On the uniqueness of Q-ball solutions to the non-linear Schrödinger equation

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We consider the non-linear Schrödinger equation

(NLS) $$(i\partial_t + \Delta)\phi - g(\phi) = 0$$

where

$$\phi\colon \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{C},\ \ g\colon \mathbb{C} \to \mathbb{C}$$

such that

$$g(zu) = zg(u)$$

for every pair (z, u) in $\mathbb{C}^2$ such that $|z| = 1$ $(z \in S^1)$.

Let $G\colon \mathbb{C} \to \mathbb{R}$ be such that for every $s \geq 0$

$$G'(s) = g(s), \quad G(0) = 0.$$

The equation (NLS) is globally well-posed in $H^1(\mathbb{R}^N;\mathbb{C})$ if

$$|g(s)| \le c(|s|^{p-1} + |s|^{q-1}), \quad 2 < p \le q < 2 + \frac{4}{N}.$$

That is, given u_0 in $H^1(\mathbb{R}^N;\mathbb{C})$, there exists only one solution

$$\phi\colon [0,+\infty) \times \mathbb{R}^N_x \to \mathbb{C}$$

to (NLS) such that

$$\phi(0,x) = u_0(x), \quad \phi(t,\cdot) \in H^1(\mathbb{R}^N).$$

The notation

$$U_t(u_0) = \phi(t,\cdot), \quad U_t\colon H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$$

is useful.

Conserved quantities

If

$$\phi\colon [0,+\infty) \times \mathbb{R}^n \to \mathbb{C}$$

be a solution to the (NLS). Then the energy

$$\mathbf{E}(t) := \int_{\mathbb{R}^N} |\nabla_x \phi(t,x)|^2 dx + \int_{\mathbb{R}^N} G(\phi(t,x))dx$$

and the charge

$$\mathbf{C}(t) := \mathrm{Re} \int_{\mathbb{R}^N} \phi(t,x)\overline{\phi}(t,x)dx$$

are constant.

Solitary waves

A solitary wave is a solution ϕ to (NLS) such that

$$\phi_v(t,x) = e^{i(\omega - |v|^2)t + iv\cdot x} u(x - tv),$$

where

$$v \in \mathbb{R}^N, \quad \omega \in \mathbb{R}, \quad u \in H^1(\mathbb{R}^N; \mathbb{R}).$$

When $v = 0$, ϕ is also called standing-wave:

$$\phi(t,x) = e^{i\omega t} u(x).$$

A standing-wave is called Q-ball if u is positive and radially symmetric.

We use the notation $H^1_{r,+}(\mathbb{R}^n)$ for the Q-balls.

The variational setting

If u is a critical point of the functional

$$E\colon H^1(\mathbb{R}^N;\mathbb{R}) \to \mathbb{R}, \quad E(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} F(u)$$

on the constraint

$$S(\lambda) = \{u \in H^1(\mathbb{R}^N;\mathbb{C}) \mid \|u\|_{L^2}^2 = \lambda\}$$

then there exists ω (positive) in $\mathbb{R}$ such that

(E) $$\Delta u - g(u) = \omega u.$$

Then, for every v in $\mathbb{R}^N$, we have solitary wave solutions

$$\phi(t,x) := e^{i\omega t}u(x), \quad \phi_v(t,x) = e^{i((\omega-|v|^2)t+v\cdot x)}u(x-tv).$$

Definition (Stable subsets of $H^1(\mathbb{R}^N;\mathbb{C})$)

A subset $S \subseteq H^1(\mathbb{R}^N;\mathbb{C})$ is stable if for $\varepsilon > 0$, there exists $\delta > 0$ such that, for every Φ in $H^1(\mathbb{R}^N;\mathbb{C})$, there holds

$$\mathrm{dist}(\Phi, S) < \delta \Rightarrow \mathrm{dist}(U_t(\Phi), S) < \varepsilon$$

for every $t \geq 0$.

$$\mathrm{dist}(\Phi, S) := \inf_{\Psi \in S} \|\Phi - \Psi\|_{H^1(\mathbb{R}^N;\mathbb{C})}.$$

Orbital stability of standing waves

If $\Phi \in H^1(\mathbb{R}^N; \mathbb{C})$, we define its orbit

$$Orb(\Phi) := \{U_t(\Phi) \mid t \geq 0\} \subseteq H^1(\mathbb{R}^N; \mathbb{C}).$$

If $\phi(t, x) = e^{i\omega t} u$ is a standing-wave,

$$Orb(u) = \{e^{i\omega t} u \mid t \geq 0\}.$$

So,

$$Orb(u) \subseteq \left\{zu(\cdot + y) \mid |z| = 1, y \in \mathbb{R}^N\right\} =: \Gamma(u)$$

A standing wave is *orbitally stable* if $\Gamma(u)$ is stable.

