SOLUTIONS OF EXERCISES 1 AND 2, PAGE 31 OF THE BOOK

Suppose that $m : \mathscr{A} \to [0, +\infty]$ is a measure function on a σ-algebra which is σ-additive. Then

(1) m is *monotone*. That is, for every $A, B \in \mathscr{A}$, there holds $m(A) \leq m(B)$
(2) if there exists $A \in \mathscr{A}$ such that $m(A) < \infty$. Then $m(\emptyset) = 0$.

Solution.

(1) We can write
$$B = A \cup (B \cap A^c).$$
Since $A \in \mathscr{A}$, $A^c \in \mathscr{A}$ and $B \cap A^c \in \mathscr{A}$. Since the two sets are disjoint from each other, we have
$$m(B) = A \cup (B \cap A^c) = m(A) + m(B \cap A^c).$$
Since $m \geq 0$, we obtain $m(B) \geq m(A)$
(2) by the σ-additivity property, we have
$$m(A) = m(A \cup \emptyset) = m(A) + m(\emptyset).$$
Then $m(\emptyset) = 0$.

□

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