SYLLABUS OF THE COURSE "SET THEORY", SPRING 2016

2016 February 29, Week 1 - Lecture 1

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Date: 2016, June 2.

Axiom 1 (Class Construction Axiom of Georg Cantor, from (6.))**.** Given a sentence $p(x)$ there exists a *set* S such that $x \in S$ if and only $p(x)$ holds is true.

Definition 1 (The set $k\mathbf{N}$, from (7.))**.** The set $k\mathbf{N}$ is the set of all the elements n such that the statement $q_k(n) : \exists m$ s.t. $n = km$ is true.

Example 1 (A set $T \in T$, from (11.))**.** There exists a set T such that $T \in T$. We consider the sentence $p(A) : \#A = \infty$. From the Class Construction Axiom 1 there exists the set $T = \{A \mid p(A)\}$. Since the set $k\mathbf{N}$ is infinite,

$$(\forall k \in \mathbf{N})\, k\mathbf{N} \in T.$$

Therefore, $\#T = \infty$, hence $T \in T$.

Example 2 (A set $H \notin H$, from (12.))**.** We consider the sentence $q(A) : \#A$ is finite. There exists a set H such that $H \in H$. From the Class Construction Axiom 1 there exists the set $H = \{A \mid q(A)\}$. For every natural number n, $\{n\} \in H$. Then H is not finite, hence $H \notin H$.

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Paradox 1 (The Russell Paradox, from (13.))**.** We define the statement

$$p(x) : x \notin x. \tag{1}$$

By the CCA, there exists a set R such that $x \in R \Leftrightarrow p(x)$. Or, using different notation,

$$R = \{x \mid p(x)\}. \tag{2}$$

Given two sets A, B, either $A \in B$ or $A \notin B$. Then $R \in R$ or $R \notin R$. Suppose that $R \in R$. By (2), $R \in R \Rightarrow p(R)$. By (1), $p(R) \Rightarrow R \notin R$. Then, we obtain a contradiction. We consider the second case, $R \notin R$. By (1), $R \notin R \Rightarrow p(R)$. By (2), $p(R) \Rightarrow R \in R$, which gives a contradiction, again. In conclusion, the sentence $R \in R$ is neither true or false, which is a paradox.

Example 3 (A1 does not hold, from (21.))**.** In the example

	C	D
C	1	1
D	0	0

$C = \{C\}$ and $D = \{D\}$. Therefore, C and D have the same elements. Then $C = D$. However, $C \in C$ but $D \notin C$.

Example 4 (A2 does not hold, from (24.))**.** In the example

	A	B	C
A	1	0	0
B	1	1	0
C	0	1	0

are three sets A, B, C and no proper classes. A1 is satisfied. However, A2 is not satisfied. In fact, $A = \{A, B\}$ and $B = \{B, C\}$ but the intersection $A \cap B = \{B\}$ does not exist. Also, the Universal Class $\mathscr{U} = \{A, B, C\}$ does not exist.

Week 3 - Lecture 1, 2016, March 14

26. A function $f\colon A \to B$ is a relation satisfying

(F1) $$(\forall x \in A)\exists y \in B \text{ s.t. } x \sim_f y$$

(F2) $$(x \sim_f y_1) \wedge (x \sim_f y_2) \Rightarrow y_1 = y_2$$

27. range of f, ran(f): $y \in \text{ran}(f) \Leftrightarrow \exists x \in A$ s.t. $f(x) = y$
28. injective functions (INJ): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
29. surjective functions (SURJ): $(\forall y \in B)\exists x$ s.t. $f(x) = y$
30. bijective functions (BIJ): injective and surjective
31. examples of functions

31.1. $f\colon \mathbf{N} \to \mathbf{N}$, $f(n) = n + 1$

31.2. identity function: $id_A\colon A \to A$, $f(x) = x$

31.3. the restriction, Definition 2

31.4. extension, Definition 3

31.5. characteristic function, χ_B, Definition 4

31.5.1. if χ_B is not SURJ, then either $B = A$ or $B = \emptyset$, Example 5

31.5.2. if $\#A = 2 \times \#B = 2$, then χ_B is injective, Example 6

32. if f is bijective, there exists g such that $f \circ g = id_A$ and $g \circ f = id_B$
33. inverse function
34. equipotent classes, Definition 6
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36. inverse image, Definition 5
37. $\check{f}(\bar{f}(C)) \neq C$, Example 7.

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38. Union of two functions, Definition 7
39. the constant function c_b, (2) of Example 2.13, page 73
40. when A and B are finite classes $\#A = n$ and $\#B = m$
40.1. $\exists f\colon A \to B$ SURJ $\Leftrightarrow n \geq m$
40.2. $\exists g\colon A \to B$ INJ $\Leftrightarrow n \leq m$
40.3. $\exists h\colon A \to B$ BIJ $\Leftrightarrow n = m$
41. Bernstein's Lemma, Lemma 1
42. ordered pairs, (3) of Definition 1.24
43. cartesian product, (4) of Definition 1.24
44. graph of a function, Definition 8
45. the Pair Axiom, Axiom 3, page 61.

Definition 2 (Restriction, from (31.3.))**.** Given two functions $f\colon A \to B$ and $g\colon C \to B$ such that $C \subseteq A$, g is a restriction of f is $f(c) = g(c)$ for every $c \in C$.

Definition 3 (Extension, from (31.4.))**.** Given two functions $f\colon A \to B$ and $g\colon C \to B$ such that $C \subseteq A$, f is an extension of g if and only if $f(c) = g(c)$ for every $c \in C$.

Definition 4 (Characteristic Function, from (31.5.))**.** Given a subclass $B \subseteq A$, the characteristic function of B, in notation $\chi_B\colon A \to \{0, 1\}$, is defined as

$$\chi_B(x) := \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

Example 5 (From (31.5.1.))**.** If χ_B is not surjective, then either $0 \notin \operatorname{ran}(\chi_B)$ or $1 \notin \operatorname{ran}(\chi_B)$. If $0 \notin \operatorname{ran}(\chi_B)$, then $B = A$. If $1 \notin \operatorname{ran}(\chi_B)$, then $B = \emptyset$.

Example 6 (From (31.5.2.))**.** Suppose that $\#A = 2$ and $\#B = 1$. Let $x_1, x_2 \in A$ such that $\chi_B(x_1) = \chi_B(x_2)$ and $x_1 \neq x_2$. Then Only one between x_1 and x_2 belongs to B, because $\#B = 1$. Therefore, if x_1 belongs to B, then x_2 does not belong to B. Then $\chi_B(x_1) = 1 \neq \chi_B(x_2) = 0$. Similarly, if $x_1 \notin B$ and $x_2 \in B$, we obtain $\chi_B(x_1) = 0 \neq \chi_B(x_2) = 1$.

Definition 5 (Direct and inverse image, from (35.) and (36.))**.** Given two classes $C \subseteq A$ and $D \subseteq B$, we define

$$\bar{f}(C) := \{y \in B \mid \exists x \in C \text{ s.t. } f(x) = y\}$$
$$\check{f}(D) := \{x \in A \mid f(x) \in D\}.$$

Given two functions $f\colon A \to B$ and $g\colon C \to B$ such that $C \subseteq A$, f is an extension of g if and only if $f(c) = g(c)$ for every $c \in C$.

Definition 6 (From (34.))**.** Two classes A, B are equipotent if there exists $f\colon A \to B$ bijective. On this case, we use the notation $A \approx B$. Equivalently, we say that "the cardinality of $\#A$" is equal to "the cardinality of B" and the notation $\#A = \#B$ is also used.

