Final Exam, 2016, June 9

Instructions.

(1) On this page, please write only your name and id. number,
(2) do not use pencils
(3) you can use any theorem or exercise contained in the syllabus,
(4) Exercises 1, 2, 3 and 4 are mandatory,
(5) choose only one exercise between 5 and 6,
(6) choose only one exercise among 7, 8, 9 and 10.
(7) In exercises 2 and 4 no explanation or proof is needed,
(8) in all the exercises you can assume that all the axioms are satisfied
(9) in exercise 6, write sets, instead of diagrams. For example,

"$A = \{0, 1\}$, $R = \{(0, 0), (1, 1), (0, 1)\}$" instead of "  ".

Do **ALL** Exercises 1-4.

Exercise 1. Is it true that $\mathscr{P}(0) = 1$?

Exercise 2. Find three infinite sets such that $\#A < \#B < \#C$.

Exercise 3. and what about $\mathscr{P}(2) = 4$?

Exercise 4. Find a maximal chain in $(\mathscr{P}(4), \subseteq)$.

Choose only one between Exercise 5 and 6.

Exercise 5. Show that there is no set y such that $y^+ = \{1, 2\}$?

Exercise 6. Find an order relation (A, R) with two minimal elements $m_1 \neq m_2$.

Choose only one among Exercises 7, 8, 9, 10.

Exercise 7. Let $\mathcal{M}$ be the set of maximal chains of $\mathscr{P}(4)$. What is $\min\{0 \leq m \mid \#\mathcal{M} \leq 10^m\}$?

Exercise 8. In (A, R), if there are two minimal elements, there are two maximal chains. [1]

Exercise 9. Given $\#A = n$ and $\#B = m$, how many maximal chains does $(F(A, B), \subseteq)$ have?

Exercise 10. Let (A, R) be such that $\#A = \#\omega$. Prove that $\#R = \#A$. [2]

Notations:

$F(A, B)$: the class of all the functions from A to B.
(A, R) an ordered class.
$\mathscr{P}(A) : B \in \mathscr{P}(A) \Leftrightarrow B \subseteq A$ and B is a set.

Date: 2016, June 13.

[1] *Hint:* Use the Hausdörff Maximum Principle

[2] *Hint:* Use Bernstein's Lemma

Solutions

Exercise 1 (28pts)**.** Is it true that $\mathscr{P}(0) = 1$?

Solution. It is true. $\mathscr{P}(0) = \mathscr{P}(\emptyset) = \{\emptyset\} = \{0\} = 1$. □

Exercise 2 (7pts)**.** Find three infinite sets such that $\#A < \#B < \#C$.

Solution. $A := \omega$, $B := \mathscr{P}(\omega)$, $C := \mathscr{P}(\mathscr{P}(\omega))$. By the Power Set Axioms, B and C are sets. Since A and B are sets, $A < \mathscr{P}(A) = B$ and $B < \mathscr{P}(B) = C$. □

Exercise 3 (28pts)**.** and what about $\mathscr{P}(2) = 4$?

Solution. It is false. For instance $3 = \{0, 1, 2\} \in 4$ but $3 \notin \mathscr{P}(2) = \{0, 1, \{1\}, 2\}$. □

Exercise 4 (22pts)**.** Find a maximal chain in $(\mathscr{P}(4), \subseteq)$.

Solution. $5 = \{0, 1, 2, 3, 4\}$ is a maximal chain. □

Choose only <u>one</u> between Exercise 5 and 6 (6 points each exercise).

Exercise 5. Show that there is no set y such that $y^+ = \{1, 2\}$?

Solution. Let y be a set such that $y \cup \{y\} = \{1, 2\}$. Then $y \in \{1, 2\}$. Then $y = 1$ or $y = 2$. If $y = 1$, then $y^+ = \{0, 1\} \neq \{1, 2\}$ because $1 \neq 2$. If $y = 2$, then $y^+ = \{0, 1, 2\} \neq \{1, 2\}$ because both 1 and 2 are non-empty. □

Exercise 6. Find an order relation (A, R) with two minimal elements $m_1 \neq m_2$.

Solution. $A = \{0, 1\}$ with $R = \{(0, 0), (1, 1)\}$. □

Choose only <u>one</u> among Exercises 7, 8, 9, 10 (9 points each exercise)

Exercise 7. Let $\mathcal{M}$ be the set of maximal chains of $\mathscr{P}(4)$. What is $\min\{0 \leq m \mid \#\mathcal{M} \leq 10^m\}$?

Solution. There are 24 maximal chains. Then $m = 2$. □

Exercise 8. In (A, R), if there are two minimal elements, there are two maximal chains.

Solution. Let m_1 and m_2 two minimal elements. We define

$$B_1 := \{x \in A \mid m_1 \leq x\}, \quad B_2 := \{x \in A \mid m_2 \leq x\}.$$

In the order relation, (B_i, R_i) there is a maximal chain C_i, where $R_i = R \cap (B_i \times B_i))$. Clearly, $m_i \in C_i$. In fact, since $C_i \subseteq B_i$, for every element of $c \in B_i$ there holds $m_i \leq c$. Then $c \in C_i$. We claim that C_1 and C_2 are maximal chains in A. Let D be a chain such that $C_i \subseteq D$. If $x \in D$, then x should be comparable to every element of C_i. Since $m_i \in C_i$, x should be comparable to m_i, that is $x \leq m_i$ or $x \geq m_i$. In the first case, we obtain $x = m_i \in C_i$ which implies $x \in B_i$; in the second case, $x \in B_i$, by definition of B_i. Then $D \subseteq B_i$. Since C_i is a maximal chain in B_i, we obtain $D = B_i$. Finally, we prove that $C_1 \neq C_2$. On the contrary, m_1 and m_2 would be comparable to each other. Then $m_1 = m_2$ because they both are minimal elements, and we obtain a contradiction with the assumption that $m_1 \neq m_2$. □

Exercise 9. Given $\#A = n$ and $\#B = m$, how many maximal chains does $(F(A, B), \subseteq)$ have?

Solution. Given two functions such that $f \subseteq g$, there holds $f = g$. Then maximal chains are singletons. Then, there are m^n maximal chains. □

Exercise 10. Let (A, R) be such that $\#A = \#\omega$. Prove that $\#R = \#A$.

Solution. Since $A \approx id_A \subseteq R$, we have $\#A \leq \#R$. Since

$$R \subseteq A \times A \approx \omega \times \omega \approx \omega \approx A$$

we have $\#R \leq \#A$. By the Bernstein's Lemma, $A \approx R$. □