SOLUTIONS OF THE EXERCISES OF THE BOOK

Exercises of page 155.

Exercise 1.

$$\oint_C (x^2-y^2)dx + 2xydy = \int_0^1 \int_{2x^2}^{2x} \nabla \times (x^2-y^2, 2xy) = \int_0^1 \int_{2x^2}^{2x} 4ydydx$$
$$= \int_0^1 \left[2y^2\right]_{2x^2}^{2x} dx = \int_0^1 (8x^2-8x^4)dx = \frac{8}{3} - \frac{8}{5} = \frac{16}{15}$$

Exercise 2.

$$\oint_C x^2ydx + 2xydy = \int_0^1 \int_{x^2}^{x} \nabla \times (x^2y, 2xy) = \int_0^1 \left(\int_{x^2}^{x} 2y - x^2dy\right) dx$$
$$= \int_0^1 \int_{x^2}^{x} 2ydydx - \int_0^1 \int_{x^2}^{x} x^2dydx$$
$$= \int_0^1 \left[y^2\right]_x^{x^2} dx - \int_0^1 x^2(x-x^2)dx$$
$$= \int_0^1 (x^4-x^2)dx - \int_0^1 (x^3+x^4)dx = \frac{1}{5} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} = -\frac{1}{12}$$

Exercise 3.

$$\oint_C (2ydx - 3xdy) = \iint_B \nabla(2y, -3x) = -5\iint_B 1 = 5\pi$$

Exercise 4.

$$\oint_C (e^{x^2}+y^2)dx + (e^{y^2}+x^2)dy = \iint_T \nabla \times (2x-2y)dydx = 2\int_0^4 \int_0^{4-x} (x-y)dydx$$
$$= 2\int_0^4 \int_0^{4-x} xdydx - 2\int_0^4 \int_0^{4-x} ydydx$$
$$= 2\int_0^4 x(4-x)dx - \int_0^4 \left[y^2\right]_0^{4-x} dx$$
$$= 2\int_0^4 (4x-x^2)dx - \int_0^4 (4-x)^2dx$$
$$= \int_0^4 (8x-2x^2-16-x^2+8x)dx$$
$$= \int_0^4 (16x-3x^2-16)dx = \left[4x^2-x^3-16x\right]_0^4 = -64$$

Exercise 11. For a region R bounded by a simple closed curve C show that the area A of R is

$$A = -\oint_C ydx = \oint xdy = \frac{1}{2}\oint_C xdy - ydx$$

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Solution. By definition,

$$A = \iint_R 1 dy dx.$$

We know that there is a vector field $\boldsymbol{X}$ such that

$$1 = \nabla \times \boldsymbol{X}.$$

This vector field is $\boldsymbol{X} = (-y, 0)$. Then

$$\iint_R 1 = \iint \nabla \times (-y, 0) = \oint_C (-y, 0) = \oint -y dx. \tag{1}$$

Another solution of the differential equation can be obtained when $P = 0$ and $Q = x$. Then

$$\iint_R 1 = \iint \nabla \times (0, x) = \oint_C (0, x) = \oint x dy \tag{2}$$

From (1) and (2), we have

$$\iint_R 1 + \iint_R 1 = \oint -y dx + \oint x dy \Rightarrow 2 \iint_R 1 = \oint (x dy - y dx)$$
$$\Rightarrow \iint_R 1 = \frac{1}{2} \oint (x dy - y dx).$$

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