1. Scalar product in Euclidean spaces

Given two vectors $v, w \in E$, we define the real number

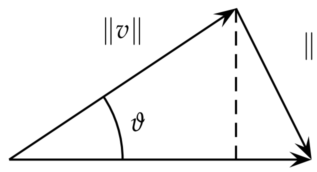
$$v \cdot w := \sum_{i=1}^{n} v_i w_i.$$

It is called *scalar product* or *dot product*. The following equalities hold for every $v, w, z \in E$ and $\lambda \in \mathbb{R}$

$$\begin{aligned} v \cdot v \geq 0 \text{ and } v \cdot v = 0 \Leftrightarrow v = 0 \\ v \cdot w = w \cdot v \\ v \cdot (w + z) = v \cdot w + v \cdot z \\ v \cdot (\lambda w) = \lambda (v \cdot w). \end{aligned}$$

Definition 1. Given $v \in E$ we define the *norm* of v as $\|v\| := \sqrt{v \cdot v}$.

The scalar product $v \cdot w$ has a natural interpretation in terms of the cosinus of the angle between v and w



In the above picture we wrote the length of each side of the triangle. By the cosinus theorem, there holds

$$\|v - w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\| \cos \vartheta$$

whence

$$\begin{aligned} \|v\|^2 + \|w\|^2 - 2v \cdot w &= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\| \cos \vartheta \\ &\Rightarrow -2v \cdot w = -2\|v\|\|w\| \cos \vartheta \\ &\Rightarrow v \cdot w = \|v\|\|w\| \cos \vartheta. \end{aligned}$$

If $\|v\|\|w\| > 0$, then

$$\cos \vartheta = \frac{v \cdot w}{\|v\|\|w\|}.$$

Proposition 1 (The Cauchy-Schwarz inequality)**.** *Given $v, w \in \mathbb{R}^n$ there holds*

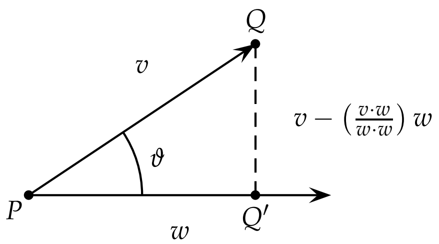
$$|v \cdot w| \leq \|v\|\|w\|.$$

If the equality holds and $w \neq 0$, then

$$v = \lambda w$$

for some $\lambda \in \mathbb{R}$.

Before giving the proof of this proposition, we illustrate a geometric interpretation of the inequality. In the picture



we see that the length of the norm of the vector $\overrightarrow{PQ'}$ is smaller than the norm of $\overrightarrow{PQ}$. That is

$$\|\overrightarrow{PQ'}\| \leq \|v\|. \tag{1}$$

The norm of $\overrightarrow{PQ'}$ is given by

$$\|\overrightarrow{PQ'}\| = \|v\||\cos\vartheta| = \|v\| \cdot \frac{|v\cdot w|}{\|v\|\|w\|} = \frac{|v\cdot w|}{\|w\|}$$

then, from (1)

$$\frac{|v\cdot w|}{\|w\|} \leq \|v\| \Rightarrow |v\cdot w| \leq \|v\|\|w\|.$$

Now, we deliver a proof based only on the definition of the scalar product without an appeal to the geometric intuition.

Proof. If $w = 0$, then the equality holds. Suppose that $w \neq 0$. The term

$$A := \left\| v - \frac{v\cdot w}{w\cdot w} w \right\|^2 \tag{2}$$

is non-negative because is the norm of a vector. We have

$$0 \leq A = \|v\|^2 + \frac{(v\cdot w)^2}{(w\cdot w)^2}\|w\|^2 - 2\frac{(v\cdot w)^2}{w\cdot w} = \|v\|^2 - \frac{(v\cdot w)^2}{w\cdot w}. \tag{3}$$

Then

$$\|v\|^2 - \frac{(v\cdot w)^2}{w\cdot w} \geq 0 \tag{4}$$

which implies

$$\|v\|^2\|w\|^2 \geq |v\cdot w|^2 \tag{5}$$

whence

$$\|v\|\|w\| \geq |v\cdot w|. \tag{6}$$

If the equality holds in (6), then the equality holds in all the previous inequalities and $A = 0$. Then

$$v = \left(\frac{v\cdot w}{w\cdot w}\right) w.$$

$\square$