The example of Cazenave and Lions, CPM, 1982

The orbit of u is contained in

$$\Gamma_1(u) := \{zu \mid z \in S^1\} \subsetneq \Gamma(u).$$

But Γ_1 is not stable. Given $w \in \mathbb{R}^N$, non-zero

$$\Phi_n := e^{ix \cdot w/n} u(x), \quad \mathrm{dist}(\Phi_n, \Gamma_1) \to 0$$

and

$$\sup_{t \geq 0} \mathrm{dist}\big(U_t(\Phi_n), \Gamma_1\big) \geq \|u(x-w) - u(x)\|_{L^2}.$$

Example due to Cazenave and Lions, CPM, 1982.

It can be shown the following

Lemma ($\Gamma(u)$ is the smallest stable and invariant subset)

If $M \subseteq H^1(\mathbb{R}^N;\mathbb{C})$ is stable, $u \in M$, and

$$U_t(M) \subseteq M \text{ for every } t \geq 0,$$

then $\Gamma(u) \subseteq M$.

Proof: apply the example of Cazenave and Lions in every direction w.

Definition (V. Benci, C. Bonanno)

Φ is orbitally stable if there exists a sub-manifold $M \subseteq H^1(\mathbb{R}^N;\mathbb{C})$ such that

$$\Phi \in M$$

and is stable, invariant and of finite dimension.

Hereafter, we will consider standing-waves

$$\phi(t,x) = e^{i\omega t} u(t,x)$$

where u is a minimum of E over $S(\lambda)$.

Given $\lambda > 0$, we define

$$\Gamma_\lambda := \{u \in H^1(\mathbb{R}^N;\mathbb{C}) \mid \|u\|_{L^2}^2 = \lambda, E(u) = \inf_{S(\lambda)} E\}.$$

It is called *ground state*.

If u is in Γ_λ, then $\Gamma(u) \subseteq \Gamma_\lambda$.

The ground state is stable

Under the assumptions

$$\exists s_0 \in (0, +\infty) \text{ such that } G(s_0) < 0$$

and

$$|g(s)| \le C(|s|^{p-1} + |s|^{q-1}), \quad 2 < p \le q < 2 + \frac{4}{N}$$

there holds:

Theorem (Bellazzini, Benci *et al.*, Adv. Nonlinear Stud., 2007)

For every $\lambda > 0$ the ground state Γ_λ is non-empty and stable.

The non-linearity we have in mind is

$$g(s) = -a|s|^{p-2}s + b|s|^{q-2}s, \quad a > 0, \; b \ge 0.$$

The proof of the stability of Γ_λ relies (in big part) on the following

Lemma (Concentrated-compactness of minimizing sequences)

If $(u_n) \subset H^1(\mathbb{R}^N;\mathbb{R})$ *is a sequence such that*

$$\|u_n\|_{L^2}^2 \to \lambda, \quad E(u_n) \to I(\lambda)$$

then, there exists a sequence $(y_n) \subseteq \mathbb{R}^N$ *and* u *such that*

$$u_n(\cdot + y_n) \to u \text{ in } H^1(\mathbb{R}^N;\mathbb{R})$$

and

$$E(u) = I(\lambda).$$

In general, minimizing sequences are not compact:

$$u_n(x) := u(x + ne_1), \quad e_1 = (1, 0 \dots, 0).$$

Properties of E

(1) For every $\lambda > 0$, E is bounded from below. We define

$$I(\lambda) := \inf_{S(\lambda)} E.$$

(2) $I(\lambda) < 0$

(3) given a weakly converging sequence $u_n \rightharpoonup u$ in H^1, there holds

$$\lim_{n \to +\infty} \big(E(u_n) - E(u_n - u) - E(u) \big) = 0$$

(4) given $0 < \mu < \lambda$, there holds

$$I(\lambda) < I(\mu) + I(\lambda - \mu).$$

The proof uses the ideas of the Concentration-Compactness Lemma (P. L. Lions, AIHPAN, 1984). Here we use a version of V. Benci and D. Fortunato for sequences in H^1 (Benci, Fortunato, Chaos Solitons Fractals, 2014).

