SOLUTIONS OF THE EXERCISES OF WEEK ONE

Exercise 1. Find the generalized union and intersection of the collection

$$G := \}[0, 1+1/n) \mid n \geq 1\}.$$

Solution. We define $G_n := [0, 1+1/n)$.

(i). Union. There holds

$$\cup G = [0, 2).$$

In fact, from $G_n \subseteq [0, 2)$ for every $n \geq 1$, it follows that

$$\cup G \subseteq [0, 2).$$

Moreover, since $[0, 2) \in G$, there also holds $[0, 2) \subseteq G$.

(ii). Intersection. We have

$$\cap G = [0, 1].$$

In fact, for every n,

$$[0, 1] \subseteq G_n$$

then $[0, 1] \subseteq \cap G$. Now, let $x \in \cap G$; then, there exists G_n such that

$$x \in G_n$$

which implies that $0 \leq x$. We claim that $x \leq 1$. Suppose that $x > 1$. Then, there exists m such that

$$\frac{1}{m} < x - 1.$$

Then $x \notin G_m$ which contradicts $x \in \cap G$. □

Exercise 2. Show that the following inclusion

$$(A - B) \cap (A - C) \subseteq A - (B \cup C)$$

holds (start with the usual sentence "Let $x \in \ldots$").

Solution. We have the following chain of implications:

$$\begin{aligned} x \in (A-B) \cap (A-C) &\Rightarrow x \in (A-B) \wedge x \in (A-C) \\ &\Rightarrow x \in (A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\ &\Rightarrow x \in A \wedge x \notin B \wedge x \notin C \Rightarrow x \in A \wedge x \notin (B \cup C) \\ &\Rightarrow x \in A - (B \cup C). \end{aligned}$$

Then $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. □

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Exercise 3. Let R be the following equivalence relation in $\mathbf{N}$

$$nRm \Leftrightarrow 2 \mid n - m^1.$$

What is $\#(\mathbf{N}/R)$?

Solution. We denote with R_k the equivalence class of $k \in \mathbf{N}$. If k is even, then there exists $m \in \mathbf{N}$ such that

$$k = 2m.$$

Then

$$k - 2 = 2m - 2 = 2(m-1) \Rightarrow k \sim 2 \Rightarrow k \in R_2.$$

Then, $R_2 = R_k$ because two equivalence class are either disjoint of equal. If k is odd, then there exists $m \in \mathbf{N}$ such that

$$k = 2m - 1.$$

Then

$$k - 1 = 2m - 2 = 2(m-1) \Rightarrow k \sim 2 \Rightarrow k \in R_1.$$

Then, $R_1 = R_k$. Clearly, $R_1 \neq R_2$ because, otherwise we would have $R_1 = R_2$, whence $2 \in R_1$ and

$$2 - 1 = 2m$$

which is false. Thus,

$$\mathbf{N}/R = \{R_1, R_2\}$$

and $\#(\mathbf{N}/R) = 2$. □

Exercise 4. Let P be the power set of the set of real numbers. We have the following function

$$f : P \to P, \quad f(A) = A \cap [0,1]$$

Is f injective? is f surjective?

Solution. The function is not injective. For instance,

$$f(\{3\}) = \{3\} \cap [0,1] = \emptyset, \quad f(\{2\}) = \{2\} \cap [0,1] = \emptyset.$$

f is not surjective either. In fact,

$$f(A) = A \cap [0,1] \subseteq [0,1].$$

Then, if $x \notin [0,1]$, there is no set A such that

$$A \cap [0,1] = \{x\}.$$

□

[1] given $n \in \mathbf{N}$, the notation $2 \mid n$ means that there exists $a \in \mathbf{Z}$ such that $n = 2a$