{Z})$ (reference [Sil09]).

This introduces one of the principal ideals we will use. Given a moduli problem, we could try to construct spaces X (whose points represent objects we want to classify) such that there exists a group action $G \times X \to X$ that conjugate points that represent the same equivalence classes. And then we move our focus to quotient X by the action of G.

In the case of elliptic curves, the quotient is given by the j invariant.

Definition 1.20 (j invariant)**.** Consider the following map:

$$\begin{aligned} j : \mathcal{H} &\to \mathbb{C} \\ \tau &\mapsto 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}, \end{aligned}$$

where:

$$g_2(\tau) = 60 \sum_{(n,m)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^4} \quad \text{and} \quad g_3(\tau) = 140 \sum_{(n,m)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^6}.$$

The series g_2, g_3 absolutely converge and the map is well defined (i.e. denominator never vanishes).
Alternatively, it can be defined from the coefficient of the elliptic curve in the form $y^2 = x^3 + ax + b$, this will allow us to define the invariant for every base field k.

It turns out this invariant classify elliptic curves:

Theorem 1.21 ($\mathbb{C}$ as moduli space of elliptic curves)**.** *$j : \mathcal{H} \to \mathbb{C}$ is the quotient map for the action of* $\mathrm{SL}_2(\mathbb{Z})$*: it is surjective and two points have the same image if and only if are conjugated by an element of the group. Moreover it is a topological quotient. We have that points of $\mathbb{C}$ represent the class of isomorphism of elliptic curves and the holomorphic structure encode how continuously these change. For this reason we can think to $\mathbb{C}$ as a moduli space.*

We do not prove these classical results, it is possible to find it again on [Sil09].

We try now to import the idea of family in this classical example. We define the following:

Definition 1.22 (Legendre family)**.** Consider elliptic curves of the forms $y^2 = x(x-1)(x-\lambda)$ for $\lambda \in \mathbb{C} \setminus \{0, 1\}$ (because the polynomial at right hand side has to be square free). This define a family with parameter λ:

$$\begin{array}{c} X = \{\text{elliptic curves of the form } y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{P}^2 \times (\mathbb{C} \setminus \{0,1\}) \\ \downarrow \phi \\ \mathbb{C} \setminus \{0,1\}\,. \end{array}$$

We write X_λ for the fibre of the family over λ.

The following remark is not difficult:

Remark 1.23. Every isomorphism class of elliptic curves is a fibre of the Legendre family, moreover:

$$X_\lambda \sim X_{\lambda'} \Leftrightarrow \lambda' \in \left\{\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}\right\}.$$

We have an action of the symmetric group S_3 where the generators act as follow:

$$\begin{aligned} S_3 \times \mathbb{C} \setminus \{0,1\} &\to \mathbb{C} \setminus \{0,1\} \\ (\tau, \lambda) &\mapsto \frac{1}{\lambda} \\ (\sigma, \lambda) &\mapsto \frac{\lambda-1}{\lambda}, \end{aligned}$$

where τ is a transposition and σ is a cycle. We avoid to verify that it is well defined.

We are instinctively inclined to quotient the space $\mathbb{C} \setminus \{0, 1\}$ in order to have a space that classify the elliptic curves. We can use the invariant j to construct the map:

$$\begin{aligned} \mathbb{C} \setminus \{0,1\} &\to \mathbb{C} \\ \lambda &\mapsto j(\{y^2 = x(x-1)(x-\lambda)\}). \end{aligned}$$

That turns out to be surjective and a topological quotient. We will deepen the implication of these results in an algebraic context in the last part of the chapter. We conclude here our rundown on the classical theory of elliptic curves. We are now ready to translate the problem into an algebraic setting.

1.4.2 The moduli space of curves of fixed genus

We already said that there is a correspondence between holomorphic and algebraic settings for elliptic curves, to better understand the parallelism we recall more precise parallelism. Notations are coherent with [Har77], Appendix B.

Given a finite type scheme over $\mathbb{C}$ X, we define X_h the *associated complex analytic space*. It is an object of $\text{Ana}_{\mathbb{C}}$ the category of analytic spaces that are locally zeros of holomorphic function in $\mathbb{C}^n$. Similarly, if $\mathcal{F}$ is a coherent sheave over X, we define $\mathcal{F}_h$ the *associated coherent analytic sheaf*. Definitions can be found in [Har77].

There is a continuous map $\phi : X_h \to X$ that send points of X_h to closed points of X and there is a natural map $\phi^{-1}\mathcal{O}_X \to \mathcal{O}_{X_h}$. This allows us to redefine $\mathcal{F}_h = \phi^*\mathcal{F}$.
We can hence define a functor:

$$\begin{aligned} h : \text{Sch}/\mathbb{C} &\to \text{Ana}_{\mathbb{C}} \\ X &\mapsto X_h \end{aligned}$$

and another one, that we call with the same name:

$$\begin{aligned} h : \text{Coh}/X &\to \text{Coh}/X_h \\ \mathcal{F} &\mapsto \mathcal{F}_h. \end{aligned}$$

We have several parallelism between these two:

- X is separated over $\mathbb{C} \Leftrightarrow X_h$ is Hausdorff;
- X is connected in the Zariski topology $\Leftrightarrow X_h$ is connected;
- X is reduced $\Leftrightarrow X_h$ is reduced;
- X is smooth over $\mathbb{C} \Leftrightarrow X_h$ is an holomorphic manifold;
- X is proper over $\mathbb{C} \Leftrightarrow X_h$ is compact.

Thanks to ϕ, we have natural maps:

$$\alpha_i : H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h).$$

The final (and fundamental) theorem that establishes the parallelism is the following. We will not use it directly but it allows us to generalize some ideas that involve cohomology.

Theorem 1.24 (Serre, GAGA-theorem)**.** *Let X be a projective scheme over $\mathbb{C}$, the functor $h : Coh/X \to Coh/X_h$ is an equivalence of category. Furthermore,*

$$\alpha_i : H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h)$$

are isomorphism for all i.

We have now that elliptic curves on $\mathbb{C}$ correspond to connected proper smooth curves of genus 1 on $\mathbb{C}$ with a fixed point. We can generalize our problem to an arbitrary field k and to an arbitrary genus.

We focus on the situation when k is algebraically closed, we define:

Definition 1.25 ($\mathcal{M}_g$)**.**

$$\begin{aligned} \mathcal{M}_g(S) = \{&\text{proper and flat families } f : X \to S \text{ whose geometric fibres are:} \\ &\text{smooth, connected, of dimension 1 and genus } g\} \end{aligned}$$

$$\mathcal{M}_g(h : T \to S) = h^* : \mathcal{M}_g(S) \to \mathcal{M}_g(T),$$

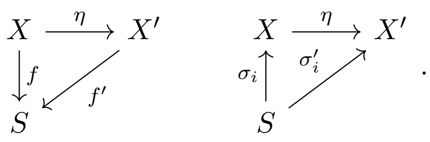
where $h^*(\left\{X \xrightarrow{f} S\right\}) = \{g^*X = X_T \to T\}$.
It is easy to check that the properties of the families are preserved by base change.

Moreover, for a more comprehensive definition, we define:

Definition 1.26 ($\mathcal{M}_{g,n}$).

$$\begin{aligned}\mathcal{M}_{g,n}(S) = &\{\text{proper and flat families } f : X \to S \text{ and } n \text{ sections } \sigma_i : S \to X.\\ &\text{Where the geometric fibres of } f \text{ are:}\\ &\text{smooth, connected, of dimension 1 and genus } g\} \,/\sim\\ \mathcal{M}_g(h : T \to S) = &h^* : \mathcal{M}_g(S) \to \mathcal{M}_g(T),\end{aligned}$$

where h^* is the pullback as before. We have moreover that two families $(f : X \to S, \sigma_1, \dots, \sigma_n)$ and $(f' : X' \to S, \sigma'_1, \dots, \sigma'_n)$ are equivalents if there exist an isomorphism $\eta : X \to X'$ such that the following diagrams commute:



It is easy to check that the properties of the family are preserved by base change.

Remark 1.27. We obviously have that $\mathcal{M}_{g,0} = \mathcal{M}_g$.

The example 1.10 is the moduli problem $\mathcal{M}_{0,4}$ because the unique curve with genus 0 is $\mathbb{P}^1$.

We can observe that:

- The reason why we specify the genus g in $\mathcal{M}_g$ (and also n in $\mathcal{M}_{g,n}$) is to resolve unbounded problem (Remark 1.16).
- The reason why we ask flat families is to avoid the "jump" phenomenon (Remark 1.17).
- The moduli space $\mathcal{M}_g$ does not admit a fine moduli space due to the existence of curves with non-trivial automorphisms (Problem 1.12).

We prove briefly the third point giving examples of curves with non-trivial automorphism of finite order. For every genus $g > 0$ we can consider the hyperelliptic curve (or elliptic if $g = 1$) of the form:

$$\mathcal{C}_g : y^2 = f(x),$$

where f is a polynomial of degree $2g+1$ with distinct roots. This define a projective curve of genus g. For this curve we have an automorphism:

$$\begin{aligned}\mathcal{C}_g &\to \mathcal{C}_g\\ (x, y) &\mapsto (x, -y),\end{aligned}$$

that is non trivial if $\operatorname{Char} k \neq 2$ (we will suppose that in the construction). Now we are in the same situation as the problem 1.12. It is possible to construct a non-trivial isotrivial families over $\mathcal{C}$, this is automatic thanks to theorem 2.6.15 in [Ser06].

We study now the example of elliptic curves from an algebraic point of view.

1.4.3 A revisited example: elliptic curves

We would like to classify now the elliptic curves with a fixed point, i.e. to study the problem $\mathcal{M}_{1,1}$. This situation is actually very similar to the classical case. It is immediate to observe that, thanks to homogeneity of $\mathcal{C}$, the classes of $\mathcal{M}_{1,1}$ are in bijection with the classes of isomorphism of elliptic curves as sets.

Theorem 1.28. *The coarse moduli space of the functor $\mathcal{M}_{1,1}$ is $\mathbb{A}^1$. Moreover to an elliptic curve $\mathcal{C}$ corresponds on $\mathbb{A}^1$ the k-point given by $j(\mathcal{C})$.*

We would like to find a strategy to emulate the classical construction via the quotient of $\mathcal{H}$ by $\mathrm{PGL}_2(\mathbb{Z})$. This is the aim of the next section.

1.5 Local universal family and action of groups

The last section of this chapter has the aim to explain the link between the study of moduli problems and the action of groups on schemes, which will be the subject of the next chapters. It is not strictly needed for the final construction and we state here some theorem that uses objects we will define in chapter 2. The reader could come back after the reading of chapters 2 and 3 for a better understanding.

We know that a universal family does not exist for a functor that is not representable. We could try to weaken the definition. Suppose that k is algebraically closed, a possibility is the following:

Definition 1.29 (Local universal family)**.** Let $\mathcal{M}$ be a moduli problem and $\mathcal{F}$ be a family over S. $\mathcal{F}$ is called a local universal family if for every family $\mathcal{G}$ on T and every t k-point of T, it exists $U \subset T$ and a map $f : U \to S$ such that $\mathcal{G}_{|U} \sim f^*\mathcal{F}$.

Next results use the action of an algebraic group on a scheme: we do not define it formally now, the reader could suppose to deal with classical notions of group and quotient.

Theorem 1.30. *Let $\mathcal{F}$ be a local universal family on a reduced scheme S. Let G act on S such that two k-points s, t of S lie in the same $G(k)$-orbits if and only if $\mathcal{F}_t \sim \mathcal{F}_s$.*

- *If a coarse moduli space exists, it is a (the) categorical quotient of S by G.*
- *Suppose that the categorical quotient exists. It exists a coarse moduli space if and only if the categorical quotient is an orbit space (at level of k-points).*

The intuition behind these results came from several classical examples. Again we use the elliptic curves, we recall the last part of section 1.4.1.

Example 1.31. Consider the Legendre family $X \to \mathbb{A}^1$ where $X \subset \mathbb{P}^2 \times \mathbb{A}^1$ in analogy with the classical case. We have that this is a local universal family. Moreover, given $G = S_3$, the action

$$S_3 \times \mathbb{A}^1 \to \mathbb{A}^1$$

(defined similarly to the classical setting) satisfied the hypothesis of the previous theorem. It is easy to prove that the quotient $\mathbb{A}^1 / S_3$ is an orbit space, hence is a coarse moduli space.

For the sake of completeness, we could state a more general version of the previous theorem:

Theorem 1.32. *Let $\mathcal{F}$ be a local universal family on S, let G act on S such that two points $f, f' : T \to S$ lie in the same $G(T)$-orbits if and only if $f^*\mathcal{F} \sim f'^*\mathcal{F}$. The followings hold:*

- *If a coarse moduli space exists, it is a (the) categorical quotient of S by G.*
- *If the categorical quotient exists. It exists a coarse moduli space if and only if the categorical quotient is an orbit space.*

Proof. We claim this bijective correspondence:

$$\{ \text{ natural transformations } \eta : \mathcal{M} \to h_M\} \leftrightarrow \{\text{G-invariant morphisms } f : S \to M\}$$
$$\eta \mapsto \eta_S(\mathcal{F}).$$

In order to prove that $\eta_S(\mathcal{F})$ is G-invariant we have to prove that the first diagram commute:

$$\begin{array}{ccc} G \times_k S & \xrightarrow{\sigma} & S \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \eta_S(\mathcal{F})} \\ S & \xrightarrow{\eta_S(\mathcal{F})} & M \end{array} \qquad \begin{array}{ccc} G(T) \times S(T) & \xrightarrow{\sigma(T)} & S(T) \\ \downarrow{\scriptstyle \pi_2(T)} & & \downarrow{\scriptstyle \eta_S(\mathcal{F})(T)} \\ S(T) & \xrightarrow{\eta_S(\mathcal{F})(T)} & M(T) \end{array}.$$

Thanks to Yoneda's lemma that's equivalent to prove the second diagram commute for every T. The good definition is now proven if for every couple of T-points f, f' in the same $G(T)$-orbit we have that $\eta_S(\mathcal{F}) \circ f = \eta_S(\mathcal{F}) \circ f'$.
Thanks to the naturality of η we have:

$$\eta_S(\mathcal{F}) \circ f = \eta_T(f^*\mathcal{F}) = \eta_T(f'^*\mathcal{F}) = \eta_S(\mathcal{F}) \circ f'.$$

Where the second equality is provided by hypothesis.
To prove the bijection we give an inverse map. Given a map $g : S \to M$ we recover a natural transformation $\eta : \mathcal{M} \to h_M$. For every $\mathcal{G}$ over $T \in \text{Sch}/k$ let's define $U_t \subset T$ an open neighbourhood of t were we can apply the local universal property: it exists $h_t : U_t \to S$ such that $\mathcal{G}_{|U_t} \sim h_t^*\mathcal{F}$. To construct the transformation we would like to glue togheter the maps $g \circ h_t : U_t \to M$. It's enought to prove that $\forall s, t$ k-points in T we have $(g \circ h_t)_{|U_t \cap U_s} = (g \circ h_s)_{|U_t \cap U_s}$.

$$(h_{t|U_t \cap U_s})^*\mathcal{F} = (h_t^*\mathcal{F})_{|U_t \cap U_s} \sim \mathcal{G}_{|U_t \cap U_s} \sim (h_s^*\mathcal{F})_{|U_t \cap U_s} = (h_{t|U_t \cap U_s})^*\mathcal{F},$$

this implies that $h_{t|U_t \cap U_s}$ and $h_{s|U_t \cap U_s}$ are element of $S(U_t \cap U_s)$ in the same $G(U_t \cap U_s)$-orbit, given that g is G-invariant.
We can glue them. We still have to prove that it is a natural transformation. We need that for every maps $v : T \to V$ the following commute:

$$\begin{array}{ccc} \mathcal{M}(V) & \xrightarrow{\eta_V} & h_M(V) \\ \downarrow{\scriptstyle \mathcal{M}(v)} & & \downarrow{\scriptstyle h_M(v)} \\ \mathcal{M}(T) & \xrightarrow{\eta_T} & h_M(T) \end{array}.$$

We omit this verification. □

Chapter 2

Group schemes and algebraic groups

In this chapter, we will give an introduction to group schemes. In particular we will start from a general point of view and moving forward in the chapter we will be more and more specific.

In the first section we will deal with general schemes and in the second one we will define algebraic groups. In the very final part we will moreover suppose that k is algebraically closed.

2.1 Group schemes

We start with some general and natural notions about group schemes. The fundamental reference for this subject is [GIT]. The general setting will be soon abandoned and we will deal with algebraic groups.

Definition 2.1 (Group scheme)**.** A group scheme over S is a structure (G, m, i, e) where G is an S-scheme and we have tree maps of S-schemes $m : G \times_S G \to G$, $i : G \to G$ and $e : S \to G$ that satisfied the followings:

$$\begin{CD} G \times_S G \times_S G @>{(\mathrm{Id},m)}>> G \times_S G \\ @V{(m,\mathrm{Id})}VV @VV{m}V \\ G \times_S G @>{m}>> G \end{CD} \qquad \begin{CD} G @>{(i,\mathrm{Id})}>> G \times_S G @<{(\mathrm{Id},i)}<< G \\ @VVV @VV{m}V @VVV \\ S @>{e}>> G @<{e}<< G \end{CD}$$

$$\begin{array}{ccccc} S \times_S G & \xrightarrow{(e,\mathrm{Id})} & G \times_S G & \xleftarrow{(\mathrm{Id},e)} & G \times_S S \\ & \underset{\simeq}{\searrow} & \downarrow{\scriptstyle m} & \underset{\simeq}{\swarrow} & \\ & & G & & \end{array} .$$

Without confusion, we say that G is an S-group.

The three maps m, i, e correspond respectively to the structure operation, the inverse map and the neutral element in the usual group theory. We could also see this scheme as his functor of points:

Remark 2.2 (Group scheme as functor)*.* Consider the functor of points $G(\_) = \mathrm{Hom}(\_, G) : \mathrm{Sch}/S \to \mathrm{Set}$ and fix an S-scheme T. Using the diagrams of the definition above and the property of fibred product we obtain:

$$\begin{array}{ccc} G(T)\times G(T)\times G(T) & \xrightarrow{(\mathrm{Id},m(T))} & G(T)\times G(T) \\ \big\downarrow{\scriptstyle (m(T),\mathrm{Id})} & & \big\downarrow{\scriptstyle m(T)} \\ G(T)\times G(T) & \xrightarrow{\quad m(T)\quad} & G(T) \end{array} \qquad \begin{array}{ccccc} G(T) & \xrightarrow{(i(T),\mathrm{Id})} & G(T)\times G(T) & \xleftarrow{(\mathrm{Id},i(T))} & G(T) \\ \big\downarrow & & \big\downarrow{\scriptstyle m} & & \big\downarrow \\ \{\star\} & \xrightarrow{\quad e(T)\quad} & G(T) & \xleftarrow{\quad e(T)\quad} & G(T) \end{array}$$

$$\begin{array}{ccccc} \{\star\}\times G(T) & \xrightarrow{(e(T),\mathrm{Id})} & G(T)\times G(T) & \xleftarrow{(\mathrm{Id},e(T))} & G(T)\times\{\star\} \\ & \searrow^{\simeq} & \big\downarrow{\scriptstyle m(T)} & \swarrow^{\simeq} & \\ & & G(T) & & \end{array} .$$

If a functor satisfies these diagrams, we say that it is a **group functor**.
We notice that these maps give to $G(T)$ the structure of a classical group, where $m(T)$ is the operation, $i(T)$ the inverse and $e(T)(\star) \in G$ is the neutral element. This situation arise also to the converse. A representable functor $\mathrm{Sch}/S \to \mathrm{Set}$ gives us a group scheme if it satisfy the diagrams.

A base change of a group scheme is still a group scheme:

Proposition 2.3 (base change of a group)**.** *Let G be an S-group and $T \to S$ a map. $G \times_S T$ is a group scheme with the following structure maps.*

$$\begin{array}{rr} \textit{Multiplication:} & G_T \times_T G_T = G \times_S G \times_S T \xrightarrow{(m,id)} G \times_S T. \\ \textit{Inverse:} & G_T = G \times_S T \xrightarrow{(i,id)} G \times_S T. \\ \textit{Neutral element:} & T = S \times_S T \xrightarrow{(e,id)} G \times_S T. \end{array}$$

Similarly to the group theory notion, we can define subgroups and actions:

Definition 2.4 (Subgroup)**.** Let G be a S-group. We say that a closed S-subscheme H of G is a subgroup of G if it is an S-group with compatible group laws.

Definition 2.5 (Action)**.** Let X be a scheme over S. An action of G over S is a map $\sigma : G \times_S X \to X$ such that:

$$\begin{array}{ccc} S\times_S X & \xrightarrow{(e,\mathrm{Id})} & G\times_S X \\ & \searrow^{\simeq} & \big\downarrow{\scriptstyle\sigma} \\ & & X \end{array} \qquad \begin{array}{ccc} G\times_S G\times_S X & \xrightarrow{(\mathrm{Id},\sigma)} & G\times_S X \\ \big\downarrow{\scriptstyle (m,\mathrm{Id})} & & \big\downarrow{\scriptstyle\sigma} \\ G\times_S X & \xrightarrow{\quad\sigma\quad} & X \end{array}$$

We can change the base of an action:

Remark 2.6. With the same notations of the previous definition. Consider a map $T \to S$, we define:

$$\sigma_T : G_T \times_T X_T = G \times_S X \times_S T \xrightarrow{(\sigma,\mathrm{Id})} X \times_S T = X_T.$$

It is easy to see that this define an action of G_T on X_T.

Looking back to basic group theory we could define the analogous of the orbit and the stabilizer of a point. In this context a point is a map $T \to X$. The following definitions are quite natural if we consider the base change of the action just defined.

Definition 2.7 (Orbit)**.** We split this definition in two parts. Let $f : S \to X$ be a S-morphism (an S-point). The set-theoretic image of

$$(id, f) : G = G \times_S S \to G \times_S X \xrightarrow{\sigma} X$$

is the *orbit* of f.
In general, let $f : T \to X$ be an S-scheme morphism, we define the *orbit* of f as the orbit of $f_T = (f, id) : T \to X_T$ under the action of σ_T.
Alternatively, we could consider $\sigma \circ (\mathrm{Id}, f) : G \times_S T \to X$ and define the orbit as the set theoretic image of:

$$\psi_f = (\sigma \circ (\mathrm{Id}, f), \pi_2) : G \times_S T \to X \times_S T.$$

We call it $o(f)$.

Definition 2.8. We define the *stabilizer* as the following fibred product:

$$\begin{array}{ccc} S(f) & \longrightarrow & T \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (f,\mathrm{Id})} \\ G \times_S T & \xrightarrow{\psi} & X \times_S T \end{array} .$$

Remark 2.9. We can notice that $S(f)$ has a structure of scheme and $O(f)$ does not in general. We will put on it anyway in suitable situations.