Example 7 (From (37.))**.** We consider the function with domain $A := \{x_1, x_2, x_3\}$, in $B := \{y_1, y_2\}$, defined as $f(x_1) = f(x_2) = y_1$ and $f(x_3) = y_2$. We set $C := \{x_2\}$. Then $\bar{f}(C) = \{y_1\}$ and $\check{f}(\bar{f}(C)) = \{x_1, x_2\} \neq \{x_2\} = C$.

Definition 7 (Union of two functions, from (38.))**.** If $f\colon B \to A$ and $g\colon C \to A$ are two functions and $B \cap C = \emptyset$ then

$$f \cup g(x) := \begin{cases} f(x) & \text{if } x \in B \\ g(x) & \text{if } x \in C \end{cases}$$

is a function.

Lemma 1 (Bernstein's Lemma, from (41.))**.** If there exists an injective function $f\colon A \to B$ and an injective function $g\colon B \to A$, then there exists a bijective function $h\colon A \to B$. That is $A \approx B$.

Definition 8 (Graph of a function, from (44.))**.** Given a function $f\colon A \to B$ its graph is the class

$$\mathrm{graph}(f) := \{(x, y) \in A \times B \mid y = f(x)\}.$$

Week 4 - Lecture 1, 2016, March 21

46. If $(a, b) = (c, d)$, then $a = c$ and $b = d$, Theorem 1.26, page 46
47. Exercise 1
48. if A2 and A3 holds, given two sets x, y, the class (x, y) exists, Remark 1
49. graphs, (1) of Definition 1.30, page 50
50. domain and range of a graph, (4) of Definition 1.30, page 50 and 51
51. inverse graph, (2) of Definition 1.30, page 50

51.1. $(G^{-1})^{-1} = G$, (2) in Theorem 1.32, page 51
51.2. $\mathrm{dom}(G) = \mathrm{ran}(G^{-1})$, (1) in Theorem 1.33, page 52
51.3. $\mathrm{ran}(G) = \mathrm{dom}(G^{-1})$, (2) Theorem 1.33, page 52

52. composite graph, (3) Definition 1.30, page 50

52.1. $(G \circ H)^{-1} = H^{-1} \circ G^{-1}$, (3) in Theorem 1.32, page 51

53. Example 8.

Remark 1 (Ordered pairs exist, from (48.))**.** Given two sets x, y the ordered pair (x, y) exists. In fact, by the Class Construction Axiom (A2), there are the classes $\{x\}$ and $\{x, y\}$. By A3, $\{x\}$ and $\{x, y\}$ are sets. By A2, again, there exists the class $\{\{x\}, \{x, y\}\}$.

Exercise 1 (From (47.))**.** Given two sets $a \neq b$, the class $\{\{a, b\}\}$ is not an ordered pair. On the contrary, there are x, y sets such that

$$\{\{a, b\}\} = \{\{x\}, \{x, y\}\}.$$

The left class is a singleton while the right one is a pair. Then

$$\{a, b\} = \{x\} = \{x, y\} \Rightarrow a = b \Leftrightarrow \neg(a \neq b).$$

Example 8 (From (53.))**.** In the following example we want to recognize sets, proper classes, the universal class, pairs, ordered pairs and graphs

	a	b	c	U
a	1	1	0	1
b	0	1	1	1
c	0	0	0	1
U	0	0	0	0

Sets:	a, b, c
proper classes:	U
Universal Class:	U
pairs:	a, b, c
ordered pairs:	$a = (a, a), b = (a, b)$
$\mathscr{U} \times \mathscr{U}$:	b
graphs:	a, b, c.

The Pair Axiom holds, because all the pairs are also sets. The Class Construction Axiom does not hold, because, for instance, $\emptyset$ does not exist. Or, we can argue like this: a and b are sets, so, if A2 holds, by Remark 1, (b, a) should exist. But it does not.

By definition, graphs are subclasses of $\mathscr{U} \times \mathscr{U} = U \times U$. In turn $U \times U = \{(a, a), (a, b)\} = \{a, b\} = b$. Subclasses of b are a, b, c. Here we also observe that

$$b = \{(a, a), (a, b)\}, \quad b^{-1} = \{(a, a)\}$$

because (b, a) does not exist. If we take the inverse graph one more time $(b^{-1})^{-1} = \{(a, a)\} = a \neq b$. Therefore, here we have a clear example of how a simple property as $(G^{-1})^{-1}$ fails if A3 and A2 are not satisfied at the same time.

Week 4 - Lecture 2, 2016, March 24

54. Solutions of the exercises of week three
55. functions, (1) of Definition 2.3
56. if $\mathrm{dom}(f) = \mathrm{dom}(g)$ then $f \subseteq g \Rightarrow f = g$, Theorem 2.10, page 71
57. composition of two functions is a function, Theorem 2.16, page 76.

Week 5 - Lecture 1, 2016, March 28

58. Remarks about the assignments:
58.1. $g \circ f = id_A$ does not imply f bijective, Remark 2
58.2. $B \neq \emptyset$ does not imply $A - B \neq A$, Remark 3
59. examples of functions
59.1. id_A, (1) of Example 2.13, page 72
59.2. characteristic functions χ_B, (4) of Example 2.13, page 73
60. injective and surjective functions, Definition 2.11, page 71
61. definitions equivalent to bijective functions, Theorem 1.

Remark 2 (From (58.1.))**.** We define $f\colon A := \{0, 1\} \to B := \{a, b, c\}$ as $f(0) = a$ and $f(1) = b$. We define $: B \to A$ as $g(a) = 0$ and $g(b) = 1$. Then $g \circ f = id_A$, but f is not surjective.

Remark 3 (From (58.2.))**.** We define $A := \{0, 1\}$ and $B = \{2\}$. Then $B \neq \emptyset$ but $A - B = A$. If $C = \{1\}$, then $A - C = \{0\} \neq A$.

Theorem 1 (From (61.))**.** *Given a function $f\colon A \to B$, the following statements are equivalent:*

(a) *f is bijective*
(b) *f is invertible*
(c) *$(f^{-1} \circ f = id_A) \wedge (f \circ f^{-1} = id_B)$*
(d) *$\exists g\colon B \to A$ s.t. $(f \circ g = id_B) \wedge (g \circ f = id_A)$.*

Proof. (a) $\Rightarrow$ (b). Suppose that f is bijective. Then

$$\mathrm{dom}(f) = A \text{ and } \mathrm{ran}(f) = B \Rightarrow \mathrm{dom}(f^{-1}) = B \text{ and } \mathrm{ran}(f^{-1}) = A.$$