Given a bounded sequence $(u_n) \subseteq H^1$, we have three cases: Concentration, Dichotomy, Vanishing.

(C) $\quad \exists (y_n) \subseteq \mathbb{R}^N$ and $u \in H^1$ such that $u_n(\cdot + y_n) \to u$ in H^1

(D) $\quad \exists (y_n) \subseteq \mathbb{R}^N$ and $u \in H^1$ such that $u_n(\cdot + y_n) \rightharpoonup u$ in H^1

and

$$0 < \|u\|_{H^1} < \lim_{n \to +\infty} \|u_n\|_{H^1}$$

(V) $$\forall (y_n) \subseteq \mathbb{R}^N : \ u_n(\cdot + y_n) \rightharpoonup 0 \text{ in } H^1.$$

The Vanishing case is ruled out

Let Q_i be an enumeration of all the cubes in $\mathbb{R}^N$ with length 1 and vertices with integral coordinates.

If (u_n) is a vanishing sequence, then

(1) $$\sup_{1\leq i} \|u_n\|^2_{L^2(Q_i)} \to 0.$$

It follows from the Rellich-Kondrachov Theorem.

Lemma (Lemma I.1 of P. L. Lions (AIHPAN, 1984))

If (1) *holds, then*

$$\lim_{n\to+\infty} \|u_n\|_{L^\alpha} = 0$$

for every $2 < \alpha < \frac{2n}{n-2}$.

Then, if (u_n) vanishes, $I(\lambda) \geq 0$, contradicting (2).

If (u_n) falls into the (D) case,

$$w_n := u_n(\cdot + y_n) \rightharpoonup u$$

we can prove that $\|u\|_{L^2}^2 = \lambda$, otherwise, by (3) and

$$\begin{aligned} I(\lambda) + o(1) &= E(u_n(\cdot + y_n)) \\ &= E(u_n(\cdot + y_n) - u) + E(u) + o(1) \\ &\geq I(\lambda - \mu) + I(\mu) + o(1) \end{aligned}$$

we obtain a contradiction with (4).

Then

$$\|u_n\|_{L^2} \to \|u\|_{L^2}$$

implying $w_n \to u$ in L^2.

Strong convergence in $L^2 \Rightarrow$ strong convergence in H^1.

$$\int_{\mathbb{R}^N} G(w_n) \to \int_{\mathbb{R}^N} G(u), \quad \|\nabla w_n\|_{L^2}^2 \geq \|\nabla u\|_{L^2}^2.$$

Since $u \in S(\lambda)$,

$$I(\lambda) + o(1) = E(w_n) \geq E(u) = I(\lambda).$$

Then

$$\|\nabla w_n\|_{L^2} \to \|\nabla u\|_{L^2} \Rightarrow \nabla w_n \to \nabla u \text{ in } L^2.$$

Then $w_n \to u$ in H^1, implying

$$\|u\|_{H^1} = \lim_{n\to+\infty} \|w_n\|_{H^1} = \lim_{n\to+\infty} \|u_n\|_{H^1}$$

and contradicting (D).

The strict sub-additivity property of I

For every $\vartheta > 1$ there holds

$$I(\vartheta\lambda) < \vartheta I(\lambda).$$

Given u in $S(\lambda)$, be such that

$$E(u) \leq I(\lambda) + \varepsilon.$$

We set

$$u_\vartheta(x) = u(\vartheta^{-1/n}x), \quad u_\vartheta \in S(\vartheta\lambda).$$

there holds

$$\begin{aligned} I(\vartheta\lambda) \leq E(u_\vartheta) &= \vartheta\Big(\vartheta^{-2/n}\|\nabla u\|_{L^2}/2 + \int_{\mathbb{R}^N} F(u)\Big) \\ &< \vartheta E(u) \leq \vartheta I(\lambda) + \vartheta\varepsilon. \end{aligned}$$

Let $0 < \mu < \lambda$

$$I(\lambda) = I\left(\mu \cdot \frac{\lambda}{\mu}\right) < \frac{\lambda}{\mu} \cdot I(\mu)$$
$$I(\lambda) = I\left((\lambda - \mu) \cdot \frac{\lambda}{\lambda - \mu}\right) < \frac{\lambda}{\lambda - \mu} \cdot I(\lambda - \mu).$$

Then,

$$\frac{\mu}{\lambda} \cdot I(\lambda) < I(\mu), \quad \frac{\lambda - \mu}{\lambda} \cdot I(\lambda) < I(\lambda - \mu).$$

Taking the sum, we obtain

$$I(\lambda) < I(\mu) + I(\lambda - \mu).$$

This is the argument of Benci, Ghimenti *et al.*, 2007.