With the following definition we add another piece to the puzzle:

Definition 2.10 (Invariant morphism)**.** Let G act on X, we say that a map $f : X \to Y$ is an invariant morphism if the following commute:

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

where π_2 is the projection on the second factor.

Definition 2.11 (Invariant locally closed subsets)**.** Let $Y \xrightarrow{i} X$ be a locally closed subset, we say that it is G-invariant if the map $\sigma \circ (\mathrm{Id}, i) : G \times Y$ factorize as in the diagram:

$$\begin{array}{ccccc} G \times_S Y & \xrightarrow{(\mathrm{Id},i)} & G \times_S X & \xrightarrow{\sigma} & X \\ & \searrow & & & \uparrow{\scriptstyle i} \\ & & & & Y \end{array} .$$

In this case it is defined an action of G on Y.

The final aim of this chapter is to quotient scheme by the action of a group scheme.
Before going on with our specific case of interest, we state a first immediate notion of quotient.

Definition 2.12 (Categorical quotient)**.** Consider a G-invariant map $\pi : X \to Y$ of S-schemes. We say that π is a categorical quotient if for any S-scheme Z and any invariant S-morphism $f : X \to Z$ it exists a unique map $g : Y \to Z$ such that the following commute:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \\ Z & & \end{array} .$$

Thanks to standard techniques of category theory, the remark below is an immediate consequence of the definition.

Remark 2.13. If a categorical quotient exists, it is unique up to a canonical isomorphism.

We have that:

Proposition 2.14. *Let (Y, ϕ) be a Categorical quotient of X by G. We have:*

- *X reduced $\implies$ Y reduced;*
- *X connected $\implies$ Y connected;*
- *X irreducible $\implies$ Y irreducible.*

Proof. We have that the map ϕ factorize by a closed immersion:

$$X \xrightarrow{\overline{\pi}} \overline{Y} \hookrightarrow Y$$

Where, respectively in the three situations, $\overline{Y}$ is:

- Y_{red} and the map factorize because X is reduced.
- The connected component that contains $\pi(X)$ (is contained in only one component because X is connected).
- The irreducible component that contains $\pi(X)$ (is contained in only one component because X is irreducible).

Consider now:

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \overline{\pi}} \\ X & \xrightarrow{\overline{\pi}} & \overline{Y} \end{array}$$

If we compose the two maps with $\overline{Y} \hookrightarrow Y$ it commutes. Closed immersions are monomorphism, hence we have that the diagram commute: this prove that $\overline{\pi}$ is G-invariant.
The universal property gives us a map $g : Y \to \overline{Y}$ such that the following commute:

$$\begin{array}{ccccc} Y & \xleftarrow{\pi} & X & \xrightarrow{\pi} & Y \\ & {\scriptstyle i}\nwarrow & \downarrow{\scriptstyle \overline{\pi}} & \swarrow{\scriptstyle g} & \\ & & \overline{Y} & & \end{array} .$$

Applying again the universal property we have that $i \circ g : Y \to Y$ is the identity. Hence we have that i and g are isomorphism (again, this follow because i is a monomorphism). This implies that Y is isomorphic to $\overline{Y}$ i.e. the thesis.

□

2.2 Algebraic groups

We do not need to work with this generality, hence we focus on the case $S = \text{Spec } k$, where k is an arbitrary field for the moment. From now on, every scheme in Sch$/k$ will be supposed to be of finite type over Spec k.

We will write $\times_k$ (or $\times$ when it is clear from context) instead of $\times_{\text{Spec } k}$, to not weigh down formulas.

Definition 2.15 (Algebraic group)**.** Let k be a field. An algebraic group G over k is an algebraic k-scheme (a finite type scheme over k) that is a group scheme over Spec k.

Remark 2.16. The map $e : \text{Spec } k \to G$ is just a point of G with residue field k. Without the chance to confuse, we will refer to e as a set-theoretic point of G.

We can improve a little the functorial interpretation given in remark 2.2:

Remark 2.17 (Algebraic groups as functor). As before, consider the functor of points $G(\_) = \text{Hom}(\_, G) : \text{Sch}/k \to \text{Set}$, we have the same diagrams we avoid to rewrite.
We restrict the functor to the subcategory of affine schemes, we obtain $G(\_) : k\text{-Alg}^F \to \text{Set}$ where $k\text{-Alg}^F$ are the category of finitely generated algebras on k (we use the same notation without confusion). A scheme over k is determined by the restriction of its functor of points to affine schemes over k (proposition VI-2, [EH00]). This allows us to see the group as a functor $k\text{-Alg}^F \to \text{Set}$.

Classically the idea of algebraic group came from well know groups whose multiplication law is defined by algebraic formulas i.e. polynomial expressions. Here a list of classical example we will work with.

Example 2.18 (Trivial group). Consider Spec k, the trivial maps:

$$\begin{aligned} m &: \text{Spec } k \times_k \text{Spec } k \to \text{Spec } k \\ i &: \text{Spec } k \to \text{Spec } k \\ e &: \text{Spec } k \to \text{Spec } k. \end{aligned}$$

Define a group structure (we omit verifications). This has only one element and it is the initial and terminal object of the category of algebraic groups.

Example 2.19 (Additive group $\mathbb{G}_a$). Let k be a field, it has an additive structure. We can can view this as $\mathbb{A}^1$ with this structure:

$$\begin{aligned} \text{m: } \mathbb{A}^1 \times_k \mathbb{A}^1 &\to \mathbb{A}^1 & \text{defined by: } k[t] &\to k[t] \otimes_k k[t] \\ (a,b) &\mapsto a+b & t &\mapsto 1 \otimes t + t \otimes 1 \\ \text{i: } \mathbb{A}^1 &\to \mathbb{A}^1 & \text{defined by: } k[t] &\to k[t] \\ a &\mapsto -a & t &\mapsto -t \\ \text{e: Spec } k &\to \mathbb{A}^1 & \text{defined by: } k[t] &\to k \\ \star &\mapsto 0 & t &\mapsto 0. \end{aligned}$$

We omit to verify that this actually is an algebraic group and that, looking to k-points, the group law corresponds to the usual sum.

Similarly we can do for multiplication.

Example 2.20 (Multiplicative group $\mathbb{G}_m$). As before, k^* can be view as the k-points of $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$. We define the group law:

$$\begin{aligned} \text{m: } \mathbb{G}_m \times_k \mathbb{G}_m &\to \mathbb{G}_m & \text{defined by: } k[t,t^{-1}] &\to k[t,t^{-1}] \otimes_k k[t,t^{-1}] \\ (a,b) &\mapsto a \cdot b & t &\mapsto t \otimes t \\ \text{i: } \mathbb{G}_m &\to \mathbb{G}_m & \text{defined by: } k[t,t^{-1}] &\to k[t,t^{-1}] \\ a &\mapsto a^{-1} & t &\mapsto t^{-1} \\ \text{e: Spec } k &\to \mathbb{G}_m & \text{defined by: } k[t,t^{-1}] &\to k \\ \star &\mapsto 0 & t &\mapsto 0. \end{aligned}$$

We omit to verify that this actually is an algebraic group and that, looking to k-points, the group law corresponds to the usual product.

An important example is the following:

Example 2.21 (Linear group GL_n). We can identify $n\times n$-matixes as k-points of $k[\{x_{i,j}\}_{1\geq i,j\geq n}]$. The general linear group are matrices whose determinant does not vanish. Define

$$p(x_{i,j}) = \det \begin{pmatrix} x_{1,1} & \dots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{N,1} & \dots & x_{N,N} \end{pmatrix},$$

it's clear that the invertible matrices are the k-points of $\mathrm{GL}_N := k[\{x_{i,j}\}_{1\geq i,j\geq n}]_{p(x_{i,j})}$. Consider now the map:

$$\begin{aligned} k[\{x_{i,j}\}_{1\geq i,j\geq n}]_{p(x_{i,j})} &\to k[\{x_{i,j}\}_{1\geq i,j\geq n}]_{p(x_{i,j})} \otimes_k k[\{x_{i,j}\}_{1\geq i,j\geq n}]_{p(x_{i,j})} \\ x_{i,j} &\mapsto \sum_{k=1}^{n} x_{i,k} \otimes x_{k,j}. \end{aligned}$$

It induced the a morphism $m : \mathrm{GL}_n \times_k \mathrm{GL}_n \to \mathrm{GL}_n$. If we considered the map at k-points, it is the usual matrix multiplication. We avoid to explicit the inverse map, we only say that it is obtained constructing the inverse with the cofactor matrix.
e is the k-point that corresponds to the identity matrix. Again, we will not write down the straightforward verifications.

As last example, we give a subgroup of GL_n:

Example 2.22 (Special linear group GL_n). As expected, SL_n is defined as the closed subscheme of GL_n defined by the ideal $I = (p(x_{i,j}) = 1)$ (i.e. determinant equal to 1). The k-points correspond to matrix in the special linear group.
The operation laws derive from the ones for GL_n. We omit again boring verifications. It is clear that SL_n is a subgroup of GL_n.

Given these examples, we could think that the k-points of an algebraic group with Zariski topology are actually a topological group, this is not true (because the topology on the product is not the usual one). The parallelism is well done in a lot of situations and, in general, the algebraic structure is obviously more rigid. For instance, a notion that came from topological groups is *homogeneity*:

Definition 2.23 (Inner automorphism)**.** Let G be an algebraic group and $g : \mathrm{Spec}\ k \to G$ a k-point. Define the map:

$$L_g : G = \mathrm{Spec}\ k \times_k G \xrightarrow{(g,\mathrm{Id})} G \times_k G \xrightarrow{m} G.$$

Given an action σ of G on X, we write without ambiguity:

$$L_g : X = \mathrm{Spec}\ k \times_k X \xrightarrow{(g,\mathrm{Id})} G \times_k X \xrightarrow{\sigma} X.$$

Remark 2.24. We have:

- L_g is an automorphism of G (with inverse $L_{g^{-1}}$).
- For every couple of k-points s, g we have neighborhoods $g \in U_g$, $s \in U_s$ such that $L_{sg^{-1}} : U_g \to U_s$ is an isomoprhism. This property is called *homogeneity.*

We will state some useful property of algebraic group.

Proposition 2.25. *Every algebraic group is separated.*

Proof. We have to prove that the diagonal subscheme $\Delta(G) \subset G \times_k G$ is closed. consider

$$G \times_k G \xrightarrow{(id,i)} G \times_k G \xrightarrow{m} G,$$

topologically $\Delta(G)$ is the inverse image of e through $m \circ (\mathrm{Id}, i)$. Hence $\Delta(G)$ is closed because e is a closed point. $\square$

Proposition 2.26. *A connected algebraic group is irreducible.*

We prove it only when $k = \overline{k}$, this is our case of interests.

Proof. Let X be a finite type scheme over k. We know that X is irreducible if and only if the restricted topological space on closed points is irreducible. Every closed point is a k-point because $k = \overline{k}$, hence we would like to prove that $G(k)$ with the restricted topology is irreducible.

Suppose it is not, there exists at least a k-point t that belong to only one irreducible component and there exists another point s that belong at least to two irreducible components. $L_{st^{-1}}$ is an automorphism that send t on s hence t and s should belong to the same number of irreducible components. This is absurd and proves that X is irreducible. $\square$

In the last results it has been useful to deal with k-points, which give us a more practical and easier interpretation. In general, it is quite convenient and more down to earth to use k-points. A suitable situation happens when k is algebraically closed and the group is reduced. The following classic proposition justifies the previous sentence:

Proposition 2.27. *Let X and Y be reduced and finite type scheme over k. A morphism of k-scheme $f : X \to Y$ is solely determined by the map induced on closed points.*

This, in some sense, implies that is easier to deal with reduced algebraic groups.

We could ask now if every algebraic group G is reduced: the answer is negative, and it is easy to construct a counterexample.

Example 2.28. Cosider $G = \operatorname{Spec} {k[t]}/{(t^p)}$ where $\operatorname{Char} k = p$, we define the following maps:

$$\begin{aligned}
&\text{m: } G \times G \to G && \text{defined by: } {k[t]}/{(t^p)} \to {k[t]}/{(t^p)} \otimes {k[t]}/{(t^p)} \\
& && \qquad t \mapsto 1 \otimes t + t \otimes 1 \\
&\text{i: } G \to G && \text{defined by: } {k[t]}/{(t^p)} \to {k[t]}/{(t^p)} \\
& && \qquad t \mapsto -t \\
&\text{e: } \operatorname{Spec} k \to G && \text{defined by: } {k[t]}/{(t^p)} \to k \\
& && \qquad t \mapsto 0.
\end{aligned}$$

It is easy to verify that it is an algebraic group.
We notice that it is a subgroup of $\mathbb{G}_a$ through the closed immersion $\operatorname{Spec} {k[t]}/{(t^p)} \hookrightarrow k[t]$.

Definition 2.29 (Group variety)**.** A group variety G over k is an algebraic group over k that is reduced. The name came from the fact that G is a variety i.e. it is reduced and separated (every algebraic group is separated).

Even if G is not reduced, G_{red} has a natural group structure if $\operatorname{Char} k = 0$ or $\overline{k} = k$. In particular it is a group variety.

Remark 2.30 (Group structure on G_{red}). We have a map

$$G_{red} \times_k G_{red} \to G \times_k G \xrightarrow{m} G.$$

Given by the closed immersion $G_{red} \to G$. If $\operatorname{Char} k = 0$ or $\overline{k} = k$, the fibred product of reduced k-schemes is reduced. The universal property of G_{red} give us a map m':

$$\begin{array}{ccc} G_{red} \times G_{red} & \xrightarrow{m} & G \\ \downarrow{\scriptstyle m'} & \nearrow & \\ G_{red} & & \end{array} .$$

Similarly, we can construct an inverse map i' and a neutral element e'. We avoid verifying that it is an algebraic group, it is an easy verification.

Example 2.31. Consider a field k such that $k = \overline{k}$ and $\operatorname{Char} k = 0$. Let $\operatorname{Spec} \frac{k[x]}{(x^p)}$ be the algebraic group constructed in the previous example. We have that G_{red} is the trivial group.

We give now a description of the orbit and stabilizer of k-points. We suppose $k = \overline{k}$ because it is our case of interests and proofs are easier.

Proposition 2.32 (Orbit of a k-point)**.** *Suppose $k = \overline{k}$. Given a k-point x, its orbit $o(x)$ is the set theoretic image of*

$$G = G \times_k \mathit{Spec}\ k \xrightarrow{(Id,x)} G \times_k X \xrightarrow{\sigma} X.$$

It is a locally closed subset of X. Therefore we can give to $o(x)$ a structure of reduced scheme, we call it $G \cdot x$.

Proof. The image of this morphism is a constructible set ([Har77]Es. 3.19). This implies that there exists an open subset $U \subset o(x)$ of $\overline{o(x)}$ (this is an easy consequence of lemma 1.1 of [An12], that is just a topological statement). For every closed point $y \in o(x)$ there exists $g \in G$ such that $y \in g \cdot U$. This implies that $\cup_{g \in G} g \cdot U \subset G \cdot x$ is a constructible set that has exactly the same closed points of $G \cdot x$. Thanks to Nullstellensatz these two coincides, hence we have the thesis. □

The proof of this proposition tells us the following.

Remark 2.33. Suppose $k = \overline{k}$. We have that $G \cdot x$ is locally closed.
This means that the closed points of the image determine completely the orbit.

Moreover we have:

Remark 2.34. Suppose $k = \overline{k}$. Consider now the action on k-points

$$G(k) \times X(k) \xrightarrow{\sigma(k)} X(k),$$

we observe that the k-points of $G \cdot x$ are the image of $G(k) \times \{p\}$ under the map $\sigma(k)$.

Proposition 2.35 (Stabilizer of k-points)**.** *Given a k-point p, we call the stabilizer G_p:*

$$\begin{array}{ccc} G_x & \longrightarrow & \mathit{Spec}\ k \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ G & \xrightarrow{\sigma \circ (Id,p)} & X \end{array} .$$

$G_x \to G$ is a subgroup.

Proof. Spec $k \xrightarrow{p} X$ is a closed immersion, it follows that $G_x \to G$ is a subgroup. □

Remark 2.36. Suppose $k = \overline{k}$. As before, consider the action on k-points:

$$G(k) \times X(k) \to X(k).$$

We have that the k-points of G_x are exactely $\mathrm{Stab}_{G(k)}\, x$.

These two results help us to see an algebraic group as simply the group of k-points with more structure. We see now a further and fundamental result in this direction.

We would like to find a more suitable condition for G-invariants map. Once again, we would like to verify the invariance only for k-points of G.

Consider a G-invariant map $f : X \to S$ and let g be a k-point of G, we have that:

$$f \circ L_g : X = \mathrm{Spec}\ k \times_k X \xrightarrow{(g,\mathrm{Id})} G \times_k X \xrightarrow{\sigma} X \xrightarrow{f} S$$

is f thanks to G-invariance. The converse is true if $k = \overline{k}$ and G is reduced:

Proposition 2.37. *Let G be a reduced algebraic group acting on X. Suppose that $\overline{k} = k$ and let $f : X \to S$ be a morphism where S is separated. If for every $t \in G(k)$ holds that $f = f \circ L_g$, we have that f is G-invariant.*

In order to prove this proposition, we recall the following lemma on the schematic image. We have that:

Lemma 2.38. *Let $f : X \to Y$ be a morphism of k-schemes. Suppose that* $\mathrm{Im}(f)$ *is quasicompact and quasiseparated. Let Z be an arbitrary k scheme and consider the product map $f \times Id : X \times Z \to Y \times Z$. Then, we have* $\mathrm{Im}(f \times Id) = \mathrm{Im}(f) \times Z$.

The lemma follows from the definition of schematic image.

Proof. (Proposition 2.37) The map of inclusion

$$h : \bigsqcup_{p \in G(k)} p \to G$$

has schematic image G (because closed points are dense and G is reduced) that is quasi-compact and separated.

Consider $h \times \mathrm{Id} : \bigsqcup_{p \in G(k)} p \times X \to G \times X$, according to the previous lemma we have:

$$\mathrm{Im}(h \times \mathrm{Id}) = G \times X.$$

Consider the diagram:

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{h} & S, \end{array}$$

the maps $h \circ \sigma$ and $h \circ \pi_2$ coincide on a closed subscheme $Z \subset G \times X$. We have that the map

$$\bigsqcup_{p \in G(k)} p \times X \longrightarrow G \times X$$

factorize by Z. By the universal property of the schematic image, we have a map

$$G \times X = \mathrm{Im}(h \times \mathrm{Id}) \longrightarrow Z \hookrightarrow G \times X.$$

This implies that $Z = G \times X$, hence the thesis. □

We cannot avoid including the hypothesis G reduced:

Example 2.39. Let k be a field such that $\operatorname{Char} k = p$. Consider the algebraic group $G = \operatorname{Spec} \frac{k[x]}{x^p}$ and the action given by the inclusion $G \subset \mathbb{G}_a$:

$$\sigma\colon G \times \operatorname{Spec} k[t] \to \operatorname{Spec} k[t] \qquad \text{defined by: } k[t] \to k[t] \otimes_k \frac{k[x]}{(x^p)}$$
$$t \mapsto t \otimes 1 + 1 \otimes x.$$

G has only a k-point, the identity e. Moreover it is easy to verify that $L_e = \operatorname{Id}$. Hence, for every map $f : X \to S$, we have $f \circ L_e = f$. It is now enough to find a map that is not G-invariant: for instance $\operatorname{Id} : X \to X$.

We fix now these notation:

Definition 2.40. Given $g \in G(k)$ we write $g \cdot f$ for the composition $f \circ L_g$.
Let $U \subset X$ be an invariant open subset, we write $\mathcal{O}(U)^G$ for the invariant functions of $\mathcal{O}(U)$.

2.2.1 Representation and coalgebra interpretation

In this subsection we will introduce the analogous for finite dimensional group's representation in algebraic context. We give particular attention to the affine case.
We fix, once for all, a k-vector space V of finite dimension. We define:

$$\begin{aligned} \operatorname{GL}(V) : k\text{-Alg}^F &\to \text{Set} \\ R &\rightsquigarrow \operatorname{Aut}_R(V \otimes_k R) \qquad (\operatorname{Aut}_R \text{ are the } R \text{ linear automorphisms}) \\ V_a : k\text{-Alg}^F &\to \text{Set} \\ R &\rightsquigarrow V \otimes_k R \end{aligned}$$

Fix a basis of V, it easy to see that:

Remark 2.41. $\operatorname{GL}(V)$ is isomorphic to the functor of points of GL_n (restricted k-Alg^F).

Thanks to remark 2.17 we can work with the category k-Alg^F instead of Sch/k, this is the reason why we avoid define $GL(V)$ on Sch/k.
There are multiple way to define a representation of V.

Definition 2.42 (Algebraic group representation)**.** Let G be an algebraic group, we can see it as a functor of points. We define a representation as:

- a homomorphism of group valued functor, $\rho : G \to GL(V)$;
- an action of G on the functor V_a, $\sigma : G \times V_a \to V_a$. Such that $\forall R \in k$-Alg^F, $G(R)$ act linearly on $V \otimes_k R$.

Where an **action of a group functor** is exactly a functor that satisfies the equivalent usual commutative diagrams in the definition of action for scheme groups.

We omit to prove the equivalence of these two: the proof is the transliteration of the classical case, with classical groups and actions.

We are now interested in another way to describe a representation of affine groups: through the co-module description. First of all, we observe that affine groups are Hopf algebras.

Let G be an **affine** group, his structure is defined by the maps m, e and i. Given that a map between affine schemes is defined by the map induced on global section, we could define G using his global section. We write Δ, ϵ, S for the dual respectively of m, e and i. We have the following diagrams:

$$\begin{array}{ccc} \mathcal{O}(G)\otimes\mathcal{O}(G)\otimes\mathcal{O}(G) & \xleftarrow{\mathrm{Id}\otimes\Delta} & \mathcal{O}(G)\otimes\mathcal{O}(G) \\ {\scriptstyle \Delta\otimes\mathrm{Id}}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathcal{O}(G)\otimes\mathcal{O}(G) & \xleftarrow{\Delta} & \mathcal{O}(G) \end{array} , \quad \begin{array}{ccccc} \mathcal{O}(G) & \xleftarrow{S\otimes\mathrm{Id}} & \mathcal{O}(G)\otimes\mathcal{O}(G) & \xrightarrow{\mathrm{Id}\otimes S} & \mathcal{O}(G) \\ \uparrow & & \uparrow{\scriptstyle \Delta} & & \uparrow \\ k & \xleftarrow{\epsilon} & \mathcal{O}(G) & \xrightarrow{\epsilon} & k \end{array} ,$$

$$\begin{array}{ccccc} k\otimes\mathcal{O}(G) & \xleftarrow{\epsilon\otimes\mathrm{Id}} & \mathcal{O}(G)\otimes\mathcal{O}(G) & \xrightarrow{\mathrm{Id}\otimes\epsilon} & \mathcal{O}(G)\otimes k \\ & \nwarrow{\scriptstyle\simeq} & \uparrow{\scriptstyle \Delta} & {\scriptstyle\simeq}\nearrow & \\ & & \mathcal{O}(G) & & \end{array} .$$

This gives a structure of Hopf algebra, but we are not interested to elaborate further.

