SOLUTIONS OF THE EXERCISES OF WEEK TWO

Exercise 1. If $v, w \in E^3$ are linearly independent, then

$$v, w, v \times w$$

are linearly independent in E^3.

Solution. Since v and w are linearly independent,

$$v \times w \neq 0. \tag{1}$$

Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\alpha v + \beta w + \gamma v \times w = 0. \tag{2}$$

We take the scalar product with the vector $v \times w$. Then

$$\gamma \|v \times w\|^2 = 0. \tag{3}$$

By (1), we have $\gamma = 0$. If we substitute $\gamma = 0$ in (2), we obtain

$$\alpha v + \beta w = 0.$$

Since α and β are linearly independent, we obtain

$$\alpha = \beta = 0 \tag{4}$$

which, together with (3), gives

$$\alpha = \beta = \gamma = 0.$$

Then v, w and z are linearly independent. □

Exercise 2. Let $v, w, z \in E^3$ be such that

$$v \cdot a = w \cdot a = z \cdot a = 0$$

for some vector $0 \neq a \in E^3$. Then v, w and z are linearly dependent.

Solution. We argue by contradiction. Suppose that v, w and z are linearly independent. Then the set

$$\{v, w, z\} \subset E^3$$

generates the linear space E^3. Then, there are $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 v + \lambda_2 w + \lambda_3 z = a.$$

Taking the scalar product with a, we obtain

$$0 = \|a\|^2 \Rightarrow a = 0$$

which contradicts the assumption $a \neq 0$. □

Exercise 3. Given $v, w \in E^n$, show that

$$|\|v\| - \|w\|| \leq \|v - w\|.$$

What is the relation between v and w if the equality holds?

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Solution. The equality is equivalent to the equality

$$|\|v\| - \|w\||^2 \leq \|v - w\|^2 \tag{5}$$

that is

$$\|v\|^2 + \|w\|^2 - 2\|v\|\|w\| \leq \|v\|^2 + \|w\|^2 - 2v \cdot w. \tag{6}$$

In turn, the inequality above is equivalent to

$$v \cdot w \leq \|v\|\|w\|.$$

The last is the Cauchy-Schwarz inequality.
If the equality

$$|\|v\| - \|w\|| = \|v - w\|$$

holds, then in (6) and (5) we have equalities

$$v \cdot w = \|v\|\|w\|$$

holds, as well. Then $v \parallel w$. $\square$