This proof goes back to the paper of Cazenave and Lions (1982) where

$$g(s) = -a|s|^{p-2}s, \quad a > 0$$

It applies to a lot problems of stability:

(1) non-linear Schrödinger equation, (Benci, Ghimenti *et al.*, 2007)
(2) non-linear Klein-Gordon equation (Benci, Bonanno *et al.*, 2010)
(3) systems NLS-KdV (Albert, Bhattarai, 2013)
(4) systems of NLS (Wang, Nguyen, 2011)
(5) systems of NLKG (G., 2012).

The ground state Γ_λ is stable

Let $(\Phi_n) \subseteq H^1(\mathbb{R}^N;\mathbb{C})$ and $\varepsilon_0 > 0$ such that

$$\mathrm{dist}(\Phi_n, \Gamma_\lambda) \to 0, \quad \mathrm{dist}(U_{t_n}(\Phi_n), \Gamma_\lambda) \geq \varepsilon_0.$$

Then

$$E(\Phi_n) \to I(\lambda), \quad C(\Phi) \to \lambda.$$

We define

$$\Psi_n := U_{t_n}(\Phi_n).$$

E and C are conserved quantities

$$E(\Psi_n) = E(\Phi_n), \quad C(\Psi_n) = C(\Phi_n).$$

We want to obtain a contradiction and prove that

$$\mathrm{dist}(\Psi_n, \Gamma_\lambda) \to 0.$$

We define

$$u_n := |\Psi_n|.$$

$$E(u_n) \leq E(\Psi_n), \quad C(u_n) = C(\Psi_n).$$

The energy inequality follows from the inequality

$$\int_{\mathbb{R}^N} |\nabla \Psi|^2 \geq \int_{\mathbb{R}^N} |\nabla |\Psi||^2$$

for $\Psi \in H^1(\mathbb{R}^N; \mathbb{C})$.

It is called "Convex Inequality for Gradients" (Lieb and Loss).

Since $E(u_n) \to I(\lambda)$ and $C(u_n) \to \lambda$

$$u_n(\cdot + y_n) \to u$$

for some sequence $(y_n) \subseteq \mathbb{R}^N$ and u in Γ_λ.

So,

$$\|u_n(\cdot + y_n) - u\|_{H^1(\mathbb{R}^N;\mathbb{C})} = \|u_n - u(\cdot - y_n)\|_{H^1(\mathbb{R}^N;\mathbb{C})} \to 0$$

and

$$\mathrm{dist}(|\Psi_n|, \Gamma_\lambda) \to 0.$$

Now, we have to show that

$$\mathrm{dist}(\Psi_n, \Gamma_\lambda) \to 0.$$

We can suppose that

$$\Psi_n(\cdot + y_n) \rightharpoonup \Psi \text{ and } |\Psi| = u.$$

$$\begin{aligned} I(\lambda) + o(1) = E(\Psi_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \Psi_n|^2 + \int_{\mathbb{R}^N} G(\Psi_n) \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \Psi|^2 + \int_{\mathbb{R}^N} G(\Psi) \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla |\Psi||^2 + \int_{\mathbb{R}^N} G(|\Psi|) = I(\lambda). \end{aligned}$$

This implies

$$\lim_{n\to+\infty} \|\nabla \Psi_n\|_{L^2}^2 \to \|\nabla \Psi\|_{L^2}$$

which means

$$\Psi_n(\cdot + y_n) \to \Psi \text{ in } H^1(\mathbb{R}^N; \mathbb{C})$$

and Ψ is in Γ_λ.

Orbital stability of $\Gamma(u)$

A characterization of the ground state

Lemma (G., 2012)

Let $\Phi \in H^1(\mathbb{R}^N; \mathbb{C})$ *be such that* $|\Phi|$ *is continuous and positive. If*

$$\int_{\mathbb{R}^N} |\nabla \Phi|^2 = \int_{\mathbb{R}^N} |\nabla |\Phi||^2$$

then there exists $c \in \mathbb{C}$ *such that*

$$\Phi(x) = c|\Phi(x)|, \quad |c| = 1.$$

A version of this lemma due to Lieb and Loss requires $\mathrm{Re}(\Psi) > 0$.