We prove F2:

$$(y_1, x), (y_2, x) \in f^{-1} \Rightarrow (x, y_1), (x, y_2) \in f \Rightarrow y_1 = y_2$$

because f is injective. Then $f^{-1}\colon B \to A$ is a function.
(b) $\Rightarrow$ (c). Since f^{-1} is a function, both compositions are functions. From (56.), it is enough to show that

$$f^{-1} \circ f \subseteq id_A$$

because $\mathrm{dom}(f^{-1} \circ f) = \mathrm{dom}(f) = A$, by (57.). If $(x, z) \in f^{-1} \circ f$ there exists y such that

$$(x, y) \in f \Rightarrow (y, x) \in f^{-1}, \quad (y, z) \in f^{-1}.$$

By F2, $x = z$. Thus $(x, z) \in id_A$.
Now, we prove that

$$f \circ f^{-1} \subseteq id_B.$$

Given $(z,x) \in f \circ f^{-1}$, there exists y such that

$$(z,y) \in f^{-1} \Rightarrow (y,z) \in f$$
$$(y,x) \in f.$$

By F2, $x = z$. Then $(z,x) \in id_B$.
(c) $\Rightarrow$ (d). It follows by setting $g := f^{-1}$.
(d) $\Rightarrow$ (a). Firstly, we show that

$$g \circ f = id_A \Rightarrow f \text{ INJ.}$$

Given $x_1, x_2 \in A$ and $y \in B$ such that

$$(x_1, y), (x_2, y) \in f.$$

Since

$$B = \operatorname{dom}(id_B) = \operatorname{dom}(f \circ g) \subseteq \operatorname{dom}(g),$$

y belongs to $\operatorname{dom}(g)$. Then, there exists $z \in A$ such that $(y,z) \in g$. Then

$$(x_1, z), (x_2, z) \in g \circ f \Rightarrow x_1 = x_2 = z.$$

We show that $f \circ g = id_B$ implies that f is surjective. In fact,

$$B = \operatorname{ran}(f \circ g) \subseteq \operatorname{ran}(f).$$

□

Week 5 - Lecture 2, 2016, March 31

62. Exercise using A2 and A3: if there are two sets, there exists a third set, Exercise 2
63. $f\colon A \to B$ INJ if and only $(\forall y \in B)\check{f}(\{y\})$ is a singleton, Proposition 1
64. generalized unions and intersections, (2) of Definition 1.39, page 55
64.1. singletons: $\cup\{A\} = \cap\{A\} = A$
64.2. Exercise 6, page 59
64.3. Remark: $\cup\mathscr{A}$ and $\cap\mathscr{A}$ are not defined when $\mathscr{A} = \emptyset$
65. subsets, Definition 1.46, page 61
66. the subsets Axiom, Axiom 4, page 61
66.1. Example 9
66.2. if A is a set and B is a class, then $A \cap B$ is a set, Consequence 1
66.3. if $C \subseteq D$ and C is a proper class, then D is a proper classes, Consequence 2
66.4. if A2 holds, then $\mathscr{U}$ is a proper class, Consequence 3 (check Remark 1.47, page 61)
66.5. if A2 holds, then $\emptyset$ is a set, Consequence 4
67. the Union Axiom, Axiom 5, page 61, Example 10
67.1. if A2, A3 and A5 hold, union of sets is a set, (2) of Theorem 1.48, page 62.

Exercise 2 (From (62.))**.** If A2 and A3 hold, and there are two sets, then there are infinitely many sets.

Solution. Let x, y be two sets such that $x \neq y$. By A2 and A3, the classes

$$a := \{x\}, \quad b := \{y\}, \quad c := \{x, y\}$$

exist and are set. Moreover, if any of these sets are equal to each other, we obtain $x = y$. □

Proposition 1 (A2, from (63.))**.** Given $f\colon A \to B$, f is injective if and only if for every $y \in B$, the class $\check{f}(\{y\})$ is a singleton or the empty class.

Proof. Suppose that f is injective and that $\check{f}(\{y\}) \neq \emptyset$. Let x_1, x_2 be elements of $\check{f}(\{y\}) \neq \emptyset$. Then $f(x_1), f(x_2) \in \{y\}$. Then $f(x_1) = y = f(x_2)$. Conversely, we can prove that f is injective. Let $x_1, x_2 \in A$ such that $y := f(x_1) = f(x_2)$. Then

$$x_1 \in \check{f}(\{y\}), \quad x_2 \in \check{f}(\{y\}) \Rightarrow \check{f}(\{y\}) \neq \emptyset.$$

From the assumptions, $\check{f}(\{y\})$ is empty or is a singleton. Since it is non-empty, it must be a singleton. Then $x_1 = x_2$. □

Example 9 (From (66.1.)). In the following example Axiom 4 is not satisfied

	x	A
x	1	0
A	0	0

Sets: x
proper classes: A
$x \subseteq A$

Consequence 1 (From (66.2.)). If A is a set and B is a class, then $A \cap B$ is a set.

Proof. $A \cap B \subseteq A$. By A4, B is a set. □

Consequence 2 (From (66.3.)). If $C \subseteq D$ and C is a proper class, then D is a proper class.

Proof. On the contrary, D is a set. Then, by A4, C is a set. But C is a proper class. □

Consequence 4 (From (66.5.)). If the empty class exists, and there exists a set, then $\emptyset$ is a set.

Proof. Let A be a set. Then $\emptyset \subseteq A$. By A4, $\emptyset$ is a set. □

Example 10 (From (67.)). In the following example the union of two sets exists, but it is not a set

	x	y	$\mathscr{U}$
x	1	0	1
y	0	0	1
$\mathscr{U}$	0	0	0

Sets: x, y
proper classes: $\mathscr{U}$
$x \cup y = \mathscr{U}$

Week 6 - Lecture 1, 2016, April 4

68. The Power Set, $\mathscr{P}(A)$ or 2^A, Definition 1.50, page 62
69. the Power Set Axiom, Axiom 6, page 62
70. Exercise 3
71. equivalence relations, (1), (3) and (4) of Definition 3.2, page 96

Exercise 3 (From (70.)). In the example,

$\in$	x	y	z	t	U
x	0	0	0	1	1
y	0	0	1	0	1
z	0	1	0	0	1
t	1	1	0	0	0
U	0	0	0	0	0

Sets: x, y, z, t
Proper Classes: U
Pairs: x, y, z, t
Ordered pairs: $x = (x, x)$ and $t = (t, t)$.

$$\nexists \mathscr{U} = \{x, y, z, t\}, \quad \nexists z \times z, \quad \nexists U \times U = \{x, t\}.$$

Axioms: if it is red, it is not satisfied. If it is blue, it is satisfied

A2: $\nexists \mathscr{U}$

A3: all the pairs are sets

A4:

subsets of x:	x (it is a set)
subsets of y:	y, t (they are sets)
subsets of z:	z (it is a set)
subsets of t:	t (it is a set)

A5:

$$\cup x = t, \quad \cup y = z \cup t = \{x, y\}, \quad \cup z = y, \quad \cup t = x.$$

A6:

$$2^x = \{x, \emptyset\} = \{x\} = t \text{ which is a set}$$
$$2^y = \{\emptyset, \{z\}, \{t\}, y\} = \{\ ,\ , x, y\} \text{ which does not exist}$$
$$2^z = \{\emptyset, z\} = \{\ , z\} \text{ which does not exist}$$
$$2^t = \{t, \emptyset\} = \{t, \dots\} = \{t\} = x$$

Week 6 - Lecture 2, 2016, April 7

72. An example of equivalence relation,

$$(n, m) \sim (h, k) \Leftrightarrow n + k = m + h$$

Example 11
73. equivalence classes, (2) of Definition 3.7, page 100
74. the quotient set, (3) of Definition 3.7, page 101
75. properties of the equivalence relation and the quotient set, Proposition 2
76. graphs are not functions, Exercise 4
77. order relations, (9) of Definition 3.2, page 96.

