EXERCISES OF WEEK FOURTEEN

Exercise 1. Suppose that $X \in \Gamma^2(\mathbb{R}^3)$, that it

$$X(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$$

and $P, Q, R \in C^2(V)$. Show that

$$\nabla \cdot (\nabla \times X) = 0.$$

Show that if $\Sigma = \partial V$ is a closed surface and X is as above, then

$$\oiint_{\Sigma} \nabla \times X = 0.$$

Solution. By definition,

$$\begin{aligned}\nabla \cdot \nabla \times X &= \partial_x(\partial_y R - \partial_z Q) + \partial_y(\partial_z P - \partial_x R) + \partial_z(\partial_x Q - \partial_y P)\\ &= \partial_x\partial_y R - \partial_x\partial_z Q + \partial_y\partial_z P - \partial_y\partial_x R + \partial_z\partial_x Q - \partial_z\partial_y P = 0\end{aligned}$$

the last equality follows from the Schwarz Theorem on the mixed derivatives. □

Exercise 2. Let $g \in C^2(V)$. Then

$$\nabla \times \nabla g = 0$$

Solution.

$$\nabla(\partial_x g, \partial_y g, \partial_z g) = (\partial_y\partial_z g - \partial_z\partial_y g, \partial_z\partial_x g - \partial_x\partial_z g, \partial_x\partial_y g - \partial_y\partial_x g)$$

which is equal to $(0,0,0)$ by the Schwarz Theorem on the mixed derivatives. □

Exercise 3. Find the derivative of the function

$$F\colon [1,+\infty) \to \mathbb{R}, \quad F(x) = \int_1^x \frac{e^{xy}}{y} dy$$

by applying the derivative formula when the endpoints are not constant functions.

Solution. We have

$$\begin{aligned}F'(x) &= \frac{e^{x\cdot x}}{x}(x)' - \frac{e^{x\cdot 1}}{1}(1)' + \int_1^x \partial_x \frac{e^{xy}}{y} dy\\ &= \frac{e^{x^2}}{x} + \int_1^x e^{xy} dy = \frac{e^{x^2}}{x} + \frac{e^{x^2}}{x} - \frac{e^x}{x}\end{aligned}$$

□

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