Finiteness, up to translations, of standing-wave solutions to a nonlinear Schrödinger equation

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Given $\omega > 0$, the equation

$$\text{(E)} \qquad \Delta u(x) - \omega u(x) - g(u(x)) = 0, \quad x \in \mathbb{R}^N$$

has a positive, radially symmetric solution in $H^1(\mathbb{R}^n)$, provided

$$\exists s_0 > 0 \text{ such that } 2G(s_0) + \omega s_0^2 < 0 \ (G' = g)$$

and

$$|g(s)| \leq C(|s|^{p-1} + |s|^{q-1})$$

where

$$2 < p \leq q < \frac{2n}{n-2} \ (\text{or } 2 < p \leq q \text{ if } n = 1).$$

Berestycki and Lions, ARMA, 1983.

Let $H^1_{r,+}$ be the set of positive, and radially symmetric functions.

We wish to answer to the following questions:

(1) ω is fixed: how may solutions do we have?

(1b) what if the L^2 norm is fixed to $\lambda > 0$?

(2) λ is fixed: how many pairs $(u, \omega) \in H^1_{r,+} \times (0, +\infty)$?

The answers can change if H^1 is replaced by

(V) $$\lim_{|x| \to +\infty} u(x) = 0.$$

The pure-power case: (1)

The result of Kwong, ARMA, 1989 completed the case

$$g(s) = -|s|^{p-2}s, \quad 2 < p < \frac{2n}{n-2} \ (2 < p \text{ if } n = 1, 2).$$

Kwong, Man Kam • Arch. Ration. Mech. Anal., 1989

There is only one u_0 in $H^{1,+}_{loc,r} \cap V$ such that

$$\Delta u_0 - u_0 + |u_0|^{p-2}u_0 = 0.$$

Since at least one solution $H^1_{r,+}$ exists, u_0 is in $H^1_{r,+}$.

The pure-power case: (1b) and (2)

Pure-power non-linearities enjoy special rescalings.

Given $\omega > 0$, if u solves

$$\Delta u - \omega u + |u|^{p-2} u = 0$$

then

$$u(x) = \omega^{1/(p-1)} u_0(\omega^{1/2} x), \quad \|u\|_{L^2}^2 = \omega^{\frac{2}{p-1} - \frac{n}{2}} \|u_0\|_{L^2}^2.$$

So, the solution is unique for every ω.

If $\|u\|_{L^2}$ is λ, there is only the pair

$$\left(\omega^{1/(p-1)} u_0(\omega^{1/2} x), \omega\right)$$

where

$$\omega = (\lambda \|u_0\|_{L^2}^{-1})^{2\alpha}, \quad \alpha := \frac{2(p-1)}{4 - n(p-1)}.$$

Serrin and Tang (IUMJ, 2000) generalized Kwong's result.

However, they require

$$2G(s) + \omega s^2$$

to have a unique zero.

Berestycki and Lions • Arch. Ration. Mech. Anal., 1983 • $n = 1$

If the first positive zero of $2G + \omega s^2$ is simple, then the solution to

$$u'' - g(u) - \omega u = 0$$

is unique.

If the H^1 is replaced by (V), the uniqueness fails:

Del Pino, Guerra, Davila • Proc. Lond. Math. Soc., 2013 • $n = 3$

For every $1 < p < 3$, there exists (a, q) such that

$$\Delta u - u + u^p + au^q = 0$$

has at least three solutions in $H^{1,+}_{loc,r} \cap V$.

We have partial answers to (1) and (1b).

Lemma (Georgiev and G., $n \geq 3$, $H^1_{r,+}$ solutions)

Suppose that g is C^1 and $g(0) = 0, g'(0) \geq 0$. Then

1. *for every $\omega > 0$, given two solutions $u_1 \neq u_2$ to*

$$\Delta u - \omega u - g(u) = 0,$$

   either $u_1 < u_2$ or $u_2 < u_1$
2. *if $\|u_1\|_{L^2} = \|u_2\|_{L^2}$, then $u_1 = u_2$*

Two of the solutions of Del Pino are vanishing, but not H^1.

We do not have an example of ω and g where two solutions

$$u_1 < u_2, \ \ u_1, u_2 \in H^1_{r,+}.$$

occur.

The motivation of problem (2)

If u is a solution to

$$\Delta u - g(u) - \omega u = 0$$

then

$$\phi(t,x) = e^{i\omega t} u(x)$$

is a standing-wave solution to the non-linear Schrödinger equation

$$i\partial_t \phi + \Delta_x \phi - g(\phi) = 0.$$

Standing-waves can be obtained as minima of the functional

$$E\colon H^1(\mathbb{R}^N;\mathbb{R}) \to \mathbb{R}$$
$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^N} G(u(x)) dx$$

on the constraint

$$S(\lambda) = \{u \in H^1(\mathbb{R}^N;\mathbb{R}) \mid \|u\|_{L^2} = \lambda\}.$$

We fix λ and define Γ_λ the set of minima and the relation

$$u_1 \sim u_2 \Leftrightarrow \exists (y, \varepsilon) \in \mathbb{R}^N \times \{-1, 1\}$$

such that

$$u_1(x) = \varepsilon u_2(x + y) \text{ for every } x \in \mathbb{R}^n.$$

We use the notation $\Gamma_\lambda(u)$ for the equivalence class of u.

Byeon, Jeanjean and Mariș • Calc. Var., 2009

$\Gamma_\lambda(u)$ has a representative in $H^1_{r,+}(\mathbb{R}^n)$.

Let P be the quotient set.

Question (2) can be restated as:

(2') What is $\#P$?

The cardinality of P is related to the orbital stability of a standing-wave

$$\phi(t,x) = u(x)e^{i\omega t}, \quad u \in H^1_{r,+}(\mathbb{R}^n).$$

If g is a pure powers, the result of Kwong implies

$$\#P = 1.$$

This fact was used to prove the orbital stability of standing-waves by T. Cazenave and P. L. Lions (Comm. Math. Phys., 1982).

If P is finite, then standing-waves are orbitally stable.

So far, we do know of an example of non-linearity g and λ where

- P is not finite
- P is finite and $\#P \neq 1$.

If λ is small, then $\#P = 1$?

The simple case $(n = 1)$ is interesting.