Example 11 (From (72.))**.** In $A = \mathbf{N} \times \mathbf{N}$ we consider the equivalence relation

$$(n, m) \sim (h, k) \Leftrightarrow n + k = m + h.$$

We prove that this is an equivalence relation.
(R).

$$(n, m) \sim (n, m) \Leftrightarrow n + m = m + n.$$

Then the reflexive property follows from the fact that the sum is commutative.
(S).

$$(n, m) \sim (h, k) \Rightarrow n + k = m + h \Rightarrow h + m = n + k \Rightarrow (h, k) \sim (n, m).$$

(T). Suppose that

$$(n, m) \sim (h, k) \wedge (h, k) \sim (a, b).$$

Then

$$n + k = m + h, \quad h + b = k + a.$$

We want to prove that $(n, m) \sim (a, b)$. That is, $n + b = m + a$. We have

$$\begin{aligned}(n + b) + h &= n + (h + b) = n + (k + a)\\ &= (n + k) + a = (m + h) + a = (m + a) + h\end{aligned}$$

which implies $n + b = m + a$.

Proposition 2. Given an equivalence relation (A, G) the following properties hold:

(i) for every $x \in A$, $x \in G_x$
(ii) given $x, y \in A$, there holds

$$(G_x \cap G_y \neq \emptyset) \Rightarrow (G_x = G_y).$$

(iii) $\cup(A/G) = A$.

Proof.
(i). Let $x \in A$. Since G is reflexive, $(x,x) \in G$. Then $x \in G_x$.
(ii). Let x and y be such that $z \in G_x \cap G_y \neq \emptyset$. Let $w \in G_x$. Since G is symmetric,

$$(w,x) \in G \Rightarrow (x,w) \in G.$$

Since $z \in R_x$, we have $(x,z) \in G$. The relation G is transitive. Then

$$(w,x) \in G \wedge (x,z) \in G \Rightarrow (w,z) \in G.$$

Since $z \in G_y$, we have $(y,z) \in G$. From (S), we have $(z,y) \in G$. Then

$$(w,z) \in G \wedge (z,y) \in G \Rightarrow (w,y) \in G.$$

Then $w \in G_y$. By switching the role of x and y, we obtain the reversed inclusion $G_y \subseteq G_x$. Then $G_x = G_y$.
(iii). If $x \in A$, from (i) we have $x \in G_x$. By definition of quotient set, $G_x \in A/G$. Then

$$x \in G_x \in A/G$$

gives $x \in \cup(A/G)$. Conversely, if $x \in A/G$, there exists $H \in A/G$ such that $x \in H$. Since H is an equivalence class, there exists $y \in A$ such that $H = G_y$. Because $G_y \subseteq A$, we can conclude that $x \in A$. ☐

Exercise 4 (From (76.)). Let $f\colon A \to A$ be a function. As a function, it is also a graph. In general, a function is not an equivalence relation unless $f = id_A$. In fact, let $(x,y) \in f$. Since $x \in A$, and f is an equivalence relation, $(x,x) \in f$. By property F2, $(x,y),(x,x) \in f$ implies $x = y$.

Week 7 - Lecture 1, 2016, April 11

78. Examples of order relations
78.1. **N** with $nRm \Leftrightarrow n \mid m$
78.2. if A is a class, $x \leq y \Leftrightarrow x \subseteq y$
79. comparable elements, (1) of Definition 4.6, page 117
80. fully ordered classes (FOC), (2) of Definition 4.6, page 117
81. chains, (2) of Definition 4.6, page 117
82. representation of the order relations

$$R_1 := \{(0,0),(1,1),(2,2),(0,1),(0,2)\}, \quad A = \{0,1,2\}$$

$$R_2 := id_B \cup \{(0,1),(1,2),(0,2),(0,3),(1,3)\}, \quad B = \{0,1,2,3\}$$

83. maximal chains, Definition 9.

Definition 9 (Maximal Chains, from (83.)). Given a partially ordered class $(A, \leq)$ a subclass $C \subseteq A$ is a maximal chain if it is a chain and for every chain D there holds

$$C \subseteq D \Rightarrow C = D.$$

Week 7 - Lecture 2016, April 14

84. The Hausdörff's Maximum Principle, Theorem 5.18, page 166
85. exercises from the past midterms and finals, Exercises 5-9

Exercise 5. Let $(\mathbf{N}, \leq)$ be the order relation defined as $x \leq y : \exists k$ s.t. $y = kx$. Prove that there exists a maximal chain.

Proof. A maximal chain is given by $M := \{2^n \mid n \geq 0\}$. In fact, let D be a chain such that $M \subseteq D$. We prove that $D = M$. Let $d \in D$. There exists are two non-negative integers a, b such that

$$2^a \leq d < 2^b$$

Since $2^a, 2^b \in D$, the element d is comparable to both of them. Then

$$2^a \mid d \mid 2^b.$$

Then, there are k_a and k_b such that

$$2^b = k_a d, \quad d = k_a 2^a.$$

Then

$$2^b = k \times 2^a, \quad k := k_a k_b.$$

Then $k = 2^{b-a}$. Since k_a and k_b are natural numbers, they are powers of 2. Since $d = k_a 2^a$, the element d is also a power of 2. Then $d \in M$. □

Exercise 6. For each of the following statements mark whether is true or false.

(i) A is a set if and only if there exists x such that $x \in A$
(ii) A is a proper class if and only if for every $b \in A$ there holds $b \neq A$
(iii) x is a set if $x \neq \emptyset$ and there exists B such that $x \in B$
(iv) x is a set if and only if $x \cap A$ is a set for every class A.

Solution.
(i). Both implications are false. The left implication is false. If A is set, by A2 and A4, and Consequence 1, $\emptyset$ is a set. However, there is no x such that $x \in A$; the right implication is false as well. For instance, from A2, A4 and Consequence 3, $\mathscr{U}$ is a proper class, and it is non-empty because $\emptyset \in \mathscr{U}$.

(ii). False. The left implication is true: if A is a proper class it is different from every set (including the sets which are elements of A), by A1. The right implication is false: for instance, if $\emptyset$ is a set, then $1 := \{\emptyset\}$ is a set, by A3, but is different from all its elements.

(iii). False. The right implication is true. The left implication is false: for instance, consider the $\emptyset$.
(iv). True. If x is a set, and A is a class, then $x \cap A$ is a set, by A2, A4 and Consequence 1 $x \cap A$ is a set. The converse implication is also true: if $x \cap A$ is a set for every class A, then $x \cap A$ is a set if $A = x$. Then $x \cap x = x$ is a set. □

Exercise 7 (A1-A6)**.** Let D be the class defined with the Class Construction Axiom

$$y \in D \Leftrightarrow \exists x (y = \{x\}).$$

Show that D is a proper class.

Proof. We argue by contradiction. Suppose that D is a set. Then, by A5, $\cup D$ is a set. However, we can show that $\cup D = \mathscr{U}$. In fact, given $x \in \mathscr{U}$, the singleton $\{x\}$ exists by A2. By A3, $\{x\}$ is a set. Then $\{x\} \in D$. If we set $y := \{x\}$, then

$$x \in y \in D \Rightarrow x \in \cup D.$$

By A5, $\cup D$ should be a set. But it is equal to $\mathscr{U}$. Then, by A1, $\mathscr{U}$ should be a set as well, giving a contradiction with Consequence 3. □

Exercise 8. True or false? explain!

(1) A2 is equivalent to: $\exists \emptyset, \mathscr{U}$
(2) for every classes X, A, either $X \in A$ or $X \in A'$
(3) $1 \in 0 \Rightarrow \mathscr{U}$ is a set

(4) let $G, H \subseteq A \times B$ graphs. Then $G \subseteq H \Rightarrow H^{-1} \subseteq G^{-1}$

Solution.

(1) False. Certainly A2 implies the existence of $\emptyset$ and $\mathscr{U}$. But the converse is not true. This is an example

$\in$	x	y	$\mathscr{U}$
x	0	0	0
y	0	1	1
$\mathscr{U}$	0	0	1

There exists $\emptyset = x$ and $\mathscr{U}$. However, there is no $\{x\}$.