Definition 2.43 (Co-module)**.** Let G be an affine algebraic group. A co-module is a map $r : V \to V \otimes \mathcal{O}(G)$ such that the followings commute:

$$\begin{array}{ccc} V\otimes\mathcal{O}(G)\otimes\mathcal{O}(G) & \xleftarrow[\mathrm{Id}\otimes r]{} & V\otimes\mathcal{O}(G) \\ {\scriptstyle r\otimes\mathrm{Id}}\uparrow & & \uparrow{\scriptstyle r} \\ V\otimes\mathcal{O}(G) & \xleftarrow[r]{} & V \end{array} , \quad \begin{array}{ccc} V\otimes k & \xleftarrow[\mathrm{Id}\otimes\epsilon]{} & V\otimes\mathcal{O}(G) \\ & \nwarrow{\scriptstyle \mathrm{Id}} & \uparrow{\scriptstyle r} \\ & & V \end{array} .$$

These diagrams are dual to the ones that define the action, we have that:

Proposition 2.44. *Let G be an affine algebraic group, there is a correspondence*

$$\{\textit{Actions of } G \textit{ on } V\} \leftrightarrow \{\textit{co-module maps } r : V \to V \otimes \mathcal{O}(G)\}$$
$$\sigma \mapsto V \xrightarrow{Id\otimes 1} V \otimes \mathcal{O}(G) \xrightarrow{\sigma(\mathcal{O}(G))(1)} V \otimes \mathcal{O}(G).$$

Proof. We omit to verify that the map is well defined. Fix now a representation: consider the functors valued on Spec R and a map f : Spec $R \to G$. We have maps:

$$\begin{array}{ccc} G(\mathcal{O}(G)) \times (V \otimes_k \mathcal{O}(G)) & \longrightarrow & V \otimes \mathcal{O}(G) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ G(R) \times (V \otimes_k R) & \longrightarrow & V \otimes R \end{array} .$$

This implies that the map $G(R) \times (V \otimes_k R) \to V \otimes R$ is completely defined by the value of $\sigma(\mathcal{O}(G))(1) : V \otimes_k \mathcal{O}(G) \to V \otimes_k \mathcal{O}(G)$. This map is $\mathcal{O}(G)$ linear, therefore it is defined by the restriction to V. This implies that the map of the thesis is injective. The surjectivity came similarly defining the action using the diagram above. We omit the details. □

The aim of the last part of this section is to study representations of the torus.

Definition 2.45 (Torus)**.** An algebraic group G is an algebraic torus if $G = \mathbb{G}_m^n$ for some $n > 0$.
Furthermore we can define

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m),$$

called **character group**.
$X^*(T)$ is an abelian group, induced by the group structure of $\mathbb{G}_m$, this explain the name.

We compute now $X^*(\mathbb{G}_m)$, it will be useful in the next theorem.

Proposition 2.46. *The morphism*

$$\begin{aligned} \theta : \mathbb{Z} &\to X^*(\mathbb{G}_m) \\ n &\mapsto (z \mapsto z^n), \end{aligned}$$

is an isomorphism of groups.

Proof. Let $\phi^* : k[t, t^{-1}] \to k[t, t^{-1}]$ be such that $\phi(t) = t^n$, this define a group morphism because $\phi \circ m = m \circ (\phi, \phi)$ (we omit the easy veritication).
θ is a group morphism because we have $\theta(a) \cdot \theta(b) = \theta(a + b)$. The map is clearly injective. The only non-trivial veritication is about surjectivity. Let $\phi : \mathbb{G}_m \to \mathbb{G}_m$ be defined by the map:

$$\begin{aligned} \phi^* : k[t, t^{-1}] &\to k[t, t^{-1}] \\ t &\mapsto \sum_{i=-m}^{m} a_i t^i. \end{aligned}$$

This define a group morphism, hence $m^*(\phi^*(t)) = \phi^*(t) \otimes \phi^*(t)$:

$$\sum_{i=-m}^{m} a_i t^i \otimes t^i = \sum_{i=-m}^{m} \sum_{j=-m}^{m} a_i a_j t^i \otimes t^j.$$

This implies clearly that at most there is only one index such that $a_i \neq 0$. This implies that the map is defined by $t \mapsto a_i t^i$. Given that $\phi \circ e = e$ we have that $a_i = 1$, hence the thesis. $\square$

We are now ready for the last result of this section:

Theorem 2.47 (Representation of a torus)**.** *Let $\rho : T \to GL(V)$ be a representation, there is a weight space decomposition*

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi,$$

where $V_\chi = \{v \in V \mid t \cdot v = \chi(t) v \ \forall t \in T\}$ are the weight spaces.

Proof. First of all we prove it for $T = \mathbb{G}_m$. Consider r, the relative co-module of a representation $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$:

$$\begin{array}{ccc} V \otimes k[t,t^{-1}] \otimes k[t,t^{-1}] & \xleftarrow{\mathrm{Id} \otimes r} & V \otimes k[t,t^{-1}] \\ {\scriptstyle r \otimes \mathrm{Id}} \uparrow & & \uparrow {\scriptstyle r} \\ V \otimes k[t,t^{-1}] & \xleftarrow[r]{} & V \end{array} , \qquad \begin{array}{ccc} V \otimes k & \xleftarrow{\mathrm{Id} \otimes \epsilon} & V \otimes k[t,t^{-1}] \\ & {\scriptstyle \mathrm{Id}} \nwarrow & \uparrow {\scriptstyle r} \\ & & V \end{array} .$$

Define the space:

$$V_m = \{v \mid r(v) = v \otimes t^m \},$$

this is a $\mathbb{G}_m$-invariant subspace of V. Moreover, we can write $r(v) = \sum f_m(v) \otimes t^m$, where $f_m : V \to V$ is a linear map. We prove in the following that f_m is a projection in V_m. Applying the first diagram to v we obtain easily that $r(f_m(v)) = f_m(v) \otimes t^m$ and $f_m \circ f_m = f_m$. Using the second diagram we have easily that $v = \sum f_m(v)$. These result implies that the linear maps f_i are projectors and decompose the space into $\mathbb{G}_m$-stable subspaces:

$$V = \bigoplus_{m \in \mathbb{Z}} V_m.$$

This prove the case $T = \mathbb{G}_m$. Consider now $T = (\mathbb{G}_m)^n$, we can generalize the proof introducing $V_{m_1,\ldots,m_n}$ in a similar way. The proof follows using the same arguments. $\square$

We immediately have that:

Corollary 2.48. *Let $\rho : \mathbb{G}_m \to GL(V)$ be a representation. There exists a basis of V such that this map can be write as*

$$\mathbb{G}_m \to \mathrm{GL}_n$$
$$t \mapsto \begin{pmatrix} t^{k_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t^{k_n}. \end{pmatrix}$$

Where k_i are integers.

We prove now a result we will use later. Despite it is not well integrated here, we prove it now because we use the co-module structure.

Proposition 2.49. *Let G act on an affine scheme Spec A. Suppose that $f \in A$ is G-invariant, thanks to definition 2.11 we hace an action on Spec A_f. It turns out that*

$$(A_f)^G = \left(A^G\right)_f.$$

Proof. Consider $\frac{a}{f^i} \in (A_f)^G$, we have by definition that the following commute:

$$\begin{array}{ccc} \mathcal{O}(G) \otimes A_f & \xleftarrow{r_f} & A_f \\ {\scriptstyle 1\otimes \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle x \to \frac{a}{f^i}} \\ A_f & \xleftarrow[x \to \frac{a}{f^i}]{} & k[x] \end{array},$$

where r_f is the co-module map of the restrinction.
We have that

$$1 \otimes \frac{a}{f^i} = r_f\left(\frac{a}{f^i}\right) = \frac{r(a)}{f^i}.$$

This implies that $1 \otimes a = r(a)$ in $\mathcal{O}(G) \otimes A_f$, i.e. there exists j such that $(r(a) - 1 \otimes a)f^j = 0 \in \mathcal{O}(G) \otimes A$. Now we have that $af^j \in A$ is G-invariant and we write $\frac{a}{f^i} = \frac{af^j}{f^{i+j}}$. Hence we have that every element of $(A_f)^G$ can be written as an G-invariant element of A up to factor f, the thesis follows easily from this. □

We finish with the following natural remark, that generalizes what introduced before:

Remark 2.50. Consider an action $G \times_k X \to X$, even if G and X are not affine, we have that $\mathcal{O}(G)$ is a Hopf algebra and that $\mathcal{O}(X)$ is a co-module.
If X is affine, we have that this co-module structure induce an action $G \times_k X \to X$ and vice versa.

We give the following definition, we will use it in the next chapter:

Definition 2.51 (Rational action)**.** Let $\sigma : G \times X \to X$ be an action, we say that G *act rationally on* X if every $f \in \mathcal{O}(X)$ is contained in a finitely generated k-subspace of $\mathcal{O}(X)$ that is G-invariant.

Chapter 3

Quotient of schemes

Once introduced the first results about algebraic groups, we are now interested in quotienting schemes by the action of a group. We already gave the definition of categorical quotients. Now we would like to push the requests further, with the final aim to get closer to an orbit space.

The last section will be devoted to Hilbert-Mumford criterion, which will be fundamental for the final construction.

The references for this chapter are the first chapters of [GIT] and [Mil17].

In the whole chapter, we will assume that k is algebraically closed.

We start with a set-theoretic proposition:

Proposition 3.1 (Orbits and invariant maps)**.** *Let $f : X \to Y$ be a G-invariant map and x be a k-point of X. We have that the set-teoretic image of $\overline{G \cdot x}$ is a closed point.*

Proof. The closed points of $G \cdot x$ maps to a unique point p. Given that these closed points are dense in $G \cdot x$, we have $f(G \cdot x) = p$. The thesis is now satisfied thanks to continuity of f. □

Recalling that we are looking for an orbit space, we define:

Definition 3.2 (Closed action)**.** We say that an action $\sigma : G \times_k X \to X$ is closed if $\forall x \in X(k)$ $G \cdot x$ is closed.

Thanks to the previous proposition, the categorical quotient can be an orbit space only if the action is closed.

This situation leads us to investigate more about orbits:

Proposition 3.3. *Let x be a k-point of X. The boundary of an orbit $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of closed points of strictly smaller dimension.*

In particular, the orbits of closed points of minimal dimension contained in $\overline{G \cdot x}$ are closed $\forall x \in X(k)$.

Proof. The set $\overline{G \cdot x} \setminus G \cdot x$ is G-invariant, hence is a union of orbits.
$G \cdot x$ is open in $\overline{G \cdot x}$ (Thanks to the proof of proposition 2.32), this implies that the orbits in the boundary are of strictly lower dimension.
Consider now the orbits of minimal dimension, if the closure of any of these is not the orbit itself we obtain an orbit of strictly lower dimension: this implies that orbit of minimal dimension are closed. □

Proposition 3.4. *Let x be a k-point of X.*

$$\dim G = \dim G \cdot x + \dim G_x.$$

Proof. The point of the proof is the flatness of:

$$\sigma_x : G \to G \cdot x.$$

First of all, we can suppose that G and X are reduced (because the dimension does not change reducing schemes). Now we have that σ_x is flat at every generic point of $G \cdot x$, hence there is an open and dense set $U \subset G \cdot x$ where σ_x is flat. Given that G act transitively on $G \cdot x$, we have that σ_x is flat. The thesis follows now from the dimension formula for a flat morphism. □

Proposition 3.5. *Consider on $X(k)$ the Zariski topology, the function*

$$\begin{aligned} X(k) &\to \mathbb{N} \\ x &\mapsto \dim G \cdot x \end{aligned}$$

is lower-semi-continuous. Hence we have also that

$$\begin{aligned} X(k) &\to \mathbb{N} \\ x &\mapsto \dim G_x \end{aligned}$$

is upper-semi-continuous.

Proof. We give only a sketch of the proof. Let P be the fibred product of the diagram:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \big\downarrow & & \big\downarrow \Delta \\ G \times X & \xrightarrow{(\pi_X, \sigma)} & X \times X \end{array} .$$

Consider the fibres $\phi^{-1}(x)$ such that $x \in X(k)$. We have that these correspond to $G \cdot x$ under the map $P \hookrightarrow G \times X$. Now we have that the dimension of the fibre is lower-semi-continuous. The second thesis is obvious thanks to the previous proposition. □

In order to understand these propositions we give an easy example:

Example 3.6. Consider $\mathbb{G}_m$ that act on $\mathbb{A}^2$ in this way:

$$\begin{aligned} &\mathbb{G}_m \times \mathbb{A}^2 \to \mathbb{A}^2 \\ &(t, (x, y)) \mapsto (t \cdot x, t^{-1} \cdot y). \end{aligned}$$

Formally, this action arise from the following morphism (we omit verifications):

$$\begin{aligned} k[x, y] &\to k[t, t^{-1}] \otimes_k k[x, y] \\ x &\mapsto t \otimes x \\ y &\mapsto t^{-1} \otimes y. \end{aligned}$$

It is easy to compute the orbits of closed points:

- $x = (\alpha, \beta)$ where $\alpha \neq 0$, $\beta \neq 0$. The orbit is the set teoretic image of:

$$\begin{aligned} \mathbb{G}_m &\to \mathbb{A}^2 \\ t &\mapsto (t \cdot \alpha, t^{-1} \cdot \beta). \end{aligned}$$

  Giving a scheme structure to the orbit we obtain the subscheme Spec $\frac{k[x,y]}{(xy-\alpha\beta)} \hookrightarrow$ Spec $k[x, y]$. This is the hiperbola $xy = \alpha\beta$.

- $p = (\alpha, 0)$ where $\alpha \neq 0$. As before we have the map:

$$\begin{aligned}\mathbb{G}_m &\to \mathbb{A}^2 \\ t &\mapsto (t \cdot \alpha, 0).\end{aligned}$$

  This time the orbit subscheme is Spec $\frac{k[x,x^{-1},y]}{(y)} \hookrightarrow$ Spec $k[x,y]$. Intuitively the punctured line $y = 0$.

- $p = (0, \beta)$ where $\beta \neq 0$. That's simmetrical to the previus case: Spec $\frac{k[x,y^{-1},y]}{(x)} \hookrightarrow$ Spec $k[x,y]$. This is the punctured line $x = 0$.

- $p = (0, 0)$. Similarly to the previous cases, the orbit is Spec $\frac{k[x,y]}{(x,y)} \hookrightarrow$ Spec $k[x,y]$ i.e. the closed point $(0,0)$.

The first and last kind of orbits are closed. The closure of $G \cdot (\alpha, 0)$ with $\alpha \neq 0$ is the line $y = 0$: Spec $\frac{k[x,y]}{(y)} \to$ Spec $k[x,y]$ i.e. $G \cdot (\alpha, 0) \cup G \cdot (0,0)$. $G \cdot (0,0)$ is closed and of minimal dimension. This agrees with proposition 3.3.
The stabilizer of the first three orbits type is the trivial group Spec $k \hookrightarrow$ Spec $k[t, t^{-1}]$. On the other hand, the stabilizer of $(0,0)$ is the whole group G_m. This agrees with proposition 3.4.

Given that not every action is closed and the categorical quotient cannot be an orbit space. We introduce now a weaker notion of orbit space, we will see later that if it exists it is categorical.

Definition 3.7 (Good quotient)**.** Let G be an algebraic group acting on X. $Y \in \mathrm{Sch}/k$ (i.e. of finite type over k) with a G-invariant map $\pi : X \to Y$ is said to be a *good quotient* if:

i) π is surjective;

ii) $\forall U \subset Y$ we have that the induced map $\pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ is the natural inclusion $\mathcal{O}_X(\pi^{-1}(U))^G \subset \mathcal{O}_X(\pi^{-1}(U))$ (Observe that the open set $\pi^{-1}(U)$ is G-invariant);

iii) if W is a closed and G-invariant subset of X, $\pi(W)$ is closed;

iv) for all W_1, W_2 disjoint, invariant and closed subsets of X, we have that $\pi(W_1) \cap \pi(W_2) = \emptyset$;

v) π is an affine morphism.

i), iii) and iv) are reasonable requests for a quotient. Request ii) introduce a new important point of view. Instead of looking only to a quotient at points level, we are now interested in regular function on the scheme. In particular, we ask that regular functions on the quotient are exactly the ones that are G invariant.

Proposition 3.8. *Let $\pi : X \to Y$ be a G-invariant map that satisfied condition from i) to v) in the definition above (everything but affine). We have that π is a categorical quotient. In particular, good quotients are categorical.*

Proof. Consider a G-invariant map $f : X \to Z$, we would like to prove that this factorize in a unique way by a map $h : Y \to Z$. We construct h covering Z by a finite (because Z is of finite type) number of affine sets $\{U_i\}_I$ and then we define $h_i : V_i \to U_i$ where V_i are

a suitable covering of Y.
Given $\{U_i\}_I$ we define:

$$V_i := Y \setminus \pi(X \setminus f^{-1}(U_i)).$$

V_i are open sets: $X \setminus f^{-1}(U_i)$ is closed and G-invariant, hence condition iii) applies. By construction we easily verify that $\pi^{-1}(V_i) \subset f^{-1}(U_i)$.
We prove now that V_i is an open cover of Y. Suppose it is not, hence $\cap_I \pi(W_i) \neq \emptyset$. Consider a k-point p in the preimage of the interseciton, We must have $G \cdot x \cap W_i \neq \emptyset$ (thanks to condition iv)). Thanks to the G-invariance of $G \cdot p$ and W_i we have that $G \cdot p \subset W_i$ for every i: this is a contradiction given that $\cap_I W_i = \emptyset$ (because $\{U_i\}$ is a cover).
We construct now $h_i : V_i \to U_i$. There exists a unique map h_i^* such that the following commute:

$$\begin{array}{ccc} \mathcal{O}_Z(U_i) & \xrightarrow{h_i^*} & \mathcal{O}_Y(V_i) \\ \downarrow f^* & & \simeq \downarrow \pi^* \\ \mathcal{O}_X(f^{-1}(U_i))^G & \longrightarrow & \mathcal{O}_X(\pi^{-1}(V_i))^G \end{array}$$

It is therefore clear that these morphisms glue togheter and define a global and unique $h : Y \to Z$. □

The request v) in the definition is needed to glue properly good quotients: it will be clear after the next proposition.

Proposition 3.9 (Local on the target)**.** *Let $\pi : X \to Y$ be a good quotient and U an open subset of X, $\pi : \pi^{-1}(U) \to U$ is a good quotient. Moreover, given a cover $\{U_i\}_{i \in I}$ such tat $\pi^{-1}(U_i) \to U_i$ is a good quotient we have that the map $\pi : X \to Y$ is a good quotient. Hence, to be a good quotient is local on the target.*

Proof. This proposition follows easily from the fact that every property that defines a good quotient is local on the target. □

Recalling the importance of closure of orbits, we have:

Lemma 3.10 (Orbits and good quotients)**.** *Let $\pi : X \to Y$ be a good quotient:*

- *if $y \in Y$ is a closed point, $\pi^{-1}(y)$ contain a unique closed orbit of a closed point.*
- *let $x_1, x_2 \in X$ closed points,*

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset \Leftrightarrow \pi(x_1) = \pi(x_2).$$

Proof. For the first point: it follows easily from requirement iv) of good quotients.
We prove now the second statement. We have that $\pi(x_1) = \pi(\overline{G \cdot x_1})$ and $\pi(x_2) = \pi(\overline{G \cdot x_2})$ because ϕ is G-invariant. Therefore, if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ we have that $\pi(x_1) = \pi(x_2)$.
The opposite direction follows from the requirement iv) of good quotients, where the closeds sets are $W_1 = \overline{G \cdot x_1}$ and $W_2 = \overline{G \cdot x_2}$. □

It is therefore natural to define:

Definition 3.11 (Geometric quotient)**.** Let $\pi : X \to Y$ be a good quotient, we say that it is a geometric quotient if $\forall p \in X$ closed point, we have that $\pi^{-1}(p)$ is a unique orbit of a closed point.

Thanks to lemma (3.10) we obviously have:

Corollary 3.12. *Let G act on X. Given a good quotient $\pi : X \to Y$, if the action is closed then π is a geometric quotient.*

3.1 Affine GIT quotient

We are now ready to construct good quotients in the affine case. Before going on we illustrate the simple case of a finite group acting on an affine scheme over k. This case will lead us to the general situation where G is an algebraic group.

Actually, a finite group can be viewed like a discrete algebraic group whose points are disconnected copies of Spec k. In this sense, we will generalize the construction.

Theorem 3.13. *Let G be a finite group acting on Spec A, where A is a finitely generated k-algebra. We have that A^G is a finitely generated k-algebra and*

$$\text{Spec } A \to \text{Spec } A^G$$

is a geometrical quotient.

Proof. The principal idea of the proof is that $A^G \subset A$ is integral. For every $a \in A$ we have that

$$\prod_{g \in G} (x - g(a)) \in A^G[x]$$

This implies that the map Spec $A \to$ Spec A^G is closed and surjective. We omit the easy verifications that are needed to conclude the proof. □

It is clear that is not possible to generalize the proof when G is an algebraic group because there are no chances to construct the polynomial we used in the previous theorem. Moreover, if G is not finite, we do not even know if A^G is a finitely generated algebra.

The latter problem is related to the 14th Hilbert problem. In general, it is not true that A^G is finitely generated, Nagata gave a counterexample. He also proves the theorem in a particular situation:

Theorem 3.14 (Nagata)**.** *Let G be a linearly reductive algebraic group acting rationally on an affine finite type scheme over k, say Spec A. A^G is a finitely generated algebra.*

Where we define:

Definition 3.15 (Linearly reductive group)**.** G is said linearly reductive if every finite-dimensional linear representation is completely reducible (i.e. it decomposes as a direct sum of irreducibles).

We do not prove this result, the reader can find it in [Nag59].

Examples of linearly reductive groups are torus and GL_n (in characteristic 0).

Asking that A^G is finitely generated is not the only requirement we have. In order to obtain a good quotient we need the following.

Lemma 3.16. *Let G be a linearly reductive group, we have that for every representation $G \to \mathrm{GL}_n$ and any non-zero $v \in \mathbb{A}^n$ G-invariant vector, there exists a non-zero $f \in k[x_1, \dots, x_n]^G$ such that $f(v) \neq 0$.*

This lemma tells us that a linearly reductive group is geometrically reductive but we are not interested in elaborating further. A reference for this lemma is [Kem00].

