EXERCISES FROM THE TEXT BOOK "SET THEORY", CHARLES PINTER

EXERCISES 2.4

Exercise 1 (check [Pin71, ex. 1, page 66])**.** Suppose that $f\colon A \to B$ is a function, $C \subseteq A$ and $D \subseteq B$.

a) Prove that $C \subseteq \bar{\bar{f}}[\bar{f}(C)]$
b) Prove that $\bar{f}[\bar{\bar{f}}(D)] \subseteq D$

Solution. a). Given $x \in C$, $f(x) \in \bar{f}(C)$. Then $x \in \bar{\bar{f}}[\bar{f}(C)]$

b). If $y \in \bar{f}[\bar{\bar{f}}(D)]$ there exists $x \in \bar{\bar{f}}(D)$ such that

$$f(x) = y$$

Since $x \in \bar{\bar{f}}(D)$, $f(x) \in D$. Then $y \in D$. □

Exercise 2 (check [Pin71, ex. 2, page 66])**.** Suppose that $f\colon A \to B$ is a function, $C \subseteq A$ and $D \subseteq B$. Then

a) If f is injective, prove that $C = \bar{\bar{f}}[\bar{f}(C)]$
b) If f is surjective, prove that $D = \bar{f}[\bar{\bar{f}}(D)]$

Solution. a). Given $x \in \bar{\bar{f}}[\bar{f}(C)]$, then

$$f(x) \in \bar{f}(C).$$

Then there exists $c \in C$ such that

$$f(c) = f(x).$$

Since f is injective, $x = c$. Then $x \in C$, which implies

$$\bar{\bar{f}}[\bar{f}(C)] \subseteq C.$$

From a) of the previous exercise, we obtain

$$\bar{\bar{f}}[\bar{f}(C)] = C.$$

b). Given $y \in D$, there exists x such that

$$f(x) = y.$$

Then,

$$x \in \bar{\bar{f}}(D).$$

Then

$$f(x) \in \bar{f}[\bar{\bar{f}}(D)] \Rightarrow y \in \bar{f}[\bar{\bar{f}}(D)].$$

Then

$$D \subseteq \bar{f}[\bar{\bar{f}}(D)].$$

From part b) of the previous exercise, we obtain

$$D = \bar{f}[\bar{\bar{f}}(D)]$$

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$\square$

Exercise 3 (check [Pin71, ex. 3, page 66])**.** Let $f\colon A \to B$ be a function. Prove the following.

a) Suppose $C \subseteq A$ and $D \subseteq A$; if f is injective $\bar{f}(C) = \bar{f}(D) \Rightarrow C = D$
b) Suppose $C \subseteq B$ and $D \subseteq B$; if f is injective $\bar{\bar{f}}(C) = \bar{\bar{f}}(D) \Rightarrow C = D$

Solution. a). We apply the results of the previous exercise. We have

$$\bar{f}(C) = \bar{f}(D) \Rightarrow \bar{\bar{f}}[\bar{f}(C)] = \bar{\bar{f}}[\bar{f}(D)].$$

Since f is injective,

$$\bar{\bar{f}}[\bar{f}(C)] = C, \quad D = \bar{\bar{f}}[\bar{f}(D)].$$

Then

$$C = D.$$

b). We have

$$\bar{\bar{f}}(C) = \bar{\bar{f}}(D) \Rightarrow \bar{f}[\bar{\bar{f}}(C)] = \bar{f}[\bar{\bar{f}}(D)]$$

Since f is surjective, from the previous exercise, we have

$$\bar{f}[\bar{\bar{f}}(C)] = C, \quad \bar{f}[\bar{\bar{f}}(D)] = D.$$

Then

$$C = D.$$

$\square$

Exercise 5 (check [Pin71, ex. 5, page 66])**.** Suppose that $f\colon A \to B$ is a function; let $C \subseteq A$.

a) Prove that $\bar{f}\{\bar{\bar{f}}[\bar{f}(C)]\} = \bar{f}$.

Solution. a). From a) of the first exercise, there holds

$$C \subseteq \bar{\bar{f}}[\bar{f}(C)]$$

From b) of the first exercise applied to $D = \bar{f}(C)$, it follows

$$\bar{f}(C) \subseteq \bar{f}\{\bar{\bar{f}}[\bar{f}(C)]\}$$

Then

$$\bar{f}(C) = \bar{f}\{\bar{\bar{f}}[\bar{f}(C)]\}.$$

$\square$

References

Pin71. Charles C. Pinter. *Set theory*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1971.