(2) False. If X is a proper class, it does not satisfy it.

(3) True. Because $1 \in 0$ is false.

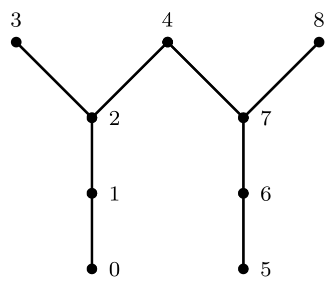
(4) False. For instance, if there are at least two sets x, y, we can define.

$$G = \{(x, x)\}, \quad H = \mathscr{U} \times \mathscr{U}.$$

Of course $G \subseteq H$. However, $G^{-1} = G$ and $H^{-1} = H$. And $H \subseteq G$ is not true, because $(x, y) \in H$ and $(x, y) \notin G$.

□

Exercise 9. The following graph represents an order relation



Find the maximal chains.

Solutions. The maximal chains are

$$\{0, 1, 2, 3\}, \quad \{0, 1, 2, 4\}, \quad \{5, 6, 7, 4\}, \quad \{5, 6, 7, 8\}.$$

□

Week 8 - Lecture 1, 2016, April 18

86. Exercises 10-14

Exercise 10 (A1 + A2 + A3)**.** Given A, B be two non-empty classes. Prove that $\cup(\cup A \times B) = A \cup B$.

Solution. Let $a \in A$ and $b \in B$ two elements. By A2 and A3, the ordered pair (a, b) exists. Then $\{a, b\} \in (a, b) \in A \times B$. Then $\{a, b\} \in \cup A \times B$. Then

$$a, b \in \{a, b\} \in \cup A \times B.$$

Therefore, $a, b \in \cup(\cup A \times B)$. Then $A \cup B \subseteq \cup(\cup A \times B)$. We prove the converse inclusion. Suppose that $x \in \cup(\cup A \times B)$. Then there exists $y \in \cup A \times B$ such that

$$x \in y \in \cup A \times B.$$

Then, there exists z such that $x \in y \in z \in A \times B$. Then, there are a and b such that $z = (a, b)$. Then $x \in y \in (a, b)$. This means that $y = \{a\}$ or $y = \{a, b\}$. If $y = \{a\}$, then $x = a \in A \subseteq A \cup B$. If $y = \{a, b\}$, then $(x = a) \vee (x = b)$ which means $x \in A \cup B$. □

Exercise 11. Let $B \neq \emptyset$ be a proper class. Show that $B \times B$ is a proper class.

Solution. Since $B \neq \emptyset$, there exists $b \in B$. Then $(b, b) \in B \times B$, so $B \times B$ is non-empty and it is possible to consider $\cup B \times B$. Since $(b, b) \in B \times B$, we have $\{b\} \in \cup B \times B$. Therefore, $\cup B \times B$ is non-empty and it is possible to define $\cup(\cup B \times B)$.

We argue by contradiction. Suppose that $B \times B$ is a set. Then, by A5, $\cup B \times B$ is a set. Again, by A5, $\cup(\cup B \times B)$ is a set. Finally, by Exercise 10, $\cup(\cup B \times B) = B \cup B = B$ and we obtain that B is a set and, thus, a contradiction. □

Exercise 12. Let $(A, \leq)$ be a partially ordered class. We consider the relation:

$$xGy \Leftrightarrow x \text{ is comparable to } y.$$

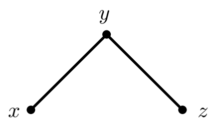
Check whether G is symmetric, reflexive and transitive.

Solution.

(R). G is reflexive: xGx if and only if x is comparable to x. Which is true, because $x = x$ (therefore $x \leq x$).

(S). G is symmetric: if xGy, then x is comparable to y, that is $(x \leq y) \vee (y \leq x)$, which is equivalent to $(y \leq x) \vee (x \leq y)$.

(R). G is not transitive: there could be x, y, z such that x is comparable to y, y is comparable to z, but x is not comparable to z, as the next example shows:



Formally, we are considering the order relation

$$A = \{x, y, z\}, \quad R = id_A \cup \{(x, y), (x, z)\}.$$

□

Exercise 13 (A1 + A2 + A3)**.** Given a class $A \neq \emptyset$, we define the order relation

$$B_1 \leq B_2 \Leftrightarrow B_1 \subseteq B_2$$

for every $B_1, B_2 \in \mathscr{P}(A)$. Suppose that $(\mathscr{P}(A), \leq)$ is a fully-ordered class. Show that A is a set.

Solution. We can prove that A is a singleton. By A3, A is a set. Let $x, y \in A$ be two elements. From A2, $\{x\}$ and $\{y\}$ exist. From A3, $\{x\}$ and $\{y\}$ are sets. Then $\{x\}, \{y\} \in \mathscr{P}(A)$. Since $\mathscr{P}(A)$ is a fully ordered class, $\{x\}$ is comparable to $\{y\}$. Then

$$(\{x\} \subseteq \{y\}) \vee (\{y\} \subseteq \{x\}).$$

In both cases, $x = y$. Since A is non-empty, A is a singleton. Then A is a set. □

Exercise 14. In the example

$\in$	x	y	z	T	U
x	0	0	0	1	0
y	0	1	1	0	0
z	1	0	1	1	0
T	0	0	0	0	0
U	0	0	0	0	0

find sets, proper classes, singletons, pairs, ordered pairs. Moreover, determine whether each of the following classes exists and to which of the classes x, y, z, T, U they correspond:

$$z \times x, \quad y \times z, \quad (y \times z) \times T, \quad y \times (z \times T)$$
$$\mathscr{P}(x), \quad \mathscr{P}(y), \quad \mathscr{P}(z), \quad \mathscr{P}(T)$$
$$\cup x, \quad \cap T$$
$$id_z \colon z \to z$$
$$\text{a bijective function from } x \text{ to } y$$
$$\text{a bijective function from } y \text{ to } x.$$

Solution.
Sets: x, y, z
proper classes: T, U
pairs: x, y, z, T
ordered pairs: $y = (y, y)$, $z = (y, z)$, $T = (z, y)$.
Now we look at the products:

$$z \times x = x, \quad y \times z = z, \quad (y \times z) \times T = z \times T = x, \quad y \times (z \times T) = y \times x = x.$$

We look at the power sets:

$$\nexists \mathscr{P}(x) = \{x\}, \quad \mathscr{P}(y) = y$$
$$\nexists \mathscr{P}(z) = \{\{y\}, \{z\}, \{y, z\}\} = \{x, y, z\}, \quad \nexists \mathscr{P}(T) = \{x\}.$$

Therefore, the Power Axiom is satisfied. Because, for every set A, either $\mathscr{P}(A)$ does not exist or $\mathscr{P}(A)$ is a set. We have

$$\cup x = z, \quad \cap T = x \cap z = x.$$

Functions. The identity id_z exists, but it is not a function. In fact, $id_z = \{(y, y), (z, z)\} = \{y\} = y$, but $\mathrm{dom}(id_z) = \{y\} = \{y, z\}$. Because x and y are singletons, there is (at most) one function from x to y, namely

$$f := \{(z, y)\} = \{T\} = \emptyset = U.$$

So, f is not a function from x to y. However, there is one bijective function from y to x which is $g := \{(y, z)\} = \{z\} = x$. Then $x \colon y \to x$ is a bijective function. Therefore, $\neg(x \approx y)$ but $y \approx x$. □

2016, April 25 - Week 9, Lecture 1

87. Solutions of the exercises of the midterm exam
88. if A, B are sets, then $A \times B$ is a set, Theorem 1.54, page 63
89. given a graph $f \subseteq A \times B$, satisfying F2, $f \colon \mathrm{dom}(f) \to B$ is a function.

