THE RUSSELL PARADOX

Below there is a famous axiom of set theory which has been widely used in Mathematics and Set Theory through the past centuries. From this axiom the existence of all the sets in mathematics follows, $\mathbb{Q}, \mathbb{Z}$ and $\mathbb{R}$ being only a few examples. It is more famous from its statement rather than its name.

Axiom (The Axiom of Unrestricted Schema of Comprehension (UCS))**.** *Let $P(x)$ designate a statement about x. Then there exists a set S such that*

$$x \in S \Leftrightarrow P(x).$$

An example is the following statement about natural numbers

$$P(n) : n \text{ is a even number.}$$

The axiom above implies that the set

$$S := \{n \mid P(n)\}$$

exists. The purpose of the Russell paradox is to show that the above device leads to a contradiction. Some remarks are needed in order to introduce this paradox.

1. **Sets and elements are the same objects.** Firstly, each set can be regarded as an element of another set: given a set S, we can consider the *singleton*

$$T := \{S\}$$

which is a set of one element: S. In turn, we can define the singleton of T

$$V := \{T\} \Rightarrow S \in T \in V$$

Such chains occurs in everyday life as well: a soccer league is a set of teams; each team is a set of players. In conclusion, sets and elements are the same objects. Then, given any two sets A and B either $A \in B$ or $A \notin B$.

2. **Sets which are elements of themselves.** Once we agreed that sets and elements are the same objects, we wonder whether there are sets A such that $A \in A$.

Unlike the section above, it is difficult to produce such an example from everyday life. We need an *ad hoc* construction. We consider the following property of sets

$$P(S) : S \text{ is infinite.}$$

From Axiom of USC there exists the set

$$T := \{S \mid S \text{ is infinite }\}.$$

For instance, $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ all are infinite sets. Then

$$\mathbb{N}, \mathbb{R}, \mathbb{C} \in T.$$

And there many more: for every integer $m \geq 1$, each of the sets

$$m\mathbb{N} := \{n \in \mathbb{N} \mid n = km \text{ for some } k \in \mathbb{N}\}$$

is infinite. Then

$$m\mathbb{N} \in T \text{ for every } m \geq 1.$$

Then there are infinitely many elements in T. Hence $T \in T$.

3. **The Russell paradox.** We define the statement

(1) $$P(T) : T \notin T.$$

By the Axiom of UCS, there exists a set $\mathscr{R}$ such that

(2) $$T \in \mathscr{R} \Leftrightarrow P(T).$$

Now, whether $\mathscr{R} \in \mathscr{R}$ or not is a meaningfull question.

First case: $\mathscr{R} \in \mathscr{R}$. By (2),

$$\mathscr{R} \in \mathscr{R} \Rightarrow P(\mathscr{R}).$$

By (1),

$$P(\mathscr{R}) \Rightarrow \mathscr{R} \notin \mathscr{R}.$$

Then, we obtain a contradiction and we are lead to consider the second case.

Second case: $\mathscr{R} \notin \mathscr{R}$. By (1),

$$\mathscr{R} \notin \mathscr{R} \Rightarrow P(\mathscr{R}).$$

By (2),

$$P(\mathscr{R}) \Rightarrow \mathscr{R} \in \mathscr{R}$$

which gives a contradiction, again. In conclusion, the sentence $\mathscr{R} \in \mathscr{R}$ is neither true or false, which is a paradox.