Traveling wave solutions to the half-wave equations

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We consider the half-wave equation

$$\text{(HW)} \qquad \big(i\partial_t - D\big)u = |u|^{p-1}u - |u|^{q-1}u$$

where

$$u\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$$

A traveling-wave solution is

$$u(t,x) = \psi(x - tv)e^{-i\omega t}$$

where ψ is a solution of the equation

$$D\psi + iv\psi' - \omega\psi = -|\psi|^{p-1}\psi + |\psi|^{q-1}\psi$$

where $2 < p < q < 4$.

Half-wave equations in dimension three and other non-linearities arise in stars collapse (Fröhlich, Jonsson and Lenzmann, Comm. Pure Appl. Math., 2007).

The existence is obtained by variational method.

We define the energy functional

$$\mathcal{E}_v(\psi) = \mathcal{H}_v(\psi) - \frac{1}{p+1}\|\psi\|_{L^{p+1}}^{p+1} + \frac{1}{q+1}\|\psi\|_{L^{q+1}}^{q+1}$$

on the constraint

$$S(\lambda) = \{\psi \in H^{1/2}(\mathbb{R}) \mid \|\psi\|_{L^2}^2 = \lambda\}$$

where

$$\mathcal{H}_v(\psi) = \frac{1}{2}\left(\|\psi\|_{\dot{H}^{1/2}(\mathbb{R})}^2 + i\int_{-\infty}^{+\infty} \overline{\psi}\nabla\psi \cdot v\right)$$

By $D\psi$ we mean the unique L^2 function such that

$$\mathscr{F}(D\psi)(\xi) = |\xi|\mathscr{F}(\psi)(\xi)$$

or

$$\text{P.V.} \int_{-\infty}^{+\infty} \frac{\psi(x) - \psi(y)}{|x-y|^2} dy$$

The term $\mathcal{H}_v(\psi)$ is real and

$$\mathcal{H}_v(\psi) \geq (1 - |v|)\|\psi\|^2_{\dot{H}^{1/2}(\mathbb{R})}$$

Define

$$I(\lambda) := \inf_{S(\lambda)} \mathcal{E}_v$$

We prove that if $|v| < 1$ and $I(\lambda) < 0$, then $\mathcal{E}_v$ achieves its infimum.

Moreover, given a minimising sequence

$$\mathcal{E}(\psi_n) \to I(\lambda)$$

there exists a sequence $(y_n) \subseteq \mathbb{R}^N$ such that

$$\psi_n(\cdot + y_n) \to \psi$$

in $H^{1/2}(\mathbb{R})$.

We have concentrated-compactness of minimising sequences.

Facts about $I(\lambda)$

1. On $S(\lambda)$ the functional $\mathcal{E}_v$ is bounded from below
2. there exists λ_* such that

$$\lambda > \lambda_* \Rightarrow I(\lambda) < 0.$$

   It follows from the rescaling $\psi_\vartheta := \vartheta^{-1/2}\psi(x\vartheta^{-1})$

3. $$I(\lambda) < I(\lambda_0) + I(\lambda - \lambda_0)$$

   for every $0 < \lambda_0 < \lambda$ (sub-additivity property of I).

Likewise problems of concentrated compactness are handled in NLS (Benci and Ghimenti, Adv. Nonlinear Stud., 2007) and HW (Guo and Huang, J. Math. Phys., 2012).

Theorem

For every $2 < p < q < 4$ *and every* $|v| < 1$

$$\mathcal{E}_v(\psi) = I(\lambda)$$

for every λ *such that* $I(\lambda) < 0$. *Given a minimising sequence* (ψ_n) *there exists a sequence* $(y_n) \subseteq \mathbb{R}^N$ *and* $\psi \in H^{1/2}$ *such that*

$$\psi_n(\cdot + y_n) \to \psi.$$

Suppose that for every sequence (y_n), $\psi_n(\cdot + y_n)$ does not converge in $H^{1/2}(\mathbb{R})$.

We still have a weak limit

$$\psi_n(\cdot + y_n) \rightharpoonup \psi$$

Define

$$\lambda_0 := \|\psi\|_{L^2}^2.$$

By the lower-semicontinuity of the norm

$$0 \leq \lambda_0 < \lambda = \liminf_{n \to \infty} \|\psi_n\|_{L^2}^2$$

$\lambda_0 > 0$ for some (y_n)

$$\begin{aligned} I(\lambda) &= o(1) + \mathcal{E}_v(\psi_n(\cdot + y_n)) \\ &= \mathcal{E}_v(\psi_n(\cdot + y_n) - \psi) + \mathcal{E}_v(\psi) + o(1) \\ &\geq I(\lambda_0) + I(\lambda - \lambda_0) + o(1) \end{aligned}$$

while the strict inequality

$$I(\lambda) < I(\lambda_0) + I(\lambda - \lambda_0)$$

holds instead. So, this case is ruled out.