2016, April 28 - Week 9, Lecture 2

90. $\#A \leq \#B$: there exists $f\colon A \to B$ injective
91. $\#B \geq \#A$: there exists $g\colon B \to A$ surjective
92. if $\#A \leq \#B$, then $\#B \geq \#A$, Theorem 2.24, page 80
93. Choice Function, Definition 5.3, page 156.

2016, May 2 - Week 10, Lecture 1

94. The Choice Axiom (A8), Axiom 8, page 158
95. an application: if B is a set, $\#B \geq \#A$ then $\#A \leq \#B$, Theorem 5.9, page 160
96. A8 is equivalent to the Hausdörff Maximum Principle Theorem 5.26, page 171
97. solution of the Exercise 15 (Homeoworks Week 7).

Exercise 15 (From (97.))**.** Prove that $\mathbf{N} \approx \mathbf{N} \times \{0,1\}$.

A bijective function is defined as

$$g(n) := \begin{cases} \left(\frac{n}{2}, 0\right) & \text{if } n \in 2\mathbf{N} \\ \left(\frac{n-1}{2}, 1\right) & \text{if } n \in 2\mathbf{N} - 1. \end{cases}$$

2016, May 9 - Week 11, Lecture 1

98. Given two sets A, B, then $\#A \leq \#B$ or $\#B \leq \#A$

Theorem 2 (A1-A6+HMP)**.** Given two non-empty sets A, B, either $\#A \leq \#B$ or $\#B \leq \#A$.

Proof. We define the S the class of the injective functions f such that $\mathrm{dom}(f) \subseteq A$ and $\mathrm{ran}(f) \subseteq B$. Since A and B are sets, $A \times B$ is a set. Since we $f \subseteq A \times B$, we have $f \in \mathscr{P}(A \times B)$. By A6 and A4, S is a set. In S we define the order relation

$$f \leq g \Leftrightarrow f \subseteq g.$$

By the Hausdörff Maximum Principle, there exists a maximal chain C. We set $h := \cup C$. Clearly,

$$(\forall f \in C) f \subseteq h. \tag{3}$$

We claim that

(i) h is an injective function
(ii) $\mathrm{dom}(h) = A$ or $\mathrm{ran}(h) = B$.

(i). Let $(a, b_1), (a, b_2)$ be two elements of h. Then there exist f_1, f_2, such that

$$(a, b_1) \in f_1 \in C, \quad (a, b_2) \in f_2 \in C.$$

Since C is a chain, the elements f_1, f_2 are comparable. Then $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$. On the first case, we have

$$(a, b_1), (a, b_2) \in f_1.$$

Since f_1 is a function, we obtain $b_1 = b_2$. We prove that h is injective. Given $(a_1, b), (a_2, b) \in h$, there are f_1, f_2, such that

$$(a_1, b) \in f_1 \in C, \quad (a_2, b) \in f_2 \in C.$$

Since C is a chain, the elements f_1, f_2 are comparable. Then $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$. On the first case, we have

$$(a_1, b), (a_2, b) \in f_1.$$

Since f_1 is injective, a function, we obtain $a_1 = a_2$.

(ii). Suppose that $\mathrm{dom}(f) \neq A$ and $\mathrm{ran}(f) \neq B$. Then there are $a_* \in A - \mathrm{dom}(f)$ and $b_* \in B - \mathrm{ran}(f)$. We define

$$h_* := h \cup \{(a_*, b_*)\} \supsetneq h. \tag{4}$$

which is injective function. From (3) and the inclusion above, $D := C \cup \{h_*\}$ is a chain. Since C is a maximal chain, $D = C$. Then $h_* \in D$. From (3), we obtain $h_* \subseteq h$ which contradicts (4).

Now, if $\mathrm{dom}(h) = A$, then $h\colon A \to B$ is an injective function and $\#A \leq \#B$.

We look at the case $\mathrm{ran}(h) = B$. If $\mathrm{dom}(h) = A$, then h is bijective and $\#A = \#B$. Suppose that $\mathrm{dom}(h) \subsetneq A$. Since $B \neq \emptyset$, there exists $b_0 \in B$. We define

$$h_0 := h \cup \{(x, b_0) \mid x \in A - \mathrm{dom}(h)\}.$$

Then $\mathrm{dom}(h_0) = A$ and $\mathrm{ran}(h_0) = B$. So, $h_0\colon A \to B$ is surjective, then $\#A \geq \#B$. By Theorem 5.9 (page 160 of the textbook), $\#B \leq \#A$, which concludes the proof. □

2016, May 12 - Week 11, Lecture 2

99. Successor of a set, Definition 6.1, page 174
100. Successor sets, Definition 6.3, page 175
101. Axiom of Infinity, Axiom 9, page 175
102. definition of ω, Definition 10
103. ω is a successor set, Proposition 3
104. definition of $\mathbf{N}$, Definition 6.6, page 176
105. if $y \in \omega$, then $y^+ \neq 0$, Theorem 6.7, page 176
106. Finite Mathematical Induction, Theorem 6,8, page 176
107. transitive sets, Definition 6.10, page 176
108. every $y \in \omega$ is a transitive set, Lemma 6.11, page 177
109. given $y, z \in \omega$, if $y^+ = z^+$, then $y = z$, Theorem 6.12, page 177
110. homeworks: prove that $\mathscr{P}(\mathscr{U}) = \mathscr{U}$.

Definition 10 (From (102.))**.** We set

$$G := \{Y \mid Y \text{ is a successor set}\}.$$

By Axiom 9, $G \neq \emptyset$. Therefore, we can define $\omega := \cap G$.

Proposition 3 (From (103.))**.** ω is a successor set.

Proof. ω is a set, because $\omega \subseteq X$ and X is a set (A4). Clearly $0 \in \omega$ because $0 \in Y$ for every $Y \in G$. Now, suppose that $y \in \omega$. Then $y \in Y$ for every $Y \in G$. Since Y is a successor set, $y^+ \in Y$ for every $Y \in G$. Then $y^+ \in \cap G = \omega$. □

2016, May 16 - Week 12, Lecture 1

111. The order relation in ω: $n \leq m$ if $n \in m$ or $n = m$, Theorem 6.28, page 187
112. least elements, (4) of Definition 4.18, page 126
113. minimal elements, (2) of Definition 4.18, page 126
114. Well-ordered classes (WOC), Definition 4.50, page 142
115. WOC $\Rightarrow$ FOC, Remark 4.51, page 142.

2016, May 19 - Week 12, Lecture 2

116. Solutions of the exercises 2, 5 of Week 7 and exercises 1, 2, 3, 4 and 5 of Week 9.

2016, May 20 - Week 12, Lecture 3

117. If $n < m$, then $n^+ \leq m$, Lemma 6.30, page 188
118. $(\omega, \leq)$ is well-ordered, Theorem 6.31, page 188
119. the notation $\#A < \#B$: there is no surjective function from A to B
120. finite sets, (1) of Definition 7.23, page 202
121. infinite sets, (2) of Definition 7.23, page 202
122. countable sets (or denumerable sets), A is countable if $\#A \leq \#\omega$
123. $\mathbf{N} \times \mathbf{N}$ is countable.
124. $\mathbf{R}$ is not countable, Theorem 7.4, page 196
125. solution of the exercise 6 of Week 9.