We consider the following equivalence relation in Γ_λ

$$\Phi_1 \sim \Phi_2 \Leftrightarrow \exists (y,z) \in \mathbb{R}^N \times S^1$$

such that

$$\Phi_1 = \varepsilon \Phi_2(\cdot + y).$$

The equivalence class is

$$\Gamma(\Phi).$$

For every Φ,

$$\Phi = c|\Phi| = u$$

Then Φ and u have the same equivalence classes

Byeon, Jeanjean and Mariș • Calc. Var., 2009

In every class there is exactly one Q-ball.

Let P be the quotient set.

There could be a sequence (Φ_n) such that

$$\mathrm{dist}(\Phi_n, \Gamma(u)) \to 0$$

but

$$\mathrm{dist}(\Phi_n, \Gamma(v)) \to 0$$

with $u, v \in H^1_{r,+}$ and $u \neq v$.

This does not happen if

$$\Gamma(u) = \Gamma_\lambda.$$

That is, if there is only one pair $(u, \omega) \in H^1_{r,+} \times \mathbb{R}$ satisfying

$$\Delta - g(u) - \omega u = 0, \quad u \in \Gamma_\lambda.$$

We wish to answer to the following questions:

(1) ω is prescribed: how may solutions?

(1b) what if the L^2 norm is also prescribed to $\lambda > 0$?

(2) λ is prescribed: how many pairs $(u, \omega) \in H^1_{r,+} \times (0, +\infty)$?

The answers can change if H^1 is replaced by

(V)
$$\lim_{|x| \to +\infty} u(x) = 0.$$

If g is a pure-power

$$g(s) = -|s|^{p-2}s, \quad 2 < p < \frac{2n}{n-2} \ (2 < p \text{ if } n = 1, 2).$$

Kwong, Man Kam • Arch. Ration. Mech. Anal., 1989

There is only one u_0 in $H^1_{loc} \cap V$ such that

$$\Delta u_0 - u_0 + |u_0|^{p-2}u_0 = 0.$$

Since at least one solution $H^1_{r,+}$ exists, u_0 is in $H^1_{r,+}$.

Pure-power non-linearities enjoy special rescalings.

Given $\omega > 0$, if u solves

$$\Delta u - \omega u + |u|^{p-2} u = 0$$

then

$$u(x) = \omega^{1/(p-1)} u_0(\omega^{1/2} x), \quad \|u\|_{L^2}^2 = \omega^{\frac{2}{p-1} - \frac{n}{2}} \|u_0\|_{L^2}^2.$$

So, the solution is unique for every ω.

If $\|u\|_{L^2}^2$ is prescribed to be λ, there is only the pair

$$\left(\omega^{1/(p-1)} u_0(\omega^{1/2} x), \omega\right)$$

where

$$\omega = (\lambda \|u_0\|_{L^2}^{-2})^{\alpha}, \quad \alpha := \frac{2(p-1)}{4 - n(p-1)}.$$

Serrin and Tang (IUMJ, 2000) generalized Kwong's result.

However, they require

$$2G(s) + \omega s^2$$

to have a unique zero.

Berestycki and Lions • Arch. Ration. Mech. Anal., 1983 • $n = 1$

If the first positive zero of $2G + \omega s^2$ is simple, then the solution to

$$u'' - g(u) - \omega u = 0$$

is unique.

If the H^1 is replaced by (V), the uniqueness fails:

Del Pino, Guerra, Davila • Proc. Lond. Math. Soc., 2013 • $n = 3$

For every $1 < p < 3$, there exists (a, q) such that

$$\Delta u - u + u^p + au^q = 0$$

has at least three solutions in $H^1_{loc} \cap V$.

We have partial answers to (1) and (1b).

Lemma (Georgiev and G., $n \geq 3$, $H^1_{r,+}$ solutions)

Suppose that g is C^1 and $g(0) = 0, g'(0) \geq 0$. Then

1. *for every $\omega > 0$, given two solutions $u_1 \neq u_2$ to*

$$\Delta u - \omega u - g(u) = 0,$$

   either $u_1 < u_2$ or $u_2 < u_1$

2. *if $\|u_1\|_{L^2} = \|u_2\|_{L^2}$, then $u_1 = u_2$.*

In fact, two of the solutions of Del Pino are vanishing, but not H^1.

If g is a pure powers, the result of Kwong implies

$$\#P = 1 \Rightarrow \Gamma_\lambda = \Gamma(u).$$

If P is finite, then standing-waves are orbitally stable.

So far, we do know of an example of non-linearity g and λ where

- P is not finite
- P is finite and $\#P \neq 1$.