$\lambda_0 = 0$ for every (y_n)

Proposition

Suppose that $(\psi_n) \subseteq H^1(\mathbb{R})$ is a bounded sequence such that

$$\psi_n(\cdot + y_n) \rightharpoonup 0$$

for every sequence $(y_n) \subseteq \mathbb{R}^N$. Then

$$\|\psi_n\|_{L^p} \to 0$$

for every $2 < p < 4$.

If that happens,

$$I(\lambda) \geq 0.$$

yielding a contradiction.

If $2 < p < q < 4$, the non-linear half-wave equation is globally well-posed.

Definition

A set $\Gamma \subseteq H^{1/2}(\mathbb{R})$ is said *orbitally stable* if and only if for every $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\mathrm{dist}(\psi, \Gamma) < \delta \Rightarrow \mathrm{dist}(u(t, \cdot), \Gamma) < \varepsilon$$

for every $t \geq 0$.

For u

$$u(0, x) = \psi(x)$$

and u solves the half-wave equation.

Theorem

Given λ and v, we define the ground state

$$\Gamma(\lambda, v) = \{\psi \in S(\lambda) \mid \mathcal{E}_v(\psi) = I(\lambda)\}$$

The proof follows from the concentrated-compactness of minimising sequences and the conserved quantities

$$\mathcal{N}(\psi) = \|\psi\|_{L^2(\mathbb{R}^N)}, \quad \mathcal{E}_v(\psi)$$

orbital stability of $\Gamma(\lambda, v)$.

By contradiction: suppose that there are sequences

$$(\psi_n) \subset H^{1/2}(\mathbb{R}), \quad (t_n) \subset \mathbb{R}$$

and $\varepsilon_0 > 0$

$$\mathrm{dist}(\psi_n, \Gamma(\lambda, v)) \to 0, \quad \mathrm{dist}(\psi_n(t_n, \cdot), \Gamma(\lambda, v)) \geq \varepsilon_0.$$

We define

$$\phi_n := \psi_n(t_n, \cdot), \quad \mathcal{E}(\phi_n) = \mathcal{E}(\psi_n), \quad \mathcal{N}(\phi_n) = \mathcal{N}(\psi_n)$$

a rescaling

$$(s_n \psi_n(t_n, \cdot)) \subseteq S(\lambda), \quad s_n \to 1$$

gives a minimising sequence in $I(\lambda)$. Then there exists $\phi \in \Gamma(\lambda, v)$ such that

$$\psi_n(t_n, \cdot + y_n) \to \phi$$

which contradicts the first assumption.

Suppose that $\mathcal{E}(\psi) = I(\lambda)$. Then

$$D\psi + i\psi' v = \omega\psi - |\psi|^{p-1}\psi + |\psi|^{q-1}\psi$$

and

$$\phi(t,x) = \psi(x+y)e^{-i\omega t}e^{i\alpha t}$$

is another traveling-wave solution; $\mathcal{N}, \mathcal{E}_v$ did not change.

So, at least the subset

$$\Gamma_{v,\lambda}(\psi) = \{z\psi(x+y) \mid y \in \mathbb{R}, z \in \mathbb{C}, |z| = 1\}$$

is contained in $\Gamma_{v,\lambda}$.

We wonder whether

$$\Gamma_{v,\lambda}(\psi) = \Gamma_{v,\lambda}$$

The orbital stability of traveling-waves

Definition

A traveling-wave is orbitally stable if $\Gamma_{v,\lambda}(\psi)$ is orbitally stable

The inclusion

$$\Gamma(\psi) \subseteq \Gamma$$

does not imply the stability of $\Gamma(\psi)$ (Cazenave and Lions, Comm. Math. Phys., 1982).

Unless

$$\Gamma = \Gamma(\psi)$$

or

$$\Gamma = \Gamma(\psi_1) \cup \cdots \cup \Gamma(\psi_k)$$

Pure power $|u|^{p-1}u$ type

The equality is related to the uniqueness of positive solutions to

$$D\psi + i\psi' v = \omega\psi - |\psi|^{p-1}\psi + |\psi|^{q-1}\psi$$

up to space translation.

When $v = 0$ and $\omega = 1$

$$(p > 1) \qquad D\psi - \psi + \psi^p = 0$$

from Frank and Lenzmann, arXiv:1009.4042.

And

$$\Delta\psi - \psi + \psi^p = 0$$

by Man Kam Kwong, ARMA, 1989 (Orbital stability of NLS and NLKG)

Combined power-type $|u|^{p-1}u - |u|^{q-1}u$

In dimension $N = 1$ (NLS, NLKG)

$$-\psi'' = f(\psi)$$

positive solutions are unique if $f(0) = 0$, $f'(0) < 0$ and the first positive zero ζ_0 is simple $f'(\zeta_0) > 0$ (Berestycki-Lions, 1983).

When the non-linearity is a combined power-type do we have finitely many (or uniqueness of) solutions to

$$D\psi = \omega\psi - |\psi|^{p-1}\psi + |\psi|^{q-1}\psi$$

up to translation and multiplication by $e^{i\alpha}$?