Lemma 3.17. *Let A be a k-algebra equipped by a rational of a lineally reductive group G. For any $I \subset A^G$ ideal we have that:*

$$IA \cap A^G = I.$$

We do not prove this result that involves Reynolds operator, a reference is again [Kem00].

Given that we will not deal with reductive groups, from now on we will omit the word "linearly" and we will write "reductive" instead of "linearly reductive".

Remark 3.18. Here A^G can be interpreted in a double way:

- $A^G = \mathcal{O}(\text{Spec } A)^G$ i.e. the G-invariant function on Spec A,
- $A^G = \{a \in A \mid \forall\, g \in G(k)\ g \cdot a = a\}$.

Thanks to proposition 2.37 there is no difference in our setting (k is algebraically closed).

We restrict now our interests to **reductive groups G that acts rationally**, to avoid that Spec A^G is not a finite type scheme over k. Similarly to the finite case we can define:

Definition 3.19 (GIT affine quotient)**.** Let X be an affine scheme of finite type over k and G be a reductive algebraic group acting ractionally on X. We define the *GIT quotient*:

$$\pi : X \to \text{Spec } \mathcal{O}_X(X)^G := X//G$$

Given by the inclusion $\mathcal{O}_X(X)^G \hookrightarrow \mathcal{O}_X(X)$. Thanks to Nagata theorem we have that $X//G$ is a k-scheme of finite type.

We prove now that the GIT affine quotient is a good quotient. We need a lemma.

Lemma 3.20 (Regular functions separate G-invariant closed subsets)**.** *Let G be a reductive group acting rationally on an affine scheme X. If W_1, W_2 are two disjoint closed subsets of X, it exists $f \in \mathcal{O}(X)^G$ such that $f_{|W_1} = 0$ and $f_{|W_2} = 1$.*

Proof. Given that W_1 and W_2 are disjoint, there exists $g \in \mathcal{O}(X)$ such that $g_{|W_1} = 0$ and $g_{|W_2} = 1$. In general, g is not G-invariant, but there exists a finite dimensional k-vector G-invariant subspace $V \subset \mathcal{O}(X)$ that contains it. Let $h_1, \dots, h_n$ be a basis of V. We easily notice that we can suppose $h_i = g_i \cdot f$. Consider now the map:

$$\begin{aligned} H : X &\to \mathbb{A}^n \\ x &\mapsto (h_1(x), \dots, h_n(x)). \end{aligned}$$

Given that $h_i = g_i \cdot f$ we have $H_{|W_1} = 0$ and $H_{|W_2} = v \neq 0$. Define now an action:

$$\begin{aligned} G &\to \text{GL}(V) \\ g &\mapsto \left(h_i \mapsto g \cdot h_i = \sum_j a_{i,j}(g) h_j \right). \end{aligned}$$

We can write it as $G \to \text{GL}_n$ were $g \mapsto (a_{ij}(g))_{i,j}$.

Obviously we have that $H : X \to \mathbb{A}^n$ is G-invariant. We apply now lemma 3.16 to the vector v (that is G-invariant) obtaining a homogeneous polynomial $P \in k[x_1, \dots, x_n]^G$ such that $P(v) \neq 0$ and $P(0) = 0$. The thesis is now easily satisfied by $f = \frac{P \circ H}{P(v)} \in \mathcal{O}(X)^G$. $\square$

We are now ready to the following:

Theorem 3.21 (GIT quotient is a good quotient)**.** *The GIT affine quotient defined above is a good quotient.*

Proof. Obviously the map $X \to X//G$ is G-invariant. We verify now the surjectivity. It is enough to prove it at level of k-points, thanks to Chevalley's Theorem. Let $y \in X//G(k)$ be defined by the ideal $m_y = (f_1, \dots, f_n)_{\mathcal{O}(X)^G} \in \mathcal{O}(X)^G$. Thanks to lemma 3.17 we have that:

$$(f_1, \dots, f_n)_{\mathcal{O}(X)^G}\mathcal{O}(X) \cap \mathcal{O}(X)^G = (f_1, \dots, f_n)_{\mathcal{O}(X)^G}.$$

This implies that $(f_1, \dots, f_n)_{\mathcal{O}(X)}$ $(\neq \mathcal{O}(X))$ contract to m_y, hence the map is surjective.
In order to prove ii) it is enough to verify it on a base of Spec $\mathcal{O}(X)^G$. Consider Y_f where $f \in \mathcal{O}(X)^G$, we have that:

$$\mathcal{O}_Y(Y)_f = \left(\mathcal{O}(X)^G\right)_f = (\mathcal{O}(X)_f)^G = \mathcal{O}_X(\pi^{-1}(Y_f))^G$$

where the second equality is given thanks to proposition 2.49. Requirement iv) is given by the previous lemma as follows. Let W_1 and W_2 be closed invariant subset, and consider f as in the thesis of the proposition. We have that $f \in \mathcal{O}(X)^G$, $f_{|\pi(W_1)} = 0$ and $f_{|\pi(W_2)} = 1$, hence $\overline{\pi(W_1)} \cap \overline{\pi(W_2)} = \emptyset$.
Moreover, this implies also requirement iii) as follow. Suppose that $\pi(W)$ is not closed, there exists a k-point $p \in \overline{(\pi(W))} \setminus \pi(W)$. Thanks to the surjectivity there exists a closed point $q \in X(k)$ such that $\pi(q) = p$: applying the previous result to the disjoint closed sets W and q the requirement iii) follows.
The last requirement is π affine: this is obvious because it is a map between affine schemes. □

It is not true that affine GIT quotients are geometric quotients. For instance in the example 3.6 this does not happen.

Example 3.22. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ as in example 3.6. In order to calculate $\mathcal{O}(\mathbb{A}^2)^{\mathbb{G}_m}$ we can apply proposition 2.37. Given $g \in \mathbb{G}_m(k) = k^*$, we have:

$$\begin{aligned} L_g^{\#} : k[x,y] &\to k[x,y] \\ p(x,y) &\mapsto p(g \cdot x, g^{-1} \cdot y). \end{aligned}$$

It is now easy to see that a polynomial $p(x,y) \in k[x,y]$ is L_g invariant $\forall t \in \mathbb{G}_m(k) = k^*$ if and only if it is a polynomial in xy.
The GIT affine quotient is

$$\begin{aligned} \pi : \mathbb{A}^2 &\to \mathbb{A}^1 \\ (\alpha, \beta) &\mapsto \alpha \cdot \beta, \end{aligned}$$

induced by the natural inclusion $k[xy] \subset k[x,y]$.
It is easy to see that this quotient is a good quotient but it is not a geometric quotient. The preimage of the k-point $(t) \in$ Spec $\mathbb{A}^1$ contains three orbits: $G \cdot (\alpha, 0)$, $G \cdot (0, \alpha)$, $G \cdot (0,0)$ (for $\alpha \neq 0$).
The counterimage of a closed point $(t-a) \in \mathbb{A}^1$ with $a \neq 0$ is Spec $k[x,y] \otimes_{k[xy]} k =$ Spec $\frac{k[x,y]}{xy-a} \hookrightarrow$ Spec $k[x,y]$. This is, as seen before, a closed orbit.
We can observe that if we restrict to the open $\mathbb{A}^1 \setminus \{0\}$ of $\mathbb{A}^1$. We have that the map

$$\begin{aligned} \mathbb{A}^2 \setminus \pi^{-1}(0) &\to \mathbb{A}^1 \setminus \{0\} \\ (\alpha, \beta) &\mapsto \alpha \cdot \beta \end{aligned}$$

induced by π is a geometric quotient.

Led by the previous example, we will prove that we can restrict to an open subset of the GIT quotient to obtain a geometric quotient. In particular, we are interested in points whose orbit is closed.

Definition 3.23 (Stable points)**.** We say that a closed point $x \in X$ is stable if $G \cdot x$ is closed and $\dim G_x = 0$.

According to the definition of stable points in [GIT], the second condition is not required. We will see in the next theorem that actually is not needed. Modern algebraic geometers often required this, with the aim to use geometrical intuition: this assumption provides $\dim G \cdot x = \dim G$ as expected.

We are now able to restrict to a good quotient:

Proposition 3.24. *Let $\pi : X \to Y$ be a good quotient. Stable points are open in the restricted topology on $X(k)$. We can define the open $X^s \subset X$ whose closed points are stable.*
X^s is G-stable and $Y^s := \pi(X^s)$ is an open subset of Y such that $\pi^{-1}(Y^s) = X^s$. Moreover $\pi : X^s \to Y^s$ is a geometric quotient.

Proof. First of all we prove that we can define X^s. Let $x \in X(k)$ be a stable point and define $X_+ = \{x \in X(k) \mid \dim G_x > 0\}$ (this is closed thanks to prop 3.5). We have that there exists $f \in \mathcal{O}(X)^G$ such that:

$$f(X_+) = 1 \qquad f(G \cdot x) = 0.$$

Given $z \in X_f(k)$ we prove now that it is stable. We have that $\dim G_z = 0$, it remain to prove that $G{\cdot}z$ is closed. Suppose there exists $w \in \overline{G \cdot z} \backslash G{\cdot}z$. We have that $f(w) = 1$, hence $w \in X_f(k)$ and $w \notin X_+$. This implies that $\dim(G{\cdot}w) = \dim G - \dim(G_w) = \dim(G{\cdot}z)$ and so this is absurd thanks to proposition 3.3. Hence every stable point has a neighbourhood whose k-points are stable. This implies the first thesis. Moreover $X_f(k)$ is G-stable, hence X^s is stable.
Given that f is G-stable, we have that $\pi(X_f) = Y_f$ and $\pi^{-1}(Y_f) = X_f$. This implies that $Y^s := \pi(X^s)$ is an open subset of Y such that $\pi^{-1}(Y^s) = X^s$. Moreover we have that $\pi : X^s \to Y^s$ is a geometric quotient. □

Recalling the previous example is easy to see that $X^s = \mathbb{A}^2 \setminus (\{y = 0\} \cup \{x = 0\})$ and $Y^s = \mathbb{A}^1 \setminus \{0\}$.

3.2 Projective GIT quotient

We have seen the construction of GIT quotient for affine schemes, looking at G-invariant sections. For projective schemes, the situation is more complicated.
First of all, we could remember that good quotients are local on the target and use it. We could try to cover X with open affine invariant subsets, applying then the previous construction and finally glueing them together. The problem is that we cannot find such open subsets a priori without extra effort because they could not exist and in general there is not a canonical way to choose them.

In this section, we focus on projective schemes. We will use this notation without recalling them every time: G is a reductive group that acts on a projective scheme X.

Thanks to the projectivity of X, there exists a homogeneous ideal $I(X)$ of $k[x_0, \dots, x_n]$ such that

$$X = \operatorname{Proj} \frac{k[x_0, \dots, x_2]}{I(X)} \hookrightarrow \mathbb{P}^n.$$

We write $R(X) = \frac{k[x_0,\dots,x_n]}{I(X)}$ a graded k-algebra finitely generated.

Suppose that we could lift the action on X to an action on $R(X)$. Given $f \in R(X)^G$ we have that $D_+(f) \subset X$ is an affine G invariant set, this allows us to glue together the GIT affine quotient.

At this purpose, we are interested to lift it directly to $k[x_0, \dots, x_n]$. We define:

Definition 3.25 (Linearization of the G-action on X)**.** Consider the linear action of GL_{n+1} on $\mathbb{P}^n$. A linear G-equivariant projective embedding is a group morphism $G \to \mathrm{GL}_{n+1}$ and a G-equivariant embedding $X \hookrightarrow \mathbb{P}^n$, where the action on $\mathbb{P}^n$ is given via the group morphism. In this situation we say that **the G-action on $X \hookrightarrow \mathbb{P}^n$ is linear**.

Consider the following diagram

$$\begin{array}{ccc} \widetilde{X} = \text{Spec } R(X) & \hookrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \pi \\ X = \text{Proj } R(X) & \hookrightarrow & \mathbb{P}^n \end{array}$$

given by the natural map Spec $A \to$ Proj A (where A is a graded ring). We call $\widetilde{X}$ and $\mathbb{A}^n$ respectively **the affine cone** of X and $\mathbb{P}^n$.

If we have a linearization of the G-action, we can obviously extend it to $\mathbb{A}^n$ and $\widetilde{X}$. Thanks to $G \hookrightarrow GL_{n+1}$, we have that G acts on $k[x_0, \dots, x_n]$ and $R(X)$ homogeneously. We have that $R(X)^G = \bigoplus_i R(X)_i^G$ and it is a graded finitely generated algebra (G is reductive). We define:

Definition 3.26 (GIT projective quotient)**.** Given a linear action on Proj $R(X) \to \mathbb{P}^n$ we define the GIT quotient as a rational map:

$$X = \text{Proj } R(X) \dashrightarrow \text{Proj } R(X)^G = X//G,$$

induced by the inclusion $R(X)^G \subset R(X)$.

Remark 3.27 ($X//G$ is of finite type). We notice that $R(X)^G$ is a finite type algebra on k. The action of G is rational because the action of G is induced by the action of GL_n and G is reductive. Hence we can apply Nagata's theorem.

Obviously, there are multiple ways to find an embedding $X \hookrightarrow \mathbb{P}^n$: these define a priori different GIT quotients. Let's suppose to have a very ample line bundle L over X and a choice of a base of the vector space $H^0(X, L)$. It is well known that these define a not degenerate closed embedding (i.e. the image is not contained in a linear subspace)

$$X \hookrightarrow \mathbb{P}^n.$$

Vice versa, given a not degenerate closed embedding, it is induced by the very ample line bundle $\mathcal{O}(1)_{|X}$ and the base $i^* x_i$.

It is easy to prove that the GIT quotient does not depend on the base up to a linear transformation. It turns out it strongly depends on chosen line bundle, we define:

Definition 3.28 (GIT quotient given by L)**.** Let L be a very ample line bundle on X and fix a base of $H^0(X, L)$, this determine a closed embedding $X \hookrightarrow \mathbb{P}^n$. Suppose that G acts linearly, with the same construction (and notations) see above we can define

$$X = \text{Proj } R(X) \dashrightarrow \text{Proj } R(X)^G = X//_L G.$$

The quotient map constructed is defined only on an open set, we define:

Definition 3.29 (Semistable points)**.** We define $X^{ss} = X \setminus V_+(R(X)^G)$ the semistable set.

X^{ss} is the biggest open set where the git quotient is defined.

It follows from standard algebraic geometry arguments that:

Remark 3.30. $x \in X^{ss}$ if and only if exists an homogeneous $f \in R(X)^G$ such that $f(x) \neq 0$ in $k(x)$ (the residue field of x).

We can now state the first result of this section:

Proposition 3.31. *Consider a linear G-action on $X \hookrightarrow \mathbb{P}^n$, the morphism*

$$\pi : X^{ss} \to X//G$$

is a good quotient.

Proof. As mentioned before, we want to cover X^{ss} by affine and G-stable open subset.

Suppose $Y = X//G$ and $f \in R(X)^G$, we have that $Y_f \subset Y$ is an open stable subset.

$\pi^{-1}(Y_f) = X_f$, we have

$$\text{Spec } R(X)_{(f)} \to \text{Spec } (R(X)^G)_{(f)}$$

given by the restriction of $R(X)^G \subset R(X)$. As proved in proposition 2.49, we have that $(R(X)^G)_f = (R(X)_f)^G$. The action is homogeneous hence we have $(R(X)^G)_{(f)} = (R(X)_{(f)})^G$. Now the map $\pi_{|X_f} : X_f \to Y_f$ is exactly the GIT affine quotient defined in the previous section. Good quotients are local on the target and

$$X^{ss} = \cup_{f \in (R(X)^G)^h} X_f,$$

where $(\_)^h$ are the homogeneous elements. We have that $\pi : X^{ss} \to X//G$ is a good quotient. □

We give now an easy example of GIT projective quotient.

Example 3.32. Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ in the following way:

$$\begin{aligned} \mathbb{G}_m \times \mathbb{P}^n &\to \mathbb{P}^n \\ (t, [x_0, \dots, x_n]) &\mapsto [t^{-1} \cdot x_0, t \cdot x_1, \dots, t \cdot x_n]. \end{aligned}$$

We avoid describing it by ring maps, it is easy to see that the action restricts to $U_i \subset \mathbb{P}^n$ coordinate open subsets.

We have that $R(X) = k[x_0, \dots, x_n]$. Similarly to the example 3.6, it is easy to see that $R(X)^G = k[x_0x_1, x_0x_2, \dots, x_0x_n]$. The GIT projective quotient is

$$\begin{array}{ccc} \text{Proj } R(X) & \dashrightarrow \text{Proj } R(X)^G \xrightarrow{\sim} & \mathbb{P}^{n-1} \\ [x_0, \dots, x_n] & \longmapsto & [x_0x_1, \dots, x_0x_n] \end{array}$$

induced by the inclusion $R(X)^G \subset R(X)$. Its easy to prove that $X^{ss} = D_+(x_0) \setminus \{[1, 0, \dots, 0]\}$. We can write the map from $X^{ss} \simeq \mathbb{A}^n \setminus \{0\}$:

$$\begin{aligned} \mathbb{A}^n \setminus \{0\} &\to \mathbb{P}^{n-1} \\ (a_1, \dots, a_n) &\mapsto [a_1, \dots, a_n]. \end{aligned}$$

This quotient actually coincide with the definition of projective space.

Emulating the same observation made for affine GIT quotient, we would define an open subset of of stable points of X^{ss} where the action of G is closed. We proceed in a slightly different way, we will find again the usual description briefly.

Definition 3.33 (Stable points)**.** A point $x \in X(k)$ is said stable if:

- $\dim G_x = 0$,
- there exists $f \in R(X)_+^G$ homogeneus such that $x \in X_f$ and the G-action on X_f is closed.

Similarly to the affine case, the condition on $\dim G_x$ is not strictly required, but it is assumed.

Remark 3.34. We have that:

- stable points are open (in $X(k)$), we can hence define $X^s \subset X$ the open subset whose closed points are stable;
- X^s is G-stable;
- $X^s \subset X^{ss}$.

Proof. The set of points in $X(k)$ with $\dim G_x < 1$ (i.e. $= 0$) is open because the map $x \mapsto \dim G_x$ is upper semi-continuous (proposition 3.5). Stable k-points are exactly the intersection of $\{x \in X(k) | \dim G_x < 1\}$ and $\cup X_f(k)$ for f as in the definition of stable points. This prove that stable k-points are open in the restricted topology and G-invariant. The second thesis follows from this last sentence.
The third thesis is now immediate. □

Similarly to the affine case, we would like to obtain a geometric quotient of X^s.

Theorem 3.35 (Geometric quotient)**.** *There exists $Y^s \subset Y$ an open subset such that:*

- $\pi^{-1}(Y^s) = X^s$,
- $\pi_{|X^s} : X^s \to Y^s$ *is a geometric quotient.*

Proof. Define as X_c the union of X_f with $f \in R(X)_+^G$ such that the G-action on X_f is closed. Moreover we define Y_c as the union of Y_f for the values of f chosen before. We have a map $\pi : X_c \to Y_c$ that is a geometric quotient.
$X^s \subset X_c$ is obtaining by intersection with the locus of points with stabilizer of dimension 0. We define $Y^s = \pi(X^s)$.
We have that $\pi^{-1}(Y^s) = X^s$ because $\pi : X_c \to Y_c$ is a geometric quotient. Moreover we have that $\pi(X_c \setminus X^s) = Y_c \setminus Y^s$ is closed in Y_c thanks to property iii) of good quotients: this implies that Y^s is open in Y_c and hence in Y. This proves the thesis. □

Before moving on we recall the last example we made. In that case, we obtain the map

$$\begin{aligned} X^{ss} = \mathbb{A}^n \setminus \{0\} &\to \mathbb{P}^{n-1} \\ (a_1, \dots, a_n) &\mapsto [a_1, \dots, a_n] \end{aligned}$$

and the stability of every k-point of X^{ss} is immediate.

The final part of this excursus on GIT projective quotient aims to state some criteria to verify the stability of k-point.

The first criterion links the definitions of stable points in the affine and projective case.

Proposition 3.36 (Criterion for stability)**.** *Let x be a semistable k-point. x is stable if and only if the followings hold:*

- *$G \cdot x$ is closed in X^{ss},*
- $\dim G_x = 0$.

Proof. First of all we prove that a stable point has a closed orbit in X^{ss}. Let x be stable and $y \in \overline{G \cdot x} \cap X^{ss}$. Suppose that $y \notin G \cdot x$, hence the orbit $G \cdot y$ has dimension strictly smaller than the orbit of x (proposition 3.3). This implies that y cannot be a stable point because $\dim G_x = \dim G - \dim G \cdot x > 0$. The contradiction arise from the fact that $y \in X^s$ (because $\pi(y) = \pi(x) \in X^s$).
Conversely, let $x \in X^{ss}(k)$ be a point with closed orbit and $\dim G_x = 0$. Consider $f \in \mathcal{O}(X)^G$ such that $x \in X_f$. Define $Z = \{z \in X_f \mid \dim G_x > 0\}$, it is closed on X_f thanks to proposition 3.5.
Thanks to lemma 3.20 we have that there exists $h \in \mathcal{O}(X_f)^G$ such that $h_{|Z} = 0$ and $h_{|G \cdot x} = 1$. Suppose that $X = \text{Proj } A$, we can write $\mathcal{O}(X_f) = \text{Spec } (A_f)_0$, hence $h = \frac{h'}{f^n}$ such that $h' \in A^G$ (thanks to proposition 2.49). We have that $X_{fh'} \cap Z = \emptyset$ and hence every orbit in $X_{fh'}$ has zero dimensional stabilizer. This implies that every orbit in $X_{fh'}$ is closed, otherwise the closure contain an orbit of strictly lower dimension, but this is impossible. □

Consider now a linear G-action on $X \hookrightarrow \mathbb{P}^n$. We link (semi)stability of points in X to the (semi)stability of a point in the cone $\mathbb{A}^{n+1}$.

Let $x \in \mathbb{P}^n$ be a k-point of X, we call $\widetilde{x} \in \mathbb{A}^{n+1}$ a **lift** of x if it is in the preimage of x under the projection $\mathbb{A}^n \to \mathbb{P}^n$. We have:

Theorem 3.37 (Topological criterion)**.** *Let $\widetilde{x} \in \mathbb{A}^{n+1}$ be a non-zero lift of x. The followings hold:*

- *x is semi stable if and only if $0 \notin \overline{G \cdot \widetilde{x}} \subset \mathbb{A}^{n+1}$,*
- *x is stable if and only if $\dim G_{\widetilde{x}} = 0$ and $G \cdot \widetilde{x}$ is closed in $\mathbb{A}^{n+1}$.*

Proof. We start with the fist statement. Suppose that x is semistable, there exists $f \in R(X)^G$ such that $x \in X_f$. We see easily that the constant function $f_{\overline{G \cdot \widetilde{x}}} \neq 0$, this conclude one implication. For the other side, we use proposition 3.20 with the closed sets $\{0\}$ and $\overline{G \cdot \widetilde{x}}$: this provides an $f \in R(X)^G$ (that can be chosen homogeneous) such that $x \in X_f$.
We prove now the second statement. We can suppose (for both directions of the proof) that x is semistable because it is obvious in both cases. We know that $x \in X^{ss}(k)$ is stable if and only if $G \cdot x \cap X^{ss}$ is closed and $\dim G_x = 0$. Let $f \in R(X)^G$ homogeneous be such that $x \in X_f$, and consider the map

$$p : Z := \left\{ z \in \tilde{X} \mid f(z) = f(\tilde{x}) \right\} \to X_f.$$

This is a finite map and $p(G \cdot \tilde{x}) = G \cdot x$, moreover the preimage of $G \cdot x$ is a finite union of orbits in $\mathbb{A}^n$ with the same dimension (hence closed and disjoint, otherwise we can use proposition 3.3). This proves the equivalence of the second thesis. □

Observe that the previous result does not depend on the lift, hence the theorem before is true for an arbitrary lift $\tilde{x}$.

