EXERCISES FROM ”SET THEORY” (CHARLES PINTER) BOOK

Exercises 2.4

Exercise 8 (check [Pin71, ex. 8, page 67])**.** Suppose that $f\colon A \to B$ is a function. Prove that

$$\bar{f}(C \cap D) = \bar{f}(C) \cap \bar{f}(D)$$

for every pair of subclasses $C \subseteq A$ and $D \subseteq A$ if and only if f is injective.

Solution. The inclusion $\subseteq$ holds even f is not injective. In fact, given

$$y \in \bar{f}(C \cap D)$$

there exists $x \in C \cap D$ such that

$$f(x) = y.$$

Then

$$x \in C \cap D \Rightarrow x \in C \Rightarrow f(x) = y \in \bar{f}(C)$$

and

$$x \in C \cap D \Rightarrow x \in C \Rightarrow f(x) = y \in \bar{f}(D).$$

We look at the include $\supset$. Suppose that f is injective.

$$y \in \bar{f}(C) \cap \bar{f}(D)$$

Then

$$y \in \bar{f}(C) \Rightarrow \exists x_1 \in C\, (f(x_1) = y)$$

and

$$y \in \bar{f}(D) \Rightarrow \exists x_2 \in C\, (f(x_2) = y).$$

Since f is injective,

$$x_1 = x_2 =: x$$

Since $x_1 \in C$, also $x \in C$. Since $x_2 \in D$, then $x \in D$. Then

$$x \in C \cap D.$$

Since $f(x) = y$, we have

$$y \in \bar{f}(C \cap D).$$

Now, suppose that the equality

$$\bar{f}(C \cap D) = \bar{f}(C) \cap \bar{f}(D) \tag{1}$$

holds for every classes C, D. We prove that f is injective. Let x_1, x_2 be such that

$$f(x_1) = f(x_2) = y.$$

We define

$$C := \{x_1\}, \quad D := \{x_2\}.$$

By (1),

$$\bar{f}(C \cap D) = \bar{f}(C) \cap \bar{f}(D) = \{y\}.$$

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Then, there exists $x \in C \cap D$ such that
$$f(x) = y.$$
Then $C \cap D$ is non-empty. But when two singletons have non-empty intersection, it follows that $C = D$. □

EXERCISES 3.2

Exercise 1 (check [Pin71, ex. 1, page 74]). State whether the following relations are equivalence relations or order relations
$$G_1 = \{(x, y) \mid x \text{ and } y \text{ are relatively prime}\}$$
$$G_2 = \{(x, y) \mid x = y \text{ and } x = -y\}$$

Solution. G_1. It is not an equivalence relation. In particular, it is not reflexive: if $x = y$, then x and y have a common divisor: x. It is not transitive either:
$$((3, 5) \in G_1) \wedge ((5, 9) \in G_1) \text{ but } (3, 9) \notin G_1.$$
It is not an order relation, because is not reflexive and it is not antisymmetric:
$$(3, 5) \in G_1 \wedge (5, 3) \in G_1 \text{ does not imply } 3 = 5.$$
G_2. It is an equivalence relation.

(Reflexive) $\forall x (x, x) \in G_2.$

It is symmetric. Suppose that $(x, y) \in G_2$. Then
$$x = y \Rightarrow (y, x) \in G_2$$
and
$$x = -y \Rightarrow y = -x \Rightarrow (y, x) \in G_2.$$
It is transitive.
$$(x, y) \in G_2, (y, z) \in G_2 \Rightarrow x = \pm y \wedge y = \pm z.$$
Then
$$x = \pm z \Rightarrow (x, z) \in G_2.$$
□

Exercise 3 (check [Pin71, ex. 3, page 74]). Show that if G is an equivalence relation in A, then
$$G \circ G = G.$$

Solution. Since G is an equivalence relation, there holds
$$G \circ G \subseteq G.$$
We prove that
$$G \subseteq G \circ G.$$
We consider $(x, y) \in G$. Since G is reflexive,
$$(x, x) \in G.$$
Then
$$(x, x) \in G \wedge (x, y) \in G.$$
Then
$$(x, y) \in G \circ G.$$
□

References

Pin71. Charles C. Pinter. *Set theory*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1971.