2016, May 23 - Week 13, Lecture 1

126. Solution of the Exercise 16 and Exercise 17
127. $\cup n \subseteq n$, Proposition 4
128. $\cup n \in \omega$ for every $n \in \omega$, Proposition 5
129. $\cup(n^+) = n$, Proposition 6

Exercise 16 (From (126.)). $\mathscr{P}(\mathscr{U}) = \mathscr{U}$.

Solution. $\mathscr{P}(\mathscr{U}) \subseteq \mathscr{U}$ because every class is a subclass of $\mathscr{U}$. We prove the converse inclusion: if $B \in \mathscr{U}$, then B is a set. Since every class is a subclass of $\mathscr{U}$, we have

$$B \text{ is a set and } B \subseteq \mathscr{U}$$

which means $B \in \mathscr{P}(\mathscr{U})$. □

Exercise 17 (From (126.)). Find a maximal chain in $(\mathscr{P}(\mathbf{N}), \subseteq)$.

Proof. We define $\mathbf{N}_k := \{1, 2, \dots, k\}$ for every $k \in \mathbf{N}$. $\mathscr{C} := \{\emptyset, \mathbf{N}_k, \mathbf{N} \mid 1 \leq k\}$. Clearly, any two sets in $\mathscr{C}$ are comparable to each other, then it is a chain. We prove that it is a maximal chain. Let $\mathscr{D}$ be a chain such that $\mathscr{C} \subseteq \mathscr{D}$. We prove that $\mathscr{D} \subseteq \mathscr{C}$. Suppose A is in $\mathscr{D}$. Then, we have two cases:
(1). $\mathbf{N}_k \subseteq A$ for every $k \geq 1$. This implies $A = \mathbf{N}$. Then $A \in \mathscr{C}$.
(2). There exists k_0 such that $\neg(\mathbf{N}_{k_0} \subseteq A)$. Since $\mathscr{D}$ is a chain, there holds $A \subsetneq \mathbf{N}_{k_0}$. Then A is a finite set. If $A = \emptyset$, then $A \in \mathscr{C}$. If $A \neq \emptyset$ then we define $k_1 := \#A$. We claim that $A = \mathbf{N}_{k_1}$. In fact, A is comparable to $\mathbf{N}_{k_1}$. Then, for instance $A \subseteq \mathbf{N}_{k_1}$. However $\#A = \#(\mathbf{N}_{k_1})$ implies $A = \mathbf{N}_{k_1}$. Then $A \in \mathscr{C}$. □

Proposition 4 (From (127.)). For every $n \in \omega - \{0\}$ there holds $\cup n \subseteq n$

Solution. If $x \in \cup n$, there exists y such that $x \in y \in n$. Since n is transitive, $x \in y \subseteq n$ which implies $x \in n$. □

Proposition 5 (From (128.)). For every $n \in \omega - \{0\}$, there holds $\cup n \in \omega$.

Solution. We define $L := \{n \in \omega - \{0\} \mid \cup n \in \omega\}$. We prove that $L = \omega$. Since $0 \in L$, we only need to prove that $n \in L \Rightarrow n^+ \in L$. If $n = 0$, then $n^+ = 1$ and $\cup(n^+) = 0 \in \omega$. Now, suppose that $n \neq 0$. Then

$$\cup(n^+) = \cup(n \cup \{n\}) = (\cup n) \cup n \subseteq n \in \omega.$$

The last inclusion follows from Proposition 4. The last membership relation follows from the simple $n \in \omega$. □

Proposition 6 (From (129.)). For every $n \in \omega$ there holds $\cup(n^+) = n$.

Solution. In fact, $\cup(n^+) = \cup(n \cup \{n\}) = (\cup n) \cup n = n$. The last equality follows from Proposition 4. □

2016, May 26 - Week 13, Lecture 2

130. For every $n \in \omega - \{0\}$ there holds $(\cup n)^+ = n$, Proposition 7
131. if $n \approx m$ then $n = m$, Proposition 8
132. $A < 2^A$, Theorem 7.5, page 196
133. definition of cardinals, Definition 8.2, page 213
134. sum of cardinals, Definition 8.5, page 214
135. products of cardinals, Definition 8.5, page 214.

Proposition 7 (From (130.))**.** For every $n \in \omega - \{0\}$ there holds $(\cup n)^+ = n$, Proposition 7.

Solution. We use the Induction Principle. We define

$$L := \{n \in \omega - \{0\} \mid (\cup n)^+ = n\}.$$

Clearly, $0 \in L$. Suppose that $n \in L$. Then $n^+ \neq 0$. From Proposition 6, $(\cup n^+)^+ = n^+$. □

Proposition 8 (From (131.))**.** If $n \approx m$ then $n = m$.

Proof. We use the induction and define

$$L := \{n \in \omega - \{0\} \mid \forall m(n \approx m \Rightarrow n = m)\} \cup \{0\}.$$

Clearly, $0 \in L$. Suppose that $n \in L$. We wish to prove that

$$\forall m(n^+ \approx m \Rightarrow n^+ = m).$$

Suppose that $n^+ \approx m$. Then $n \approx \cup m$. From Proposition 5, $\cup m \in \omega$. Since $n \in L$ then $n = \cup m$. Then $n^+ = (\cup m)^+$. From Proposition 6, $n^+ = m$. □

2016, May 30 - Week 14, Lecture 1

136. If $f: A \to B$ is bijective and $C \subseteq A$, then $A - C \approx B - \bar{f}(C)$, Proposition 9
137. given a class A and $a, b \in A$, then $A - \{a\} \approx B - \{b\}$, Proposition 10
138. if $n, m \in \omega$, then $n \approx m$, then $n = m$, Proposition 11
139. if $C \approx C'$, $D \approx D'$ and
$$C \cap D = C' \cap D' = \emptyset$$
then $C \cup D \approx C' \cup D'$, Proposition 12
140. if $C \approx C'$ and $D \approx D'$, then $C \times D \approx C' \times D'$, Proposition 13
141. exponentiation of cardinals, Definition 8.8, page 215
142. the Bernstein's Lemma, Theorem 8.14, page 219
143. Exercise 18,
144. Exercise 19.

Proposition 9 (From 136.)**.** If $f: A \to B$ is bijective and $C \subseteq A$, then $A - C \approx B - \bar{f}(C)$

Proof. We consider the restriction of f to the subclass $A - C$. We show that $\bar{f}(A - C) \subseteq B - \bar{f}(C)$. In fact, given $y \in \bar{f}(A - C)$, there exists $x \in A - C$ such that $f(x) = y$. We claim that $y \notin \bar{f}(C)$. Otherwise, there exists $x' \in C$ such that $f(x') = y$. Since $x \notin C$ we have $x' \neq x$ but $f(x) = f(x')$. However, this contradicts the fact that f is injective. Then

$$f: A - C \to B - \bar{f}(C)$$

is a function. We prove that f is surjective. In fact, given $y \in B - \bar{f}(C)$, there exists $x \in A$ such that $f(x) = y$. Since $y \notin \bar{f}(C)$, clearly, $x \notin C$. Then $x \in A - C$. ☐

Proposition 10 (From 137.)**.** Given a class A and $a, b \in A$, then $A - \{a\} \approx B - \{b\}$.

