A NON-MEASURABLE SET: THE VITALI'S SET

In the closed interval $I := [0,1]$ we consider the equivalence relation:

$$xGy: \ x - y \in \mathbf{Q}.$$

Let X be the quotient set of this equivalence relation. By the Choice Axiom, there exists a Choice Function $\phi\colon 2^I \to I$ such that $\phi(x) \in x$ for every $x \in I$. We define

$$S := \bar{\phi}(X).$$

The set S is called *Vitali's Set*. The set S has the following properties:

(1) $S \cap G_x \neq \emptyset$ for every $x \in I$

(2) $S \cap G_x \approx \mathbf{N}_1$.

In other words, S intersects every equivalence class in a single point.

Proposition 1.

(i) Given $q_1, q_2 \in Q$ such that $q_1 \neq q_2$, we have $(q_1 + S) \cap (q_2 + S) = \emptyset$.

(ii) $\bigcup_{q \in Q \cap [-1,1]} (q + S) \supseteq I$

(iii) $q + S \subseteq [-1,2]$ for every $q \in Q \cap [-1,1]$.

Proof.

(i) Suppose that $(q_1 + S) \cap (q_2 + S) \neq \emptyset$. Then there exists $r \in \mathbf{R}$ such that

$$r = q_1 + s_1 = q_2 + s_2 \Rightarrow s_1 - s_2 \in \mathbf{Q} \Rightarrow s_1 G s_2.$$

By (2), $s_1 = s_2$ which implies $q_1 = q_2$.

(ii) given $x \in I$, we have $x \in G_x$, its equivalence class. By (1), $\exists s \in S \cap G_x$. Then $x = q + s$ for some $q \in Q$. Since $s, x \in I$, then $q \in [-1,1]$ and $x \in q + S$

(iii) since $S \subseteq I$ and $q \in Q \cap [-1,1]$, we have $q + S \subseteq [-1,2]$.

□

Since $\mathbf{Q} \cap [-1,1] \subseteq \mathbf{Q}$, the set $\mathbf{Q} \cap [-1,1]$ is countable. Then, there exists a bijective function q from $\mathbf{N}$ to $\mathbf{Q} \cap [-1,1]$. We write

$$\mathbf{Q} \cap [-1,1] = \{q_n \mid 1 \leq n\}.$$

Theorem 1. S is not measurable.

Proof. If $S \in \mathcal{M}$ then $q_n + S \in \mathcal{M}$ for every $n \geq 1$ and $m(q_n + S) = m(S)$. Since $S \subseteq [0,2]$, S has finite measure, by the monotonicity property of the measure. By (ii) and (iii) of Proposition 1

$$[0,1] \subseteq \bigcup_{n=1}^{\infty} (q_n + S) \subseteq [-1,2].$$

Then

$$1 \leq m\Big(\bigcup_{n=1}^{\infty} (q_n + S)\Big) \leq 3.$$

From (i) all the sets $q_n + S$ are disjoint from each other. By the σ-additivity of the outer measure on $\mathcal{M}$, we have

$$1 \leq \sum_{n=1}^{\infty} m(q_n + S) = \sum_{n=1}^{\infty} m(S) \leq 3.$$

Since we are taking the series of a constant sequence, we should have $m(S) = 0$. However, this contradicts

$$1 \leq \sum_{n=1}^{\infty} m(S).$$

□

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