This last proposition will be very useful in the next section:

Proposition 3.38. *Let x and $\tilde{x}$ be as before. x is stable if and only if $\dim G \cdot x = 0$ and $\phi_{\tilde{x}} : G \to \mathbb{A}^n$ is proper.*

Proof. We omit this proof, a reference is [GIT], proposition 2.2.
We only say that this proposition holds because there is an equivalence between closed orbits and proper orbit maps. □

3.3 The criterion for stability

The criteria for (semi)stability we stated are still difficult to handle. In this section, we focus on a criterion that will help us to verify (semi)stability. The setting of this section is as before: X is projective with a G-action equipped with a linearization.

The ideas in this section are inspired by the generalization of an easy situation: the linear action of a subgroup of GL_{n+1} on $\mathbb{P}^n$.

We fix again a notation we will use in this section: σ is the action of G on a projective scheme over k $X \subset \mathbb{P}^n$. We fix an embedding $X \hookrightarrow \mathbb{P}^n$ and a linearization given by a map $G \to \mathrm{GL}_{n+1}$. We assume that G is reductive, sometimes we will repeat that on the statements to underline the importance of this hypothesis.

We will see that the stability of a k-point can be verified by looking at the induced action of one-parameter subgroups. This idea came from differential geometry and the next remark, before that we introduce a straightforward definition.

Definition 3.39 (Induced action)**.** Let $H \xrightarrow{\eta} G$ be a homomorphism of algebraic groups. The action of G on X induces an action of H:

$$H \times X \xrightarrow{(\eta,\mathrm{Id})} G \times X \xrightarrow{\sigma} X.$$

Moreover, if $G \to \mathrm{GL}_{n+1}$ is a linearization, it induces a linearization $H \xrightarrow{\eta} G \to \mathrm{GL}_{n+1}$:

$$\begin{array}{ccccccc} H \times \mathbb{A}^{n+1} & \xrightarrow{(\eta,\mathrm{Id})} & G \times \mathbb{A}^{n+1} & \longrightarrow & \mathrm{GL}_{n+1} \times_k \mathbb{A}^{n+1} & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H \times X & \xrightarrow{(\eta,\mathrm{Id})} & G \times X & \longrightarrow & \mathrm{GL}_{n+1} \times X & \longrightarrow & X \end{array},$$

where the omitted maps are the obvious ones.

We can now state:

Remark 3.40. Let x be a k-point of X. Consider a subgroup $H \overset{i}{\hookrightarrow} G$, if x is a (semi)stable point for the action of G we have that it is a (semi)stable point under the action of H.

Proof. If x is semistable respect to G, it exists $f \in R(X)^G$ homogeneous such that $f(x) \neq 0$. x is hence semistable respect to H.

Let x be stable respect to G, using the caracterization given on corollary 3.38 we have that $\phi_{\tilde{x}}$ is proper and $\dim G_{\tilde{x}} = 0$, where $\tilde{x}$ is a lift of x.
Consider the following:

$$\begin{array}{ccccc} H \times_G G_{\tilde{x}} & \xrightarrow{\pi_2} & G_{\tilde{x}} & \longrightarrow & \operatorname{Spec} k \\ \downarrow{\scriptstyle \pi_1} & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \tilde{x}} \\ H & \overset{i}{\hookrightarrow} & G & \xrightarrow{\phi_{\tilde{x}}} & \mathbb{A}^n \end{array},$$

we notice that $\phi_{\tilde{x}} \circ i$ is the orbit map of $\tilde{x}$ respect to the action of H, this show that $H_{\tilde{x}} = H \times_G G_{\tilde{x}}$. i is a closed immersion hence proper, this implies that $\phi_{\tilde{x}} \circ i$ is proper. It remain to prove that $\dim H_{\tilde{x}} = 0$. This is true because π_2 is a closed immersion, hence $\dim H_{\tilde{x}} \leq \dim G_{\tilde{x}} = 0$. □

We could now ask if it is possible to find an implication in the inverse sense. Precisely, we ask if it is enough to verify (semi)stability of point for suitable subgroups $H \subset G$ to prove the stability with respect to G.

It turns out we could verify stability via one-parameter subgroups.

Definition 3.41 (One parameter subgroup)**.** Let G be an algebraic group. A group map map $\lambda : \mathbb{G}_m \to G$ is called a one-parameter subgroup.
For brievity we will simple refer to **1-Ps**.
We say that λ is **trivial** if it is constant with value e.

This definition raise from the analogy between S^1 and $\mathbb{G}_m$. In differential geometry, the homomorphism $S^1 \to G$ are the one-parameter subgroups that are not injective.

Observe that, as in differential geometry, it is not true in general that λ is injective, we will see that a non-trivial 1-Ps factorize by a closed immersion. This is the reason why we call them *subgroups.*

Proposition 3.42. *Let $\lambda : \mathbb{G}_m \to G$ be a non-trivial 1-Ps we have that exists an homomorphism $\lambda' : \mathbb{G}_m \to G$ that is a closed immersion such that:*

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda} & G \\ \downarrow{\scriptstyle n\cdot} & \nearrow_{\lambda'} & \\ \mathbb{G}_m & & \end{array}$$

where $n\cdot$ is the map $\mathbb{G}_m \to \mathbb{G}_m$ given, at level of k-points, by $t \to t^n$.

Proof. First of all, let's prove that the image is closed. Call Z the image of λ. We have that Z is a constructible set and similarly to proposition 2.32 we can prove that it is locally closed. Moreover, Z has a structure of algebraic group compatible with G. Z is reduced (because $\mathbb{G}_m$ is reduced) and $\overline{Z}$ (with the reduced structure) is a subgroup of G (we omit the easy verification with group laws). The inclusion map $Z \hookrightarrow \overline{Z}$ is an open immersion. Consider the union of the non-trivial cosets of Z in $\overline{Z}$: we have that Z is the complementary of this open set in $\overline{Z}$, hence it is closed in $\overline{Z}$. This proves that Z is closed in G.
Consider now the map $\mathbb{G}_m \to Z$, it is a map between reduced groups on a closed field and it is surjective. Hence Z is a quotient of $\mathbb{G}_m$.
Let $Y \subset \mathbb{G}_m$ be the kernel of this morphism, this is on the form Spec $\frac{k[t,t^{-1}]}{(f(t))}$, beacuse $\mathbb{G}_m$ has dimension 1. Consider the k-points of Y, these are all solutions of a polynomial whose roots are a multiplicative group. This implies that every root has finite order, hence f can only be a product of factors of the following kind: $t^i - 1$, for $i \in \mathbb{Z}$. Using compatibility with multiplication is not difficult to show that $f = t^n - 1$. It is easy to verify that

$$0 \to \text{Spec } \frac{k[t,t^{-1}]}{(t^n - 1)} \hookrightarrow \text{Spec } k[t,t^{-1}] \xrightarrow{n\cdot} \text{Spec } k[t,t^{-1}] \to 0$$

is an exact sequence, for more details see [Mil17], chapter 7. This implies the thesis. □

Definition 3.43. Let $\lambda : \mathbb{G}_m \to G$ be a 1-Ps, we write $\lambda_x : \mathbb{G}_m \to X$ for the orbit map of a closed point $x \in X$ under the action induced by λ.
With a slightly abuse of notation we write $\lambda_{\tilde{x}} : \mathbb{G}_m \to \mathbb{A}^{n+1}$ for the action induced by a fixed linearization (clear from context) of σ.

Example 3.44. We recall the corollary 2.48. This implies that a 1-Ps of GL_n, up to a base change, is defined by:

$$\begin{aligned} \mathbb{G}_m &\to \mathrm{GL}_n \\ t &\mapsto \begin{pmatrix} t^{k_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t^{k_n} \end{pmatrix}. \end{aligned}$$

Where k_i are integers. It is easy to see that the image of this map is closed. Define $k = \gcd(k_i)$, we have that the map factorize by $k\cdot$. With the same notation of the previous proposition can define a 1-Ps that is a closed immersion:

$$\begin{aligned} \lambda' : \mathbb{G}_m &\to \mathrm{GL}_n \\ t &\mapsto \begin{pmatrix} t^{\frac{k_1}{k}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t^{\frac{k_n}{k}} \end{pmatrix}. \end{aligned}$$

Proposition 3.45. *Let G be a linearly reductive group, X be a projective scheme and suppose we have a linear action $G \curvearrowright X$. For all $x \in X(k)$, we have that:*

$$x \text{ is } G\text{-semistable} \Rightarrow x \text{ is } \lambda\text{-semistable } \forall \text{ non-trivial 1-Ps } \lambda \text{ of } G;$$
$$x \text{ is } G\text{-stable} \Rightarrow x \text{ is } \lambda\text{-stable } \forall \text{ non-trivial 1-Ps } \lambda \text{ of } G.$$

Proof. Let λ be a 1-Ps, recalling the setting of the previous proposition, we have that λ' is a closed immersion. Hence by remark 3.40 x is λ'-(semi)stable.
To conclude the proof it is enough to notice that the map $\mathbb{G}_m \xrightarrow{n\cdot} \mathbb{G}_m$ is surjective (it corresponds to an integer extension of rings). Hence x is λ (semi)stable if and only if it is λ' (semi)stable. □

We would prove the vice versa, this is the aim of the next sections.

3.3.1 Limits and one parameter subgroups

Fix a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$. Let x be a closed point of X, in order to verify (semi)stability we have to deal with the closure of the orbit $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)}$. For this purpose we define a notion of limit, his utility will be soon clear.

As in the previous subsections, we will assume k is algebraically closed.

Definition 3.46. Let Y and Z be finite type schemes over k. Suppose that Z is separated, $y_0 \in Y(k)$ is a closed but not open point (i.e. is not connected to the rest of the scheme) and that we have a map $f : Y \setminus \{y_0\} \to Z$ of k-schemes. We write

$$\lim_{y \to y_0} f(y) = z_0$$

where $z_0 \in Z$, if there exists a dashed arrow in the following diagram:

$$\begin{array}{ccc} Y \setminus \{y_0\} & \xrightarrow{f} & Z \\ \Big\downarrow i & \overset{\bar{f}}{\nearrow} & \Big\downarrow \\ Y & \longrightarrow & \mathrm{Spec}\ k \end{array},$$

such that $\overline{f}(y_0) = z_0$.
If such z_0 does not exist we say that the **limit does not exist**.
We can notice that, since Z over k is separated, if z_0 exists it is unique. Furthermore z_0 is closed i.e. a k-point (because in these hypotheses the image of a closed point is closed).

Remark 3.47. With the same notation of the definition, we notice that if Z over k is proper, this limit exists.

Remark 3.48. Suppose y_0 as in the definition above, we have that $y_0 \in \overline{Y \setminus \{y_0\}}$.
Let f be as below, for topological reasons we have that

$$\lim_{y \to y_0} f(y) \in \overline{f(Y)}.$$

The reason why we have introduced the notion of limit will be clear with the following example.

Consider the open immersion $\mathbb{G}_m \subset \mathbb{P}^1$, where $\mathbb{P}^1 \setminus \mathbb{G}_m$ are two closed points that we call 0 and ∞. Fix $x \in X(k)$, we have:

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda_x} & X \\ \big\downarrow i & \overset{\overline{\lambda_x}}{\nearrow} & \big\downarrow \\ \mathbb{P}^1 & \longrightarrow & \operatorname{Spec} k \end{array} .$$

X over k is proper, hence we can extend the map to 0 and ∞ in a unique way.

Given that $\mathbb{P}^1/k$ is proper and X/k is separated, we have that the image of $\mathbb{P}^1$ under the diagonal map is closed. $\lambda_x(\mathbb{G}_m) \subset \overline{\lambda_x}(\mathbb{P}^1)$, we have two possibilities:

- $\lim_{t\to 0} \lambda_x(t) \in \lambda_x(\mathbb{G}_m)$ and $\lim_{t\to\infty} \lambda_x(t) \in \lambda_x(\mathbb{G}_m)$. This means that $\lambda_x(\mathbb{G}_m) = \overline{\lambda_x}(\mathbb{P}^1)$ i.e. the orbit of x is closed,
- at least one between $\lim_{t\to 0} \lambda_x(t)$ and $\lim_{t\to\infty} \lambda_x(t)$ does not belong to $\lambda_x(\mathbb{G}_m)$. This means that the orbit of x is not closed.

Consider now the situation where, thanks to a linearization of G, the action of $\mathbb{G}_m$ extend to $\mathbb{A}^{n+1}$. We know from corollary 2.48 that there exists a base $e_0, \dots, e_n \in k^{n+1}$ such that the action (defined at level of k-points) is the following:

$$\mathbb{G}_m \times \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$$
$$(t, \sum a_i e_i) \mapsto \sum t^{r_i} a_i e_i.$$

Where $r_i \in \mathbb{Z}$ are called **weights**.
$\mathbb{A}^{n+1}$ is not proper, hence is not true that the orbit map $\phi_{\tilde{x}} : \mathbb{G}_m \to \mathbb{A}^{n+1}/k$ extends to $\mathbb{P}^1$ as before.
Anyway, It remains true the following:

Proposition 3.49. *Let λ be a 1-Ps of G. Any point in the boundary $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)} \setminus \lambda_{\tilde{x}}(\mathbb{G}_m)$ is equal to $\lim_{t\to 0} \lambda_{\tilde{x}}(t)$ or $\lim_{t\to\infty} \lambda_{\tilde{x}}(t)$.*

Proof. Consider an affine chart $\mathbb{A}^{n+1} \to \mathbb{P}^{n+1}$, by composition we have a map $\mathbb{G}_m \to \mathbb{P}^{n+1}$. This extends to a map $\mathbb{P}^1 \to \mathbb{P}^{n+1}$: the image of this map is closed and hence $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)} \setminus \lambda_{\tilde{x}}(\mathbb{G}_m) \subset \mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}$ is the image of an element of $\mathbb{P}^1$. This conclude the proof. □

We define now an important quantity for stability of points.

Definition 3.50 (Hilbert-Mumford weight)**.** Let $\tilde{x}$ be a lift of $x \in X(k)$. Using the same notation as above, we define:

$$\mu(x, \lambda) := -\min\{r_i : x_i \neq 0\}.$$

This weight depends on the chosen linearization of the action, we have fixed it at the beginning of the section. But it is clear that does not depend on the lift of x.

We have that:

Theorem 3.51. *Let $\lambda : \mathbb{G}_m \to G$ be a 1-Ps and $\tilde{x}$ be a lift of x; then:*

- $\mu(x, \lambda) < 0 \Leftrightarrow \lim_{t\to 0} \lambda_{\tilde{x}}(t) = 0$.
- $\mu(x, \lambda) = 0 \Leftrightarrow \lim_{t\to 0} \lambda_{\tilde{x}}(t)$ *exists and is not* 0.
- $\mu(x, \lambda) > 0 \Leftrightarrow \lim_{t\to 0} \lambda_{\tilde{x}}(t)$ *does not exists.*

Proof. Consider the action:

$$\begin{aligned} \mathbb{G}_m \times \mathbb{A}^{n+1} &\to \mathbb{A}^{n+1} \\ (t, \sum a_i e_i) &\mapsto \sum t^{r_i} a_i e_i. \end{aligned}$$

Fix $\tilde{x} = (a_0, \dots, a_n)$, the limit in 0 exists if the map

$$\begin{aligned} k[x_0, \dots, x_n] &\to k[t, t^{-1}] \\ x_i &\mapsto t^{r_i} e_i \end{aligned}$$

factorize by $k[t]$, in this case the limit is given by the contraction of $0 \in \operatorname{Spec} k[t]$. From these observation the thesis follows easily. □

The union of these two results give us a criterion for stability under 1-Ps.

Theorem 3.52 (Critereon for stability for 1-Ps)**.** *Fix a linearization of the action $G \curvearrowright X$. Let $\lambda : \mathbb{G}_m \to G$ be a 1-Ps; then:*

- *$x \in X(k)$ is semistable for the action of $\mathbb{G}_m$ $\Leftrightarrow \mu(x, \lambda) \geq 0$ and $\mu(x, \lambda^{-1}) \geq 0$.*
- *$x \in X(k)$ is stable for the action of $\mathbb{G}_m$ $\Leftrightarrow \mu(x, \lambda) > 0$ and $\mu(x, \lambda^{-1}) > 0$.*

Where $\lambda^{-1} : \mathbb{G}_m \xrightarrow{i} \mathbb{G}_m \xrightarrow{\lambda} G$ (here i is the inverse group map of $\mathbb{G}_m$).

Proof. It follows easily from the topological criterion. □

We are now ready for the general criterion.

3.3.2 Hilbert-Mumford criterion

In this section we will prove the following:

Theorem 3.53 (Hilbert-Mumford criterion)**.** *Suppose that $k = \overline{k}$. Let G be a linearly reductive group, X be a projective k-scheme and suppose we have a linear action $G \curvearrowright X$. For all $x \in X(k)$, we have that:*

$$\begin{aligned} x \text{ is } G\text{-semistable} &\Leftrightarrow \mu(x, \lambda) \geq 0 \ \forall \text{ non-trivial 1-PS } \lambda \text{ of } G. \\ x \text{ is } G\text{-stable} &\Leftrightarrow \mu(x, \lambda) > 0 \ \forall \text{ non-trivial 1-PS } \lambda \text{ of } G. \end{aligned}$$

We start by introducing a bit of notation useful in the proof of the criterion.
We will refer to:

$$
\begin{array}{ll}
R := k[[t]] & \text{i.e. the formal power series} \\
K := k((t)) = Frac(k[[t]]) & \text{i.e. the formal laurent series.}
\end{array}
$$

The introduction of R could lead us to the definition of "formal neighbourhood", we skip this because is not useful for our purpose, anyway everything is well explained in [Har77].

The natural inclusion $R \subset K$ induces a set teoretic map:

$$G(R) = \mathrm{Hom}_k(\mathrm{Spec}\ R, G) \longrightarrow G(K) = \mathrm{Hom}_k(\mathrm{Spec}\ K, G),$$

where G is an algebraic group. **This map is a homomorphism of groups and an inclusion**. The former is obvious, to prove the latter consider the diagram:

$$
\begin{array}{ccc}
\mathrm{Spec}\ K & \longrightarrow & G \\
\big\downarrow i & \nearrow & \big\downarrow \\
\mathrm{Spec}\ R & \longrightarrow & \mathrm{Spec}\ k.
\end{array}
$$

Thanks to the valuative criterion of separateness (G is always separated, prop. 2.25), we have that given a K-points it exists at most a unique R-points such that the diagram commute. Consider the map $\mathrm{Spec}\ k \to \mathrm{Spec}\ R$ given by the map $R \to k$ that send t to 0, this define another map ω (by composition):

$$\omega : G(R) \longrightarrow G(k)$$

Given $f : \mathrm{Spec}\ K \to G$ that is also an R-point of G, it is easy to see that

$$\lim_{t \to s} f(t) = \omega(f),$$

where s is the closed point in Spec R.

Definition 3.54 ($< \lambda >$)**.** Let λ be a 1-Ps, we define:

$$< \lambda >: \mathrm{Spec}\ K \to \mathbb{G}_m \xrightarrow{\lambda} G$$

where the first map is induced by inclusion $k[t, t^{-1}] \subset K$.

We state now a theorem we will not proof, it will be central in the proof of the criterion.

Theorem 3.55 (Cartan-Iwahori decomposition)**.** *Let G be a reductive group over k, where* Char $k = 0$*. We have that for every $p \in G(K)$ there exists λ and $b_1, b_2 \in G(R)$ such that:*

$$p = b_1 \cdot < \lambda > \cdot b_2$$

viewed in the group $G(K)$.

Before going on we give an easy example of this theorem:

Example 3.56. Suppose $G = \mathrm{GL}_n$, the R-points are matrices with coefficients in R and determinant in R^*. Similarly, the K-points are matrices with coefficients in K and determinant not 0.

Let A be a K-point of GL_n, there exists an element $\delta \in R$ such that $\delta \cdot A$ has coefficients in R.

R is a principal ideal domain: thanks to Smith decomposition, there exist S and T R-points such that

$$S\,\delta A\,T = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_n \end{pmatrix},$$

where $\alpha_i | \alpha_{i+1}$. There exist $x_i \in R^*$ such that $\delta \cdot \alpha_i = t^{k_i} \cdot x_i$. We call X the diagonal matrix with coefficients x_i, we have:

$$A = S^{-1}X \begin{pmatrix} t^{k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t^{k_n} \end{pmatrix} T^{-1} = S^{-1}X < \lambda > T^{-1}.$$

Where λ is the 1-Ps specified by k_i in the given base. $S^{-1}X$ and T^{-1} are R-points: this conclude the example.

We state now a slightly powerful version of the valuative criterion of properness.

Lemma 3.57. *Let $f : X \to Y$ be a morphism of finite type, with X noetherian. f is proper if for every complete discrete valuative ring A forming a commutative diagram*

$$\begin{array}{ccc} \mathit{Spec}\ Frac(A) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathit{Spec}\ A & \longrightarrow & Y \end{array}$$

there exists a unique dashed arrow making the diagram commutative.

This lemma is proven in [EGAII] proposition 7.3.9 and 7.3.8.

Corollary 3.58. *With the same notations of the previous lemma, suppose that Y is a k-scheme. We have that f is proper if and only if for every square diagram*

$$\begin{array}{ccc} \mathit{Spec}\ K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathit{Spec}\ R & \longrightarrow & Y \end{array}$$

there exists a unique dashed arrow making it commutative.

Proof. We have that R has to be a complete DVR that contain the field k. Using the Cohen structure theorem, it is not difficult to see that R has to be $k[[t]]$; this proves the corollary. □

In order to prove the criterion, we will use corollary 3.38. The following result is the core of the proof.