Proof. A bijective function is given by $g := \{(x, x) \mid x \notin \{a, b\}\} \cup \{(b, a)\}$. ☐

Proposition 11 (From 138.)**.** Given $n, m \in \omega - \{0\}$, there holds $n \approx m \Rightarrow n = m$.

Proof. We use the induction. We define

$$L := \{n \in \omega - \{0\} \mid n \approx m \Rightarrow n = m\} \cup \{0\}.$$

Clearly, $0 \in L$. Suppose that $n \in L$ and $n^+ \approx m$. We want to prove that $n^+ = m$. Then there exists a bijective function $f\colon n^+ \to m$. We define $a := f(n)$. By Proposition 9, applied with $C = \{n\}$, we have $n^+ - \{n\} \approx m - \{a\}$. By Proposition 10, $m - \{a\} \approx m - \{\cup m\}$. Then

$$n^+ - \{n\} \approx m - \{\cup m\}.$$

Since $n \notin n$, the left set is equal to n. From Proposition 7, the right set is equal to $\cup m$. Therefore, $n \approx \cup m$. Since $n \in L$, we have $n = \cup m$. Then $(n)^+ = (\cup m)^+$. By Proposition 7, $(\cup m)^+ = m$. ☐

Proposition 12 (From 139.)**.** If $C \approx C'$, $D \approx D'$ and $C \cap D = C' \cap D' = \emptyset$, then $C \cup D \approx C' \cup D'$.

Proof. Let $f\colon C \to C'$ and $g\colon D \to D'$ be two bijective functions. Then $h := f \cup g\colon C \cup D \to C' \cup D'$ is a bijective function. ☐

Proposition 13 (From 140.)**.** If $C \approx C'$ and $D \approx D'$, then $C \times D \approx C' \times D'$.

Proof. Let $f\colon C \to C'$ and $g\colon D \to D'$ be two bijective functions. Then $h(c, d) = (f(c), g(d))$ is a bijective function from $C \times D$ to $C' \times D'$. ☐

Exercise 18 (From 143.)**.** There is no set y such that $y^+ = \{2\}$.

Solution. On the contrary, we have $y \in y^+ = \{2\}$. Then $y \in \{2\}$ implies $y = 2$. This implies $2^+ = \{0, 1, 2\} = \{2\}$ which implies $0 = 1 = 2$, which is not possible because $0 = \emptyset$ and $2 \neq \emptyset$. ☐

Exercise 19 (From 144.)**.** There exists a proper class $\mathscr{A}$ such that $\mathscr{A} \neq \mathscr{U}$ and

(i) $0 \in \mathscr{A}$
(ii) $y \in \mathscr{A} \Rightarrow y^+ \in \mathscr{A}$.

Solution. For instance, there is $\mathscr{A} := \mathscr{U} - \{\{2\}\}$. Clearly, $0 \in \mathscr{A}$, so (i) is satisfied. Suppose that $y \in \mathscr{A}$. By Exercise 19, it is not possible that $y^+ = \{2\}$. That is $y^+ \in \mathscr{A}$. ☐

2016, May 30 - Week 14, Lecture 2

145. Exercises 20, 21, 22, 23, 24, 25, 26, 27, 28.

Exercise 20. Is it true that for every class $\mathscr{A}$, there holds $\mathscr{A} \subseteq \cup\mathscr{A}$? Is it true that for every class $\mathscr{A}$, there holds $\cup\mathscr{A} \subseteq \mathscr{A}$?

Proof. Both inclusion are false, and there is a counterexample for both. Consider $\mathscr{A} := \{1\}$. Then $\cup\mathscr{A} = 1$. Clearly, $\{1\} \nsubseteq 1 = \{0\}$, because $0 \neq 1$. Similarly, $\{0\} \nsubseteq \{1\}$ for the same reason. ☐

Exercise 21. $\omega \notin \omega$.

Proof. If $\omega \in \omega$, then $n := \omega$ is a natural number. Thus, $n \in n$, which is not possible. $\square$

Exercise 22. There is no set A such that $A^+ = \omega$.

Proof. No, there is not. Otherwise, $A \cup \{A\} = \omega$ gives $A \in \omega$. Since ω is a successor set, $A^+ \in \omega$. Then $\omega \in \omega$, which contradicts Exercise 21. $\square$

Exercise 23. Let B be a proper class. Show that $B \times B$ is a proper class.

Proof. We argue by contradiction. Suppose that $B \times B$ is a set. By Exercise 1 of Week Seven, $\cup(\cup B \times B) = B \cup B = B$. By A5, $\cup(\cup B \times B)$ is a set. Then B is a set, and we obtain a contradiction. $\square$

Exercise 24. If $A - B \approx B - A$, then $A \approx B$

Proof. Let f be a bijective function from $A - B$ to $B - A$. We define $g := f \cup id_{A \cap B}$. $\square$

Exercise 25. Let $C \subseteq A$ and $D \subseteq B$ such that $A \approx B$ and $C \approx D$. Is it true that $A - C \approx B - D$?

Proof. It is false. For instance, consider $A := \omega$, $C = 2\omega$, $B = \omega$ and $D = \omega - \{0\}$. Then

$$A = B, \quad C = 2\omega \approx \omega \approx \omega - \{0\} = D.$$

However, $A - C = 2\omega - 1$ is not equipotent to $B - D = \{0\}$. $\square$

Exercise 26. For each of the following statements say whether it is true or false.

(i) $A = \{x \,|\, \exists y(y^+ = x)\}$ is a successor class

(ii) $B = \{y \,|\, y \text{ is transitive}\}$ is a successor class.

Proof. A is not a successor class, because $0 \notin A$; B is a successor class. In fact, $0 \in B$. Morevoer, if y is transitive, then we can show that y^+ is transitive. In fact, given $x \in y^+$, either $x \in y$, implying $x \subseteq y \subseteq y^+$, because y is transitive, or $x = y \subseteq y^+$. $\square$

Exercise 27. Show that $(\omega \times \omega) \times \omega \approx \omega$.

We apply the relation $\omega \times \omega \approx \omega$ two times:

$$(\omega \times \omega) \times \omega \approx \omega \approx (\omega) \times \omega \approx \omega.$$

Exercise 28. If A is not finite, then $\omega \leq A$.

Proof. We consider the class

$$S := \{f \subseteq \omega \times A \mid \mathrm{dom}(f) \in \omega \text{ and } f \text{ INJ}\}.$$

From Exercise 6 of Week 9, S is a set. We consider the order relation $f \leq g : f \subseteq g$. By the Hausdörff Maximum Principle, there exists a maximal chain $\mathscr{C} \subseteq S$. We define $f_* := \cup\mathscr{C}$.

$f_* \subseteq \omega \times A$ is an injective function. We claim that $\mathrm{dom}(f_*) = \omega$. On the contrary, $\mathrm{dom}(f_*) \subsetneq \omega$. We set

$$B := \{m \in \omega \mid m \notin \mathrm{dom}(f_*)\}.$$

We set $n := \min(B)$. Then $\mathrm{dom}(f_*) = \min(B)$. Moreover, there exists $a \in A - \mathrm{ran}(f_*)$. Otherwise $f_* \colon n \to A$ would be bijective and $A \approx n$ which contradicts the assumption that A is not finite. Then, we define

$$f_{**} := f_* \cup \{(n, a)\}$$

which is injective. Then $\mathscr{D} := \mathscr{C} \cup \{f_{**}\} \supsetneq \mathscr{C}$ contradicts the fact that $\mathscr{C}$ is a maximal chain.

Then $\mathrm{dom}(f_*) = \omega$. Then $f_* \colon \omega \to A$ is an injective function. $\square$