Proposition 3.59. *Suppose that $k = \overline{k}$. Fix a linearization of the action of a reductive group G on a projective variety X. Let x be a k-point of X and fix a lift $\tilde{x}$. We have that:*

- *$\phi_{\tilde{x}}$ is not proper $\Rightarrow$ there exists a non-trivial 1-Ps λ of G such that $\mu(x, \lambda) \leq 0$.*
- *$0 \in \overline{G \cdot \tilde{x}}$ $\Rightarrow$ there exists a non-trivial 1-Ps λ of G such that $\mu(x, \lambda) < 0$.*

Proof. Given that $\phi_{\tilde{x}}$ is not proper, there exist maps η, μ such that the following diagram commute (thanks to the previous corollary)

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\eta} & G \\ \downarrow & \nearrow & \downarrow \phi_{\tilde{x}} \\ \text{Spec } R & \xrightarrow{\mu} & \mathbb{A}^{n+1} \end{array} ,$$

but it does not exist a dotted arrow. To avoid an heavy notation, we will write $\phi_{\tilde{x}} \circ \eta$ instead of μ because we have the inclusion $G(R) \subset G(K)$.
Due to the proposition 3.55 there exist ρ_1, ρ_2 in $G(R)$ and a non-trivial 1-Ps λ such that

$$\eta = \rho_1 \cdot < \lambda > \cdot \rho_2.$$

Define $b_i = \lim_{t \to s} \rho_i(t)$ for $i = 1, 2$, these are two k-points.
Consider the map $b_2^{-1} \cdot \lambda \cdot b_2 : \mathbb{G}_m \to G$ given by composition with right and left multiplication. We can write:

$$\eta = (\rho_1 \cdot b_2) \cdot (b_2^{-1} \cdot < \lambda > \cdot b_2) \cdot (b_2^{-1} \cdot \rho_2) = (\rho_1 \cdot b_2) \cdot (< b_2^{-1} \cdot \lambda \cdot b_2 >) \cdot (b_2^{-1} \cdot \rho_2),$$

where the second equality follows immediately by the definition of $< \lambda >$. The reason why we conjugate by b_2 will be clear later.
For every k-scheme T we have an action at level of T points $G(T) \times X(T) \to X(T)$. Moreover, given $T \to S$ a map of k-schemes, by functoriality we have maps $X(S) \to X(T)$ and $G(S) \to G(T)$ compatible with action. This allow us to see any S-point as a T-point. For this reason the k-point $\tilde{x}$ can be seen as a R or K-point, thanks to the obvious map. It is easy to verify that $\phi_{\tilde{x}} \circ \eta = \tilde{\sigma}(\eta, \tilde{x})$ where $\tilde{x}$ is seen as K-point, from now on we will avoid to specify $\tilde{\sigma}$ and we will simply use $\cdot$. Merging this equation with the previous one we have:

$$(\rho_1 \cdot b_2)^{-1} \cdot (\phi_{\tilde{x}} \circ \eta) = (< b_2^{-1} \cdot \cdot \lambda \cdot b_2 >) \cdot (b_2^{-1} \cdot \rho_2) \cdot \tilde{x}.$$

We choose now suitable coordinartes for $\mathbb{A}^{n+1}$, such that the action of the 1-Ps $(b_2^{-1} \cdot \lambda \cdot b_2)$ is diagonalizable.
Moreover, a Spec A-point f of $\mathbb{A}^{n+1} = \text{Spec } k[x_0 \ldots, x_n]$ is defined by the image of x_i in A, we call these $X_i(f) \in A$. Hence we have

$$X_i((\rho_1 \cdot b_2)^{-1} \cdot (\phi_{\tilde{x}} \circ \eta)) = X_i((< b_2^{-1} \cdot \cdot \lambda \cdot b_2 >) \cdot (b_2^{-1} \cdot \rho_2) \cdot \tilde{x}),$$

for all $i \in 0, \ldots, n$.
We remember now that a diagonalized action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ is defined by the ring map

$$\begin{gathered} k[x_0, \ldots, x_n] \to k[x_0, \ldots, x_n] \otimes_k k[t, t^{-1}] \\ x_i \mapsto x_i \otimes t^{r_i}, \end{gathered}$$

where r_i are the weight of the action. Thanks to the choice of the basis we have

$$X_i((\rho_1 \cdot b_2)^{-1} \cdot (\phi_{\tilde{x}} \circ \eta)) = t^{r_i} X_i((b_2^{-1} \cdot \rho_2) \cdot \tilde{x}),$$

for all $i \in 0, \ldots, n$. The left hand side is an element of R (there is an R-point of G that act on an R-point of $\mathbb{A}^{n+1}$), hence $X_i((b_2^{-1} \cdot \rho_2) \cdot \tilde{x}) \in t^{-r_i} R$.

Consider the k-algebra map $R \to k$ that send t to 0, it sends ρ_2 to b_2, hence it sends the R-point $(b_2^{-1} \cdot \rho_2) \cdot \tilde{x}$ to $\tilde{x}$ (thanks to the compability of the action). This implies that

$$X_i((b_2^{-1} \cdot \rho_2) \cdot \tilde{x}) = X_i(\tilde{x}) + tz_i,$$

where $z_i \in R$, for every $i \in 0, \dots, n$.
Hence we have $r_i \geq 0$ whenever $X_i(\tilde{x}) \neq 0$, this implies that

$$\mu(x, b_2^{-1}\lambda b_2) = -\min\{r_i | r_i \text{ weight such that } X_i(\tilde{x}) \neq 0\} \leq 0.$$

The second point follows similarly, suppose $0 \in \overline{G \cdot \tilde{x}}$, there exists a K-point of G η such that

$$\lim_{t \to s} \phi_{\tilde{x}} \circ \eta(t) = 0.$$

With the same steps of the previous point, we have again that

$$X_i((\rho_1 \cdot b_2)^{-1} \cdot (\phi_{\tilde{x}} \circ \eta)) = t^{r_i} X_i((b_2^{-1} \cdot \rho_2) \cdot \tilde{x}).$$

This time, the R-point at left hand side has coordinate in tR. Hence $X_i((b_2^{-1} \cdot \rho_2) \cdot \tilde{x}) \in t^{-r_i+1}R$, we conclude as before. □

We recall the criterion:

Theorem 3.60 (Hilbert-Mumford criterion)**.** *Suppose that $k = \overline{k}$ and* $\operatorname{Char} k = 0$*. Let G be a linearly reductive group, X be a projective k-scheme and suppose we have a linear action $G \curvearrowright X$. For all $x \in X(k)$, we have that:*

$$x \text{ is } G\text{-semistable} \Leftrightarrow \mu(x, \lambda) \geq 0 \ \forall \text{ non-trivial 1-PS } \lambda \text{ of } G.$$
$$x \text{ is } G\text{-stable} \Leftrightarrow \mu(x, \lambda) > 0 \ \forall \text{ non-trivial 1-PS } \lambda \text{ of } G.$$

Proof. It follows immediately from the propositions 3.45 and 3.59. □

Chapter 4

Intermezzo: Hilbert and Picard schemes

This chapter is devoted to the introduction of Hilbert and Picard schemes. We will outline the construction of Hilbert scheme and in particular we will describe the immersion in the Grassmannian. Moreover, we will define Picard scheme. This chapter aims to state some classical results we will use in the next chapter. For this reason proofs are substantially omitted. A reference for the constructions is [FGAexp]. First of all, we start recalling the example of Grassmannian in chapter 1.

4.1 Plücker embedding

We recall briefly what the Grassmannian functor is:

$$\begin{aligned}\mathcal{G}r(S) &:= \{\text{exact sequences of locally free sheaves on } S \\ &\qquad 0 \to \mathcal{K} \to \mathcal{O}_S^n \to \mathcal{Q} \to 0\} \,/\sim \\ \mathcal{G}r(h : S \to T) &:= h^* : \mathcal{G}r(T) \to \mathcal{G}r(S).\end{aligned}$$

Before stating the embedding, we define:

Definition 4.1 (Projective space)**.** let V be a vector space on k, we define:

$$\mathbb{P}(V) = \text{Proj Sym}(V).$$

Where $\text{Sym}\, V$ has a natural structure of algebra on k.
In such a way, k-points of $\mathbb{P}(V)$ are equivalence classes of nonzero linear forms on V.

In a certain sense, the algebraic projectivization $\mathbb{P}(V)$ is dual to the standard one. We define:

Proposition 4.2 (Algebraic Plücker embedding)**.** *Consider the map defined at level of k-points as:*

$$\begin{aligned}\iota : Gr(d, V) &\to \mathbb{P}\left(\bigwedge^d V\right) \\ (q : V \to W) &\mapsto [\bigwedge^d q : \bigwedge^d V \to \bigwedge^d W],\end{aligned}$$

it is an element of the dual because $\bigwedge^d W \simeq k$.
This map is a closed embedding between schemes.

4.2 The Hilbert scheme

The Hilbert scheme is roughly a scheme that parametrizes the closed subset of $\mathbb{P}^n$, it is a fine moduli space of a functor. We give a brief overview of the construction. The Hilbert scheme is a particular case of Quot scheme, we avoid handling the general definition despite it is easy to generalize the construction.

Definition 4.3 (Hilbert functor)**.** Fix a base scheme S, consider the functor $\mathcal{H}ilb$: Sch$/k \to$ Set:

$$\begin{aligned} \mathcal{H}ilb(T) &:= \{\text{Closed } S\text{-subschemes } Y \subset \mathbb{P}^n_S \times_S T \mid Y \to T \text{ is flat}\} \\ \mathcal{H}ilb(g : T \to V) &:= g^*. \end{aligned}$$

The functor is well defined because base change of closed immersion and flat maps are still respectively closed immersion and flat maps.
Alternatively, it is possible to define the functor as

$$\mathcal{H}ilb(T) = \left\{\text{Coherent quotient sheaf } q : \mathcal{O}_{\mathbb{P}^n_S \times T} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is flat over } T\right\} / \sim,$$

where $q : \mathcal{O}_{\mathbb{P}^n_S \times T} \to \mathcal{F}$ is equivalent to $q' : \mathcal{O}_{\mathbb{P}^n_S \times T} \to \mathcal{F}'$ if there exist an isomoprhism of $\mathcal{O}_T$-sheaves $\alpha : \mathcal{F} \to \mathcal{F}'$ such that

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^n_S \times T} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle \alpha} & & \\ \mathcal{O}_{\mathbb{P}^n_S \times T} & \longrightarrow & \mathcal{F}' & \longrightarrow & 0 \end{array}$$

commutes.

It is not difficult to check that the two descriptions are equivalent.

We are interested in the case $S = \operatorname{Spec} k$, from now on we will consider only schemes over k.

We remember the following definition:

Definition 4.4 (Hilbert polynomial)**.** Let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}^n$, we can define the Hilbert polynomial as:

$$\Phi(m) = \chi(\mathcal{F}(m)) = \sum_{i=0}^{n} \dim_k H^i(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(m)),$$

where the dimensions are finite because $\mathcal{F}$ is coherent over a projective scheme.

We have that the following holds:

Theorem 4.5 (Invariance of Hilbert polynomial by flat families)**.** *Let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}^n_T$ and suppose that $\mathcal{F}$ is flat over T.*
Consider now the map $p_t : \mathbb{P}^n \times_k \operatorname{Spec} k(t) \to \mathbb{P}^n \times_k T$ where t is a point of T. $\mathcal{F}_t = p_t^* \mathcal{F}$ is a coherent sheaf over $\mathbb{P}^n$ that has an Hilbert polynomial $\Phi_t(x)$. We have that the function

$$\begin{aligned} T &\to \mathbb{Q}[x] \\ t &\mapsto \Phi(m) \end{aligned}$$

is locally constant.

This say that the functor naturally decompose:

$$\mathcal{H}ilb_n = \coprod_{p(x)\in\mathbb{Q}[x]} \mathcal{H}ilb_n^{p(x)}.$$

Our focus shift now to represent $\mathcal{H}ilb_n^{p(x)}$. A subscheme of $\mathbb{P}^n$ is simply defined by an homogeneous ideal sheaf $I(X) \subset k[x_0,\dots,x_n]$, we can notice that there exists a big enought m such that $I(X)_m$ generate $I(X)_{l\geq m}$ as $k[x_0,\dots,x_n]$-module. This implies that the subscheme X it is uniquely defined by a subspace of $k[x_0,\dots,x_n]_m$ (i.e. homogeneous polynomials of degree n). The next theorem is a stronger result in this sense.

Theorem 4.6 (m-regularity)**.** *Fix a polynomial p. There exists an m_0 such that for every $m \geq m_0$ and every $X \subset \mathbb{P}^n$ closed subset with Hilbert polinomial p, we have that:*

- *$I(X)_m$ generate $I(X)_{\geq m}$ as $k[x_0,\dots,x_n]$-module.*
- *$H^i(X, I(X)(m)) = 0$ and $H^i(X, \mathcal{O}_X(m)) = 0$ for all $i > 0$.*

Let $X \subset \mathbb{P}^n$ be a closed subset with Hilbert polynomial p, the map

$$\mathrm{res}_{X,m} : H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(\mathbb{P}^n, i^*\mathcal{O}_X(m))$$

is surjective and $\dim_k H^0(\mathbb{P}^n, i^*\mathcal{O}_X(m)) = p(m)$.

The latter follows from the second point of the last theorem, the former follows from the long exact sequence of $0 \to I(X) \to \mathcal{O}_{\mathbb{P}^n} \to i^*\mathcal{O}_X \to 0$.

This lead us to define an embed $\mathcal{H}ilb_n^{p(x)}$ in the Grassmannian functor $\mathcal{G}r(l,s)$. Where $l = \dim_k k[x_0,\dots,x_n]_m = \binom{m+n}{m}$ and $s = p(m)$ for a big enough m.

We need to generalize this idea to families, the details can be found in [FGAexp], section 5.5.

Proposition 4.7. *Consider a family given by the quotient $q : \mathcal{O}_{\mathbb{P}^n_S\times T} \to \mathcal{F}$ where $\mathcal{F}$ is flat on T and the Hilbert polynomial of $\mathcal{F}$ is p. Let $\mathcal{G}$ be the kernel of q and $\pi_T : \mathbb{P}^n_T \to Spec\ T$ the natural map. We have the following exact sequence:*

$$0 \to \pi_{T*}\mathcal{G}(m) \to \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) \to \pi_{T*}\mathcal{F}(m) \to 0$$

of locally free sheaves with $\mathrm{rk}\, \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) = \binom{m+n}{m}$ *and* $\mathrm{rk}\, \pi_{T*}\mathcal{F}(m) = p(m)$*, for an uniform m big enough.*
Moreover, via pullback π_T^* it is possible to recover the initial $\mathcal{F}$ and $\mathcal{G}$ because there is a natural map of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_T^*\pi_{T*}\mathcal{G}(m) & \longrightarrow & \pi_T^*\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) & \longrightarrow & \pi_T^*\pi_{T*}\mathcal{F}(m) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}(m) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n_T}(m) & \longrightarrow & \mathcal{F}(m) & \longrightarrow & 0 \end{array}.$$

Thanks to the previous proposition we have easily:

Proposition 4.8 (Embedding in $\mathcal{G}r$)**.** *We have a natural transformation of functors:*

$$\alpha(T) : \mathcal{H}ilb_n^{p(x)}(T) \to \mathcal{G}r\left(p(m), \binom{m+n}{m}\right)(T)$$
$$\{family\ q : \mathcal{O}_{\mathbb{P}^n_T} \to \mathcal{F}\} \mapsto \{\pi_{T*}(q(m)) : \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) \to \pi_{T*}\mathcal{F}(m)\ over\ T\}$$

The morphism is injective thanks to the second part of the previous proposition.

We know that $\mathcal{G}r$ is representable. We defined this map in order to prove the representability of the Hilbert scheme exploiting the fact that $\mathcal{G}r$ is representable. In particular we will have that a closed subscheme of the Grassmannian represent it.

With this purpose we introduce:

Definition 4.9 (Relatively representable)**.** Given two contravariant functors $F, G : \mathcal{C} \to$ Set we say that a map $\beta : F \to G$ is representable if the following holds. For every $U \in Ob(G)$ and any element $\xi \in G(U)$ such that the functor $\mathrm{Hom}(\_, U) \times_G F$ is representable, where $\times$ is the fibred product of functors.
If the map β is clear from the context, we say that F is relatively representable over G.

We would use the following:

Proposition 4.10. *If $\beta : F \to G$ is representable and G is a representable functor, we have that F is a representable functor.*

In our case, we have that:

Proposition 4.11. $\alpha : \mathcal{H}ilb_n^{p(x)} \to \mathcal{G}r\left(p(m), \binom{m+n}{m}\right)$ *is representable, and there exists a locally closed k-subscheme*

$$Hilb_n^{p(x)} \subset Gr\left(p(m), \binom{m+n}{m}\right)$$

for big enough m.

Prove this proposition would take us too far afield, a reference is [FGAexp], section 5.5.6. We will briefly state that is a closed embedding.

From the first section, we know the Plücker embedding:

$$\mathrm{Gr}\left(p(m), \binom{m+n}{m}\right) \hookrightarrow \mathbb{P}(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m)$$

Thanks to the valuative criterion for properness it is possible to prove that Hilbert schemes are proper, hence we have:

Proposition 4.12. *The map*

$$e : Hilb_n^{p(x)} \hookrightarrow \mathbb{P}(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m)$$

is a closed immersion, hence Hilbert schemes are projective.

The last question we answer in this section is:

Problem 4.13. *Given a subscheme of $\mathbb{P}^n$ with Hilbert polynomial $p(x)$, what is the correspondent k-point of $Hilb_n^{p(x)}$ through the embedding e?*

At level of k-points we have:

$$\mathcal{H}ilb_n^{p(x)}(\mathrm{Spec}\ k) \to \mathcal{G}r\left(p(m), \binom{m+n}{m}\right)(\mathrm{Spec}\ k)$$

$$\{X \subset \mathbb{P}^n \text{ with Hilbert polynomial } p(x)\} \mapsto \left\{\mathrm{res}_{X,m} : H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(\mathbb{P}^n, i^*\mathcal{O}_X(m))\right\}$$

Hence we have:

$$\mathcal{H}ilb_n^{p(x)}(\mathrm{Spec}\ k) \to \mathbb{P}(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m)$$

$$\{X \subset \mathbb{P}^n \text{ with Hilbert polynomial } p(x)\} \mapsto \left\{\bigwedge^{p(m)} \mathrm{res}_{X,m} : \bigwedge^{p(m)} H^0(\mathbb{P}^n, \mathcal{O}(m)) \to k\right\}.$$

4.3 The Picard scheme

In this section we will introduce the Picard scheme, we will use it in the first part of the next chapter. The standard reference for the construction is [FGA]. Another reference is [GIT], section 0.4 (d).

Again we suppose to work on Sch/k.

Definition 4.14 (Picard functor)**.** Fix X an S-scheme. Define the functor $\mathcal{P}ic_X : \mathrm{Sch}/S \to$ Set as:

$$\mathcal{P}ic_{X/S}(T) := \frac{\{\text{Group of invertible sheaves } L \text{ on } X \times_S T\}}{\{\text{Subgroup of } \pi_2^*(K)\text{, where } K \text{ is an invertible sheaf on } T\}}$$

$$\mathcal{P}ic_{X/S}(g : T \to T') := g^*.$$

Where g^* is the pullback and $\pi_2 : X \times_S T \to T$ is the projection. We omit to verify that the functor is well defined.

It easy to see that:

Remark 4.15. $\mathcal{P}ic_{X/S}$ is a group functor, with the operation of tensoring sheaves.

We have a more suitable description of $\mathcal{P}ic$ functor in a particular situation:

Proposition 4.16. *Suppose that there exists $\sigma : S \to X$ a section of $\pi : X \to S$, we can write the Picard functor as follows:*

$$\mathcal{P}ic_{X/S}(T) \simeq \Big\{(L, \Psi) \,|\, L \textit{ is an invertible sheaf on } X \times_S T \textit{ and } \Psi : (\sigma \circ f, 1_T)^*(L) \xrightarrow{\sim} \mathcal{O}_T\Big\}$$

$$\mathcal{P}ic_{X/S}(g : T \to T') = g^*.$$

Where g^* is the pullback and where $f : T \to S$.

This last formulation causes the automorphism group of (L, Ψ) to be trivial. It turns out that if such a section exists, the functor is representable. Moreover if such a section does not exist, we can still recover the following:

Theorem 4.17. *Let $X \to S$ be flat and projective, whose geometric fibres are varieties. There is a unique $Pic_{X/S}$ with a natural transformation $\phi : \mathcal{P}ic_{X/S} \to \mathrm{Hom}_S(\_, Pic_{X/S})$, such that:*

- *$\phi(T)$ is injective for all T.*
- *$\phi(T)$ is surjective when $X \times_S T \to T$ admits a section.*

Chapter 5

Construction of $\mathcal{M}_g$ for $g \geq 2$

In this chapter we will perform the final construction. finding a coarse moduli space for the functor $\mathcal{M}_g$ that we recall:

$$\begin{aligned} \mathcal{M}_g(S) = \{&\text{proper and flat families } f : X \to S \text{ whose geometric fibres are:} \\ &\text{smooth, connected, of dimension 1 and genus } g\} \\ \mathcal{M}_g(g : T \to S) = g^*&. \end{aligned}$$

We use the construction of Hilbert scheme we recall in chapter 4. The idea of the construction follows what was done by Mumford and then Gieseker. In particular, we will verify the stability of curves using Gieseker's criterion.

The chapter is naturally split into two parts. In some sense, the first part verifies that the scheme that should be the coarse moduli space verify the universal property (the second request in definition 1.13). The second part aims to verify the first request in definition 1.13, this is equivalent to verifying that a particular quotient is geometric, i.e. verify the stability of closed points.

The two fundamental references we use are [GIT] and [Gie82], moreover we will use a result in [Mum77]. The reader can found the construction also in [HM98], chapter 5. We will use classical results about algebraic curves, a reference is [Har77].

In the whole chapter k will be an algebraically closed field and $\operatorname{Char} k = 0$.

5.1 The idea of the construction

In the first chapter we constructed the moduli space of elliptic curves with the help of a marked point. We did not specify why fixing points on curves is useful to construct moduli space but, in general, the question is strictly related to line bundles.

Usually points are fixed to define a specific very ample line bundle that gives an embedding in $\mathbb{P}^n$. This allows us to see the curve as a point in a suitable Hilbert scheme. This is the standard technique to construct $\mathcal{M}_{g,n}$.

In our case, the embedding is provided in a different way. Let C be a connected, proper, smooth curve of genus $g \geq 2$ on k (from now on, simply a **curve**). We use the canonical bundle that is ample, hence a big enough power defines an embedding.

Before going on we recall some essential classical results.

Theorem 5.1 (Riemann-Roch and Serre duality)**.** *Let L be a line bundle on C, a curve over k. The Riemann-Roch formula state that:*

$$\dim H^0(C, L) - \dim H^1(C, L) = 1 - g + \deg L.$$

The serre duality say that there is an isomorphism $H^1(X, L) \simeq H^1(X, K_C \otimes L^*)^*$ *(where* L^* *is the dual of L), these two implies that:*

$$\dim H^0(C, L) - \dim H^0(C, K_C \otimes L^*) = 1 - g + \deg L.$$

Where $K_{\mathcal{C}}$ *is the canonical bundle on* $\mathcal{C}$*.*

Theorem 5.2. *Let* C *be a connected projective curve on Spec* k *of genus* g*. Suppose* $L \to C$ *a line bundle. Fix a basis* $\sigma_0, \dots, \sigma_n$ *of* $H^0(C, L)$ *(of dimension* $n+1$*). It defines a map:*

$$\begin{aligned} \phi_L : C \to \mathbb{P}^n = \mathbb{P}(H^0(C, L)) \\ x \mapsto (\sigma_0(x), \dots, \sigma_n(x)). \end{aligned}$$

If $\deg(L) \geq 2g+1$*, the map is a non-degenerate embedding (i.e. L is very ample).*

Proposition 5.3. *We have that* $\deg(K_C) = 2g-2$*. Hence if* $g \geq 2$ *the line bundle* $K_C^{\otimes v}$ *is very ample for all* $v \geq 3$*.*

Proof. $\deg K_C^{\otimes v} = v \cdot (2g-2) \geq 6g-6 \geq 2g+1$ (if $g \geq 2$). □

In this way we can choose a line bundle that provides an embedding without fixing points.

Let $\mathcal{C}$ be a curve of genus $g \geq 2$. The embedding provided by $K_C^{\otimes v}$ in $\mathbb{P}^n$ has degree $\deg(K_C^{\otimes v}) = v(2g-2)$ and the dimension of the projective space is $n = \dim H^0(\mathcal{C}, \Omega^v_{\mathcal{C}/k}) - 1 = (2v-1)(g-1) - 1$.

The degree and genus determine the Hilbert polynomial of a curve, that is:

$$p(x) = (2xv - 1)(g-1).$$

Hence, fixed a basis of $H^0(\mathcal{C}, K_C^{\otimes v})$, we can view the curves as points in the Hilbert scheme $\mathrm{Hilb}_n^{p(x)}$. Changing the basis of $K_C^{\otimes v}$ we obtain a different curve in $\mathbb{P}^n$ that is conjugated to the previous one under the natural action of GL_{n+1} on $\mathbb{P}^n$.

Consider now the subscheme $H_v \subset \mathrm{Hilb}_n^{p(x)}$ that parametrize smooth curves embedded in $\mathbb{P}^n$ via $K_{\mathcal{C}}^v$ (it is not obvious that this exists, we will prove that in the next section). Every curve of genus g is a k-point of H_v by construction. Moreover, two k-points of $\mathrm{Hilb}_n^{p(x)}$ represent the same curve if and only if they are conjugated by an element of GL_{n+1}.

The best candidate for our moduli space M_g is hence a quotient of H_v by GL_{n+1}. We ask if every point of H_v is stable under the action of GL_{n+1} because we need an orbit space.

The answer to this question concern the second part of this chapter. To verify the stability we will follow the proof made by Gieseker.

This approach cannot work in dimensions 0 and 1 because the canonical bundle has a non-positive degree, hence it is not ample.

5.2 The action of GL_{n+1} on Hilbert schemes

The functor $\mathcal{GL}_{n+1} : \mathrm{Sch}/k \to \mathrm{Set}$ act on $\mathcal{H}ilb$ by an affine base change, as follow:

$$\begin{aligned} \sigma(T) : \mathcal{GL}_{n+1}(T) \times \mathcal{H}ilb_n^{p(x)}(T) \to \mathcal{H}ilb_n^{p(x)}(T) \\ \{f \in \mathrm{Hom}(T, \mathrm{GL}_{n+1})\}, \left\{\mathcal{O}_{\mathbb{P}^n \times T} \xrightarrow{h} \mathcal{F}\right\} \mapsto \left\{\overline{f}^* \mathcal{O}_{\mathbb{P}^n \times T} \xrightarrow{\overline{f}^* h} \overline{f}^* \mathcal{F}\right\}, \end{aligned}$$

where $\overline{f}$ is defined as follows. f induce a map $\mathbb{A}_T^{n+1} \to \mathbb{A}_T^{n+1}$ that factorize through $\mathbb{P}_T^n$: in this way we have $\overline{f} : \mathbb{P}_T^n \to \mathbb{P}_T^n$. We omit to verify that it is well defined and that it is an action.

Given that the two functors are both representable, we have that it induces an action on $\mathrm{Hilb}_n^{p(x)}$.

In order to explicit it we extend the morphism to the functor

$$\mathrm{Hom}\left(\_, \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \dots, x_n]_m\right)\right),$$

we omit the easy verifications. It is clear that the action is (we describe it at level of k-points):

$$\begin{aligned} \mathrm{GL}_{n+1} \times \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \dots, x_n]_m\right) &\to \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \dots, x_n]_m\right) \\ (g, [q_1 \wedge \cdots \wedge q_{p(m)}]) &\mapsto [g \cdot q_1 \wedge \cdots \wedge g \cdot q_{p(m)}]. \end{aligned}$$

5.3 The functor $\mathcal{M}_g$ as quotient of $\mathcal{H}_v$

We would like to find a functor $\mathcal{H}_v$ that represent smooth curves not-degenerately embedded in $\mathbb{P}^n$ via the canonical bundle at some power v. Then define a transformation that forgot about the immersion in $\mathbb{P}^n$:

$$\begin{aligned} p : \mathcal{H}(S) &\to \mathcal{M}_g(S) \\ \{\text{a family } \Gamma \subset S \times \mathbb{P}^n\} &\mapsto \{\Gamma \to S\}. \end{aligned}$$

If such a functor exists, we can also define an action of $\mathcal{GL}_{n+1}$ on $\mathcal{H}_v$ such that the forgetful transformation results invariant. Moreover, we would like that $\mathcal{H}_v$ is representable i.e. of the form $\mathrm{Hom}(\_, H_v)$. We fix $p(x) = (2xv-1)(g-1)$ as before and $n = (2v-1)(g-1)-1$, we have:

Theorem 5.4. *Fix $v \geq 3$. There exists a locally closed immersion $H_v \subset Hilb_n^{p(x)}$ such that any map $f : S \to Hilb_n^{p(x)}$ factor through H_v if and only if the followings are satisfied:*

1. *the closed subscheme $i : \Gamma \subset S \times \mathbb{P}^n$ induced by the pullback of the universal family is that $\pi : \Gamma \to S$ is smooth, (obviously proper,) and whose geometric fibre are connected curves of genus g,*
2. *$i^*\mathcal{O}_{S\times\mathbb{P}^n}(1)$ is isomorphic to $\left(\Omega_{\Gamma/S}\right)^v \otimes \pi^*(L)$. Where L is an invertible sheaf of S,*
3. *for every k-point of S, we have that $\Gamma_s \subset \mathbb{P}^n$ is not degenerate.*

Hence we define $\mathcal{H}_v$ to be the functor determined by the three condition: it is representable by H_v.

We give a sketch of the theorem's proof, a reference is [GIT].

Proof. Basically we add the conditions one for time and we verify if there exists a subscheme of $\mathrm{Hilb}_n^{p(x)}$ such that the map factorize.
We start with the smoothness of $\Gamma \to S$: thanks to the proposition 1.1 and 2.1 of [SGA2] we have that there exists an open subset $U_1 \subset \mathrm{Hilb}_n^{p(x)}$ that satisfy the property.
Moreover, the locus of connected geomeric fibres is open, thanks to [Stacks, Tag 03GX].

This concludes the first request.
Consider $\Gamma_1 \subset U_1 \times \mathbb{P}^n$ induced by the pullback of the universal family. It is smooth, proper and with connected geometric fibres: thanks to theorem 4.17 we can construct the Picard scheme $\mathrm{Pic}_{\Gamma_1/U_1}$. The maps $\mathrm{Hom}(U_1, \mathrm{Pic}_{\Gamma_1/U_1})$ are in bijecition with the line bundles on Γ_1 up to a pullback of a line bundle on U_1. Consider the line bundle induced on Γ_1 by $\mathcal{O}_{\mathbb{P}^n}(1)$, it corresponds to a map $\lambda : U_1 \to \mathrm{Pic}_{\Gamma_1/U_1}$. Similarly $(\Omega_{\Gamma_1/U_1})^v$ corresponds to $\omega^v : U_1 \to \mathrm{Pic}_{\Gamma_1/U_1}$. We would find now a subscheme $U_2 \subset U_1$ such that the maps $U_2 \hookrightarrow U_1 \xrightarrow{\omega^v} \mathrm{Pic}_{\Gamma_1/U1}$ and $U_2 \hookrightarrow U_1 \xrightarrow{\lambda} \mathrm{Pic}_{\Gamma_1/U1}$ are equal, i.e. the line bundle induced by $\mathcal{O}_{\mathbb{P}^n}(1)$ and $(\Omega_{\Gamma_1 \times U_2/U_2})^v$ are isomorphic up to a pullback of a line bundle on U_1. The scheme we need is just the following fibred product (where Δ is the diagonal map):

$$\begin{array}{ccc} U_2 & \longrightarrow & U_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\lambda,\omega^v)} \\ \mathrm{Pic}_{\Gamma_1/U_1} & \xrightarrow{\Delta} & \mathrm{Pic}_{\Gamma_1/U_1} \times_{U_1} \mathrm{Pic}_{\Gamma_1/U_1} \end{array} ,$$

Given that $\mathrm{Pic}_{\Gamma_1/U_1}$ is separated (we refer to [FGAexp] for a proof of this), Δ is a closed immersion, hence U_2 is a closed subscheme of U_1. The subscheme U_2 is therefore what we request for the second condition and $\eta : \Gamma_2 = \Gamma_1 \times_{U_1} U_2 \to U_2$ is the pullback of the universal family.
It remains the third request. Let L be the line bundle induced on Γ_2 by $\mathcal{O}_{\mathbb{P}^n}(1)$, we have a natural map of $\mathcal{O}_{U_2}$ sheaves:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{U_2} \xrightarrow{h} \eta_* L.$$

These sheaves have dimension $n+1$, call $\mathcal{F}$ the cokernel of h.
The base change with a k-point give us a map of k vector spaces of dimension $n+1$:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \xrightarrow{h_s} H^0(\Gamma_s, \Omega^v_{\Gamma_s/k}),$$

where the base change of $\eta_* L$ is $H^0(\Gamma_s, \Omega^v_{\Gamma_s/k})$ because $H^1(\Gamma_s, \Omega^v_{\Gamma_s/k}) = 0$. The thesis follows considering the open set where $\mathcal{F}$ restrincts to the sheaf 0. □

The subfunctor $\mathcal{H}_v$ is invariant under the action of $\mathcal{GL}_{n+1}$ because the requirements in the previous proposition are invariant. Hence we can restricts the action to $\mathcal{H}_v$:

$$\sigma : \mathcal{GL}_{n+1} \times \mathcal{H}_v \to \mathcal{H}_v.$$

Given that the functor $\mathcal{H}_v$ is representable, we have an action of schemes $\mathrm{GL}_{n+1} \times H_v \to H_v$.

We define the forgetful transformation $p : \mathcal{H}_v \to \mathcal{M}_g$ as said before: obviously it is well defined and we have that the following commutes:

$$\begin{array}{ccc} \mathcal{GL}_{n+1} \times \mathcal{H}_v & \xrightarrow{\sigma} & \mathcal{H}_v \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p} \\ \mathcal{H}_v & \xrightarrow{p} & \mathcal{M}_g \end{array} ,$$

where p_2 is the projection on the second factor.

It turns out that the the quotient functor is not our moduli space $\mathcal{M}_g$.

Remark 5.5 (quotient functor). We define

$$p' : \mathcal{H}_v \to \mathcal{M}'_g,$$

the quotient of the functor $\mathcal{H}_v$ via the action of $\mathcal{GL}_{n+1}$.
Hence there exists a map I such that

$$\mathcal{H}_v \xrightarrow{p'} \mathcal{M}'_g \xrightarrow{I} \mathcal{M}_g$$

is the map p.

We avoid proving this remark, it follows by easy set theory. Despite I is not an isomorphism of functor, we have that $\mathcal{M}_g$ is a sort of sheafification of $\mathcal{M}'_g$:

Proposition 5.6. *The map $I(S) : \mathcal{M}'_g(S) \to \mathcal{M}_g(S)$ is injective for every $S \in Sch/k$. For every S and every $\alpha \in \mathcal{M}_g(S)$ there exist a open cover $\{U_i\}$ of S such that $\alpha_{|U_i} \in \mathcal{M}_g(U_i)$ is in the image of $I(U_i)$ for every i.*

Proof. Consider $\phi_1, \phi_2 \in \mathrm{Hom}(S, H_v)$ and suppose that the induced families $\Gamma_1 \subset S \times \mathbb{P}^n$ and $\Gamma_2 \subset S \times \mathbb{P}^n$ are isomorphic over S:

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\Psi} & \Gamma_2 \\ & {\scriptstyle\omega_2}\searrow \quad \swarrow{\scriptstyle\omega_1} & \\ & S & \end{array}$$

Where Ψ is an isomorphism. Consider the diagram

$$\begin{array}{ccccc} \Gamma_1 & \hookrightarrow & S \times \mathbb{P}^n & \xrightarrow{p_2} & \mathbb{P}^n \\ & {\scriptstyle\omega_1}\searrow & \downarrow{\scriptstyle p_1} & & \\ & & S & & \end{array},$$

and the analogous one for Γ_2. We have natural maps of sheaves over S:

$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S \xrightarrow{h_1} \omega_{1*}(\mathcal{O}_{\Gamma_1} \otimes \mathcal{O}_{\mathbb{P}^n}(1))$$
$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S \xrightarrow{h_2} \omega_{2*}(\mathcal{O}_{\Gamma_2} \otimes \mathcal{O}_{\mathbb{P}^n}(1)).$$

The second condition in the previous proposition assures that the sheaves at right-hand sides are locally free sheaves. Moreover, the third condition says that, under base change with a k-point s, $(h_i)_s$ are isomorphisms: this implies that h_i are isomorphisms.
Recall the notation $\mathbb{P}(\mathcal{F}) = \mathrm{Proj}\ (\mathrm{Sym}\,\mathcal{F})$. The map $\Psi^* : (\Omega_{\Gamma_2/S})^v \to (\Omega_{\Gamma_1/S})^v$ give us an isomorphism:

$$S \times \mathbb{P}^n = P\left(H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S\right) = P\left(w_{2*}((\Omega_{\Gamma_2/S})^v)\right)$$
$$\simeq P\left(w_{2*}((\Omega_{\Gamma_1/S})^v)\right) = P\left(H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S\right) = S \times \mathbb{P}^n.$$

The map has to be induced by the action of GL_{n+1}: it is defined by an element $\beta \in \mathrm{Hom}(S, \mathrm{GL}_{n+1})$. From this follows that $\beta \cdot \phi_1 = \phi_2$.
We give only a sketch for the second thesis. Consider $\omega : \Gamma \to S$ and fix an open cover $\{U_i\}_I$ of S such that $\omega_*((\Omega_{\Gamma/S})^v)_{|U_i}$ are free sheaves for all i.
This implies that we can focus on the situation where $\omega_*((\Omega_{\Gamma/S})^v)$ is free. We call for brevity $\mathcal{E}_v = \omega_*((\Omega_{\Gamma/S})^v)$. We need a closed immersion:

$$\begin{array}{ccc} \Gamma & \hookrightarrow & \mathbb{P}(\mathcal{E}_v) \\ & \searrow & \downarrow \\ & & S \end{array},$$

If we base change with a k-point $s \in S$ the thesis is a matter of classical algebraic geometry because $v \geq 3$. The point is that we can extend it in a neighbourhood of s. It remains to verify that it defines a closed immersion and that is a U_i valued point of H_v. For the details we refer to [GIT] proposition 5.2. □

We are now ready to state the last theorem of this section:

Theorem 5.7. *For every $v \geq 3$, a geometric quotient of H_v by GL_{n+1} is a coarse moduli space of $\mathcal{M}_g$.*

Proof. Thanks to Yoneda's lemma, there is a canonical bijection between natural transformations $\psi' : \mathcal{M}'_g \to \mathrm{Hom}(\_, N)$ where $N \in \mathrm{Sch}/k$ and GL_{n+1}-invariant maps $f : H_v \to N$ such that:

$$\begin{array}{ccc} \mathrm{Hom}(\_, H_v) & & \\ \downarrow{\scriptstyle p'} & \searrow^{f_*} & \\ \mathcal{M}'_g & \xrightarrow{\psi'} & \mathrm{Hom}(\_, N) \end{array}$$

commutes.
We would like that a canonical bijection holds also between GL_{n+1}-invariant map $f : H_v \to N$ and $\mathcal{M}_g \to \mathrm{Hom}(\_, N)$: this is a straightforward application of the previous proposition. Consider the diagram:

$$\begin{array}{ccc} \mathrm{Hom}(\_, H_v) & & \\ \downarrow{\scriptstyle p} & \searrow^{f_*} & \\ \mathcal{M}'_g & \xrightarrow{\psi'} & \mathrm{Hom}(\_, N) \\ \downarrow{\scriptstyle I} & \nearrow_{\psi} & \\ \mathcal{M}_g & & \end{array} .$$

Given Ψ there exists a unique f again for Yoneda's lemma, and obviously it is GL_{n+1} invariant. Moreover the converse holds: for every invariant map f there exists a unique natural transformation ψ such that the diagram below commutes. Uniqueness follows from the injectivity of I and to construct it is enough to apply the previous proposition with $S = H_v$ and glue the morphisms $U_i \to N$.
This implies that a categorical quotient $\pi : H_v \to Q$ satisfy the second request in definition 1.13 and we have $\eta : \mathcal{M}_g \to \mathrm{Hom}(\_, Q)$. It is now obvious that $\eta(\mathrm{Spec}\ k)$ is an isomorphism if and only if π is an orbit space at level of k-points: this is satisfied if π is a geometrical quotient. □

5.4 Smooth curves are stable points

To conclude the construction we will use theorem 5.7. It is enough to study the stability of closed points of H_v in order to obtain that the categorical quotient is geometric.

In the definition 3.33 we require that stabilizers of stable points have to be zero-dimensional. In our case GL_{n+1} act on projective space, so the diagonal matrixes act trivially. This is not a problem because we can simply consider the action of the subgroup SL_{n+1}.

Remark 5.8 (Action of SL_{n+1}). Consider the action on $\mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m\right)$ defined by the subgroup $i : \mathrm{SL}_{n+1} \hookrightarrow \mathrm{GL}_{n+1}$:

$$\mathrm{SL}_{n+1} \times \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m\right) \xrightarrow{\sigma \circ (i, \mathrm{Id})} \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m\right).$$

The categorical quotient over GL_{n+1} exists if and only if there exists the categorical quotient over the subgroup SL_{n+1}, in that case they coincide.
Hence the quotient by GL_{n+1} is geometrical if and only if that holds for for the action of SL_{n+1}.

Proof. The action of the diagonal matrices is trivial. From this and the hypothesis $k = \overline{k}$ follows the thesis. □

From now on we will consider the action of SL_{n+1}. We define:

Definition 5.9 (Hilbert stability)**.** Let $X \subset \mathbb{P}^n$ be a closed scheme, for every m it represents a point $[X]_m \in \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m\right)$ defined by res^m_X.
We say that X is Hilbert-stable if there exists M such that $[X]_m$ is a stable k-point under the action of SL_{n+1} for $m \geq M$.

We will use the Hilbert-Mumford criterion to verify the stability of points. The purpose is therefore verifying the stability via 1-Ps: this is a matter of weight of $\mathbb{G}_m$-representation. In particular, fixed a k-point x of $\mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m\right)$, we would like to compute $\mu(x, \lambda)$ for every 1-Ps λ.

Proposition 5.10 (1-Ps of SL_n)**.** *Let $\lambda : \mathbb{G}_m \to \mathrm{SL}_n$ be a 1-Ps. There exists a base change such that this map can be write as*

$$\begin{aligned} \mathbb{G}_m &\to \mathrm{SL}_n \\ t &\mapsto \begin{pmatrix} t^{k_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & t^{k_n} \end{pmatrix}, \end{aligned}$$

where k_i are integers such that $\sum_i^n k_i = 0$.

Proof. A 1-Ps of SL_n is, by composition, a 1-Ps of GL_n. Thanks to corollary 2.48 and asking that the determinant of matrices is 1 we conclude easily. □

Fix a 1-Ps of SL_{n+1}, we would like to find a suitable base of the cone space

$$\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m \to \mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m\right).$$

First of all we perform a change of basis in such a way that the action of $\mathbb{G}_m$ on $\mathrm{Spec}\, k[x_0, \ldots, x_n]$ is diagonal. We call w_i the weigh of x_i for $1 \geq i \geq n$. There is a natural weighted basis for $\bigwedge^{p(m)} k[x_0, \ldots, x_n]_m$.

$$B_m = \left\{ \prod x_i^{r_i} \mid \text{where} \sum r_i = m \right\}$$

is a basis for $k[x_0, \ldots, x_n]_m$. Define $Y_I := \prod x_i^{r_i}$ where $I = (r_0, \ldots, r_n)$ is a multi index.

$\mathbb{G}_m$ act diagonally on $k[x_0, \dots, x_n]_m$ with weights $w'_I = \sum w_i \cdot r_i$ respect to the basis B_m.

Fix an arbitrary total order on indexes I (e.g. a monomial one, but is not relevant), we define:

$$Z_{m,p(m)} = \left\{ Y_{I_1} \wedge \dots \wedge Y_{I_{p(m)}}) \mid Y_{I_k} \in B_m \text{ are distinct and indexes are ordered} \right\}.$$

This is obviously a basis of $\bigwedge^{p(x)} k[x_0, \dots, x_n]_m$. Again the action of $\mathbb{G}_m$ is diagonal, we call $w''_{I_1,\dots,I_{p(m)}} = \sum_i w'_{I_i}$ the weights of $Y_{I_1} \wedge \dots \wedge Y_{I_{p(m)}})$.

Starting from a weighted basis of $\mathbb{A}^{n+1}$ we have constructed a weighted basis for $\bigwedge^{p(x)} k[x_0, \dots, x_n]_m$.

Fix a closed subscheme $X \subset \mathbb{P}^n$, this provide a k-point $[X]_m = [Q]$ of

$$\mathbb{P}\left(\bigwedge^{p(m)} k[x_0, \dots, x_n]_m \right),$$

that corresponds to a map $\mathrm{res}^m_X : k[x_0, \dots, x_n]_m \to H^0(X, \mathcal{O}_X(m))$. Consider a lift Q and write in in the basis $Z_{m,p(m)}$. From chapter 3 we know that the stability is verified looking at the weights of the elements of $Z_{m,p(m)}$ that show up in Q.

The Plücker embedding says that an element $z \in Z_{m,p(m)}$ show up in Q if and only if the images of the monomials contained in z through res^m_X are a basis of $H^0(X, \mathcal{O}_X(m))$. If such a situation occurs, we say that the set of monomials is a B-base and the weight of z is the B-weight. This gives a new description for the stability of a point:

Lemma 5.11. *$[X]_m$ is stable (semistable) if and only if for every choice of a weighted basis of $\mathbb{A}^{n+1}$ (i.e. a choice of a 1-Ps λ of SL_n) there is a B-base with negative (non positive) B-weight.*

Proof. The proof is a straightforward application of Hilbert-Mumford criterion. □

We can reformulate this proposition in a more suitable way, introducing filtrations.

Definition 5.12 (Weighted filtration)**.** Let V be a vector space, a weighted filtration F is a collection of $\{(V_i, w_i)\}_{0 \geq i \geq \dim V - 1 = n}$ such that

$$\begin{aligned} V = & V_0 \supsetneq V_1 \supsetneq \dots \supsetneq V_n \supsetneq 0 \\ & w_0 \geq w_1 \geq \dots \geq w_n, \end{aligned}$$

where $w_i \in \mathbb{Q}$.

A weighted basis define a filtration:

Definition 5.13 (Compatible filtration)**.** Let $e_0, \dots, e_n$ be a basis of V and fix a weight $w_i \in \mathbb{Q}$ for every element of the basis, up to a permutation we suppose that w_i are ordered. Consider $w \in \mathbb{Q}$, we define

$$U_w = \mathrm{span}\left\{ e_i \mid w_i \leq w \right\}.$$

Observe that $U_w \subset U_v$ if $v > w$. Obviously for $w = w_i$ there is a "jump" in dimension (i.e. $\dim U_{w_i} - \dim U_w > 0$ where $w < w_i$). Suppose that w_i are all distinguished: the jumps are of height 1, this define the filtration

$$\begin{aligned} V = U_{w_0} \supsetneq U_{w_1} \supsetneq \dots \supsetneq U_{w_n} \supsetneq 0 \\ w_0 \geq \ \ w_1 \geq \dots \geq w_n. \end{aligned}$$

We say that the weight of an element $v \in V$ (we write $w(v)$) is the minimum w_i such that $v \in V_i$.

Suppose now that the w_i are not all distinct. There are jumps of dimension bigger than 1, we can define a filtration making an arbitrary choice on ordering elements of the basis with the same weight. We could define the weighted filtration:

$$\begin{array}{ccccccc} V = U_{w_0} = \text{span } \{e_i \mid i \geq 0\} & \supsetneq & U_{w_1} = \text{span } \{e_i \mid i \geq 1\} & \supsetneq & \ldots \supsetneq U_{w_n} \supsetneq 0 \\ w_0 & \geq & w_1 & \geq & \ldots \geq w_n. \end{array}$$

The arbitrary choices we made are not relevant in our treatment. Again, we define the weight of $v \in V$ as done above.

We can moreover define:

Definition 5.14 (Weight of a filtration)**.** Given a weighted filtration F of V, we define the total weight of F as the sum of weights in the filtration (counted with multiplicity).

The next remark gives the idea of why having a weighted filtration on a space is not as strong as a weighted base.

Remark 5.15. Suppose that we have a weighted filtration (even with distinct weights) induced by a weighted basis. From this, we cannot recover the initial weighted basis we started with because there exist multiple weighted bases that are compatible.

The benefit of this new formalization is that filtration go down to the quotient in a canonical way.

Definition 5.16 (Weighted filtration on the quotient)**.** let $\phi : V \to W$ be a surjective map o vector space and suppose to have a weighted filtration V_i on V. Consider

$$W = \phi(V_0) \supseteq \phi(V_1) \supseteq \cdots \supseteq \phi(V_n) \supseteq 0,$$

Up to delete repeated spaces, this define a weighted filtration were the weight of an element $w \in W$ as $\min_{v \in \phi^{-1}(w)} w(v)$.

Coming back to our situation, we have that a weighted base of $\mathbb{A}^{n+1}$ (seen as vector space) define a weighted filtration F on it. This induces a weighted filtraiton on $k[x_0, \ldots, x_n]_m$ and hence on $H^0(X, \mathcal{O}_X(m))$ for every projective variety X of Hilbert polinomial $p(x)$. We define $w_F(m)$ (where the dipendence on m is explicit) as the total weight of the induced filtration on $H^0(X, \mathcal{O}_X(m))$.

We reformulate the stability with the new formalism.

Proposition 5.17 (Numerical criterion)**.** *let $X \in \mathbb{P}^n$ be a closed subscheme with Hilbert polynomial $p(x)$.*
$[X]_m$ is stable (semistable) respect to the action of SL_{n+1} if and only if for every filtration F of $\mathbb{A}^{n+1}$ we have:

$$w_F(m) < m \cdot \alpha_F \cdot p(m) \qquad \textit{respectively} \leq \textit{for semistability},$$

where α_F is the average of weights of F.

Proof. This is simply a matter of reformulation. $[X]_m$ is stable if and only if for every weighted basis of $\mathbb{A}^{n+1}$ with total weight 0 there exists a B-basis with negative weight. This is true if and only if for every weighted filtration of $\mathbb{A}^{n+1}$ with total weight 0 and integer weights, we have that $w_F(m) < 0$.
Given a filtration with weights w_i, we can rescale the weights $w_i' = \beta(w_i - \alpha)$ where α and $\beta > 0$ are chosen such that $w_i' \in \mathbb{Z}$ and the average of w_i' is 0.
From an easy computation we have that $w_F'(m) = \beta w_f(m) - \beta\alpha m p(m)$: this implies the thesis for stable point. The thesis for semistability follows in the same way. □

We face a choice: we could continue with the original proof by Gieseker or we could use a slightly different method that avoids some calculations. We will follow the latter, which can be found on [HM98]. Anyway, a reference for the original proof is [Gie82].

The construction uses the following reasonable theorem which, in some sense, is a generalization of the basic idea that the dimension of homogeneous components of a graded finite module on a finitely generated graded k-algebra is polynomial (for big enough grades).

Proposition 5.18. *Fixed a weighted filtration F on $\mathbb{A}^{n+1}$ and a closed subscheme $X \subset \mathbb{P}^n$, we have that $w_F(m)$ is a polynomial function of degree* $\dim X$ *for m enough big.*

Moreover there exists a uniform bound on coefficients of $w_F(m)$.

Proposition 5.19. *Fix a Hilbert polynomial $p(x)$, there exist M, C such that for every weight filtration F there exists a constant e_F that satisfy:*

$$\left| w_F(m) - e_F \frac{m^{r+1}}{(r+1)!} \right| < Cm^r,$$

for every $m \geq M$, projective scheme X with Hilbert polynomial $p(x)$ and dimension r.

The proofs of the previous results can be found on [Mum77], theorem 2.9.

From this, we have:

Proposition 5.20. *Let X be a projective scheme of dimension r and degree d and fix a weighted filtration F.*

$$e_F < \alpha_F \cdot d \cdot (r+1) \Rightarrow X \textit{ is Hilbert stable with respect to } F.$$

The converse also holds:

$$e_F > \alpha_F \cdot d \cdot (r+1) \Rightarrow X \textit{ is not Hilbert semistable with respect to } F.$$

Proof. Thanks to classical theory we have that $p(x) = \frac{dm^r}{r!}$ + terms of lower degree, this inplies that:

$$\begin{aligned} w_F(m) - m\alpha_F p(m) &= e_F \frac{m^{r+1}}{(r+1)!} - m\alpha_F \frac{dm^r}{r!} + \text{terms of lower degree} \\ &= \frac{(e_F - \alpha_F(r+1)d)\, m^{r+1}}{(r+1)!}. \end{aligned}$$

The thesis follows now from proposition 5.17. □

We have an uniform version of the previous result:

Theorem 5.21 (Asymptotic numerical criterion)**.** *Fix a locally closed $H \subset Hilb_n^{p(x)}$, where $p(x)$ has degree r. Suppose that there exists $\delta > 0$ such that*

$$e_F < \alpha_F \cdot d \cdot (r+1) - \delta$$

for all weighted filtraitons F associated to a k-point of H.
There is an M such that $[X]_m$ is stable for all $m \geq M$ and all X k-point of H.

Proof. Thanks to proposition 5.19 we have the following.
Then, there exist M' and $C' > 0$ such that for every $m \geq M'$, $X \in \mathrm{Hilb}_n^{p(x)}$:

$$\begin{aligned} w_F(m) - m\alpha_F p(m) = &\frac{(e_F - \alpha_F(r+1)d)\, m^{r+1}}{(r+1)!} + \\ &+ \left\{\text{term in } m^r, \text{ with coefficient } \leq C'\right\} + \{\text{terms of lower degree}\} \end{aligned}$$

From this follows the thesis. □

5.4.1 The idea of Gieseker

The strategy is now to find an auxiliary filtration of the space $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(m))$ where $\mathcal{C}$ is a curve and m a big integer. This method works also with some curves that are not smooth, but we are not interested in that case.

Fix now a smooth curve $\mathcal{C}$ that is embedded by a linear bundle L of degree d:

$$\mathcal{C} \hookrightarrow \mathbb{P}(H^0(\mathcal{C}, L)) = \mathbb{P}^N.$$

Where the embedding is not degenerate, hence $L = i^*\mathcal{O}_{\mathbb{P}^N}(1)$, and with Hilbert polynomial p. For the moment we forgot that the embedding is given by the v-canonical, it is not useful for the moment.

Fix a weighted filtration F on $\mathbb{A}^{N+1}$ with weights w_i:

$$\begin{aligned} V = & V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_N = 0 \\ & w_0 \geq w_1 \geq \cdots \geq w_N, \end{aligned}$$

where $V = H^0(\mathcal{C}, L)$. Consider now an integer $h \leq N$ and an ordered subsequence of $(1, \ldots, N)$: $0 = j_0 < \cdots < j_h = N$. Define the space:

$$W^n_{k,l} = \mathrm{Sym}^n \left(V \cdot \mathrm{Sym}^{p-l}\left(V_{j_k}\right) \cdot \mathrm{Sym}^l\left(V_{j_{k+1}}\right)\right) \subset \mathrm{Sym}^{n(p+1)}(V) = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n(p+1)))$$

where the $\cdot$ is the symmetric product. The spaces define a filtration in k, l where the index k runs from 0 to $h-1$ and, for each k, $l \in 0, \ldots, p$. Similarly to what done before, we can assigns weights to this filtration: $W^n_{k,l}$ has weight $w^n_{k,l} = n(w_0 + (p-l)w_{j_k} + l w_{j_{k+1}})$ (we omit the easy computation).

Via the map $\mathrm{res}^{n(p+1)}_{\mathcal{C}}$ we obtain a weighted filtration $U^n_{k,l}$ of $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n(p+1)))$, where an element of $U^n_{k,l}$ has weight at most $w^n_{k,l}$. This filtration is coarser than the one induced by F, hence it gives us an upper bound on the total weight $w_F(n(p+1))$. Hence we have the inequality:

$$\begin{aligned} w_F(n(p+1)) \leq & \left(\sum_{k=0}^{h-1} \sum_{l=0}^{p-1} \left(\dim\left(U^n_{k,l}\right) - \dim\left(U^n_{k,l+1}\right)\right) w_{k,l} \right) + \dim\left(U^n_{h,0}\right) w_{h,0} = \\ & = \dim\left(U^n_{0,0}\right) w_{0,0} + \left(\sum_{k=0}^{h-1} \sum_{l=1}^{p-1} \dim\left(U^n_{k,l}\right)\left(w_{k,l} - w_{k,l-1}\right)\right). \qquad (5.1) \end{aligned}$$

In order to obtain a bound we need an uniform estimation of $\dim\left(U^n_{k,l}\right)$.

Proposition 5.22. *Fix an Hilbert polynomial $p(x)$ and integers p, N. Define d_i as the degrees of the subsheaves of L generated by global sections in V_i. Suppose moreover that* $\deg L \geq 2g$.
There is an M depending on these three choices but not on the weighted filtration F such that

$$\dim\left(U^n_{k,l}\right) = n\left(d + (p-l)d_{j_k} + l d_{j_{k+1}}\right) - g + 1$$

holds for every $n \geq M$ and for every k and l.
Moreover, we can choose M that works for every smooth curve $\mathcal{C} \subset \mathbb{P}^N$ with Hilbert polynomial $p(x)$ and embedded by a line bundle of degree grather than $2g$.

Proof. Let L_i be the line bundle generated by the sections in $V_i \subset H^0(\mathcal{C}, L) = H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(1))$. $U^1_{k,l}$ generate

$$M_{k,l} = L \otimes \left(L_{j_k}\right)^{p-l} \otimes \left(L_{j_{k+1}}\right)^l.$$

Moreover, $U^n_{k,l}$ is a sublinear series of $H^0(\mathcal{C}, (M_{k,l})^n)$.
We observe that $U^1_{k,l}$ is a very ample linear series on $\mathcal{C}$ (because has V_0 as sublinear series), hence it defines an immersion:

$$\mathcal{C} \hookrightarrow \mathbb{P}(W^1_{k,l}).$$

Therefore we have the map:

$$\phi^n_{k,l} : H^0(\mathbb{P}(W^1_{k,l}), \mathcal{O}(n)) \to H^0(\mathcal{C}, (M_{k,l})^n).$$

This is surjective for big enough n and hence $\dim(U^n_{k,l}) = \dim H^0(\mathcal{C}, M^n_{k,l})$. We have $\deg(M^n_{k,l}) \geq \deg(L) \geq 2g$, hence $H^1(\mathcal{C}, M^n_{k,l}) = 0$. Thanks to Riemann-Roch we have:

$$\dim H^0(\mathcal{C}, M^n_{k,l}) = 1 - g + \deg(H^0(\mathcal{C}, M^n_{k,l})) = n\left(d + (p-l)d_{j_k} + l d_{j_{k+1}}\right) - g + 1.$$

Fix now k and l, thanks to theorem 4.6 we have that there exists $M_{k,l}$ such that for every $n \geq M_{k,l}$ and every curve $\mathcal{C}$ that satisfy the hypothesis, $\phi^n_{k,l}$ is surjective for $n \geq M_{k,l}$. The numbers k and l can assume only a finite number of values, this implies the thesis. □

We can now put together the last results to obtain upper bound to e_F.

Lemma 5.23. *Fix again $p(x)$, p and n, we have that for every weighted filtration F, every smooth curve $\mathcal{C}$ and every choice of the sequence $j_0, \ldots, j_h$ holds:*

$$e_F \leq \sum_{k=0}^{h-1} \left(e_{j_k} + e_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right),$$

where $e_j = d - d_j$.

Proof. This is simply a manipulation. We know $\dim\left(U^n_{k,l}\right)$ for big enough n, hence we have an inequality for $w_F(n(p+1))$ given by 5.1. We know that $W_F(m)$ is polynomial of degree $2 = 1 + \dim \mathcal{C}$ in m and the in the inequality we have that n and p appear with power at most 2. This make possible to bound e_F with the coefficient of n^2p^2 (because the inequalities holds for big n and p). This bound is exactely the thesis. For a complete calculation we refer to [HM98], chapter 4. □

Thanks to this calculation we can reformulate proposition 5.20 in a way that does not require fixing a filtration:

Proposition 5.24. *Let $\mathcal{C} \subset \mathbb{P}^N$ be a smooth curve with Hilbert polinomial $p(x)$ (that is uniquely determined by the degree d and the genus g). Suppose that ϵ_i are upper bound for $e_i = d - d_i$, i.e. the codegree of every subspace of dimension i on every filtration. We define*

$$\epsilon_{\mathcal{C}} = \max_{\substack{w_0 \geq \cdots \geq w_N = 0 \\ \sum_{i=0}^N w_i = 1}} \left(\min_{0 < j_0 < \cdots < j_h = N} \left(\sum_{k=0}^{h-1} \left(\epsilon_{j_k} + \epsilon_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right) \right) \right).$$

$\mathcal{C}$ is Hilbert stable if

$$\epsilon_{\mathcal{C}} < \frac{2d}{N+1}.$$

We have also a uniform version, thanks to theorem 5.21:

Proposition 5.25 (Gieseker's criterion). *Fix $p(x)$ a Hilbert polynomial and a locally closed $H \subset Hilb_N^{p(x)}$ whose k-points are smooth curves (Notice that in this case $p(x)$ is determined by degree d and genus g). Suppose that there exists $\delta > 0$ such that*

$$\epsilon_{\mathcal{C}} < \frac{2d}{N+1} - \delta$$

to every curve that is a k-point of H.
There is an M such that $[\mathcal{C}]_m$ is stable for all $m \geq M$ and all $\mathcal{C}$ k-point of H.

We do not prove these two because are suitable reformulations of what we already proved. We only notice that we write min on subsequences j_k in the formula because we can choose them freely.

We came now to the final piece of our puzzle.

Theorem 5.26 (Uniform stability of smooth curves). *Consider the smooth curves of genus $g \geq 2$ in $\mathbb{P}^N$ that are embedded by a very ample line bundle L of degree $d \geq 2g$.*
These are Hilbert stable and there exists M such that $[\mathcal{C}]_m$ with $m \geq M$ is stable for all such curves.

In order to prove the theorem we need the following:

Theorem 5.27 (Clifford). *Let L be a line bundle on a smooth projective curve $\mathcal{C}$ of degree d such that $H^1(\mathcal{C}, L) \neq 0$. We have that*

$$\dim\left(H^0(\mathcal{C}, L)\right) \leq \frac{d}{2} + 1.$$

For the proof of this theorem, we refer to [Arb+85]. We are now ready for the final proof.

Proof. We split the proof in four steps.
Step 1: we prove that the subsequence j_k that realize the bound of $\epsilon_{\mathcal{C}}$ is $0, 1, 2, \ldots, N$.
Fix the weights w_i. Consider the plot of the points (ϵ_i, w_i) on the Cartesian plane. Given a subsequence $\{j_k\}$, the number we want to minimize is

$$\sum_{k=0}^{h-1} \left(\epsilon_{j_k} + \epsilon_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right).$$

It represent the double of the area between axes x, y and the piecewise linear curve defined by the points $(\epsilon_{j_k}, w_{j_k})$. Therefore, the area is minimized when the subsequence represent the lower convex envelope (we call it E).
Fix now the weight w_i and the subsequence j_k that realize the bound for $\epsilon_{\mathcal{C}}$. Obviously it does not exists an index i such that $(\epsilon_i, w_i) \in E^\circ$ (in the interior of E) because otherwise E is not the lower convex envelope. Suppose now that there exists an index i such that (ϵ_i, w_i) is not in E. We could decrease w_i until the point is on the border of E, in this way the sum $\sum_{i=0}^{N} w_i < 1$ and we can rescale the weights in such a way the bound itself is rescaled, hence bigger: that is not possible. Therefore, w_i is on the border of E. This implies that $(\epsilon_i, w_i) \in \partial E$ for every $0 \leq i \leq N$. Hence we can suppose that the subsequence j_i is the whole sequence $0, \ldots, N$.
Moreover, the piecewise linear curve that join (ϵ_i, w_i) is convex.
Step 2: we prove that we can take

$$\epsilon_i = \left(\frac{d}{N} - \delta\right) i.$$

Consider the points $(\deg(U), \dim(U))$ where $U \subset H^0(\mathcal{C}, L)$ is a linear system, we write L_U the line bundle generated by U, we have two cases:

- $H^1(\mathcal{C}, L_U) = 0$, this implies that $\dim(U) = H^0(\mathcal{C}, L_U) - 1 = \deg(U) - g$ by Riemann-Roch theorem.
- $H^1(\mathcal{C}, L_U) \neq 0$, this implies that $2\dim(U) \leq \deg(U)$.

We plot these information on the $\deg(U), \dim(U)$ graph. In particular the subspaces of the filtration on $H^0(\mathcal{C}, L)$ corresponds to the points $(d - e_i, N - i)$.
For the complete linear system we have that $H^1(\mathcal{C}, L) = 0$ (this follows from Serre duality and $d \geq 2g$). This implies that the slope of the segment that link a point $(d - e_i, N - i)$ (for $i \neq 0, N$) to (d, N) is strictly bigger that the slope of the segment that link the origin to (d, N). Furthermore consider the case $i = N$, the line bundle generated by the unique element of V_N cannot be of degree 0, otherwise we should have that L is the trivial bundle, hence the point $(d_N, 0) \neq (0, 0)$ and the strict inequality holds also for $i = N$. This implies

$$\frac{e_i}{i} < \frac{d}{N} \qquad i = 1, \dots, N.$$

Given that there are only finitely many choices for the point $(d - e_i, N - i)$, we have that there exists δ (that depends only on N and d) such that we can take

$$\epsilon_i = \left(\frac{d}{N} - \delta\right) i.$$

Notice that for $i = 0$ we have that $\epsilon_i = 0$ is coherent with the previous equation.
Step 3: Let ϵ_i be as before and fix w_i that realize the bound. The function defined by the piecewise curve that links $P_i = (\epsilon_i, w_i)$ is convex, consider the polygon defined by the convex hull of these points. The midpoint of the segment P_0P_N has the same abscissa than the barycenter of the polygon (because $\epsilon_i - \epsilon_{i-1}$ is constant). Moreover the barycenter belongs to the polygon, hence we have an inequalities on the ordinate:

$$\frac{\sum_{i=0}^{N} w_i}{N+1} \leq \frac{w_0 + w_N}{2} = \frac{w_o}{2}.$$

Step 4: we compute the bound on $\epsilon_{\mathcal{C}}$. Thanks to first step j_i is the sequence $0, \dots, N$. Suppose that w_i maximize the bound, we have:

$$\begin{aligned}
\epsilon_{\mathcal{C}} &\leq \sum_{i=0}^{N-1} (\epsilon_i + \epsilon_{i+1})(w_i - w_{i+1}) \\
&\leq \left(\frac{d}{N} - \delta\right) \sum_{i=0}^{N-1} (i + (i+1))(w_i - w_{i+1}) = \\
&= \left(\frac{d}{N} - \delta\right)\left(w_0 + \sum_{i=1}^{N-1} 2w_i\right) = \left(\frac{d}{N} - \delta\right)(2 - w_0).
\end{aligned}$$

Using the bound in step 3:

$$\epsilon_{\mathcal{C}} \leq \left(\frac{d}{N} - \delta\right)\left(2 - \frac{2}{N+1}\right) = \frac{2d}{N+1} - \frac{2\delta N}{N+1},$$

this is the thesis. ☐

Hence we have the final theorem:

Theorem 5.28. *Given $v \geq 3$ and $g \geq 2$, we have that the k-points of H_v are stable. Hence the GIT-quotient $H_v//SL_{N+1}$ is a geometrical quotient and therefore it is the coarse moduli space for the problem $\mathcal{M}_g$.*

Proof. The theorem follows applying the previous theorem and theorem 5.7. ☐

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