2. SEPARABLE VARIABLES DIFFERENTIAL EQUATIONS

2.1. **Functions with zero derivatives on intervals.** In the next two proposition, we characterize (two-variable) functions defined on (product of intervals) intervals with zero (partial derivative) derivative.

Proposition 2.1. *Let $y\colon J \to \mathbb{R}$ be a derivable function on an interval such that $y'(x) = 0$ for every $x \in J$, then y is a constant function.*

Proof. The proof of this fact follows from the Mean Value Theorem: let us fix $x_0 \in J$. Then, given $x \in J$,

$$[x_0, x] \subseteq J$$

because J is an interval. There exists ϑ between x_0 and x such that

$$y(x) - y(x_0) = y'(\vartheta)(x - x_0) = 0.$$

Then $y(x) = y(x_0)$. In conclusion, for every x in J, $y(x) = y(x_0)$. Then y is a constant function. □

An analogous result applies for partial derivatives:

Proposition 2.2. *Let $g\colon J_1 \times J_2 \to \mathbb{R}$ be a continuous function, derivable with respect to x on $J_1 \times J_2$ such that $\partial_x g = 0$. Then g does not depend on x.*

Proof. For every y in J_2, we consider x_1 and x_2 in J_1. We define the function

$$h(t) := g(t, y).$$

By definition of partial derivative,

$$h'(t) = \partial_x g(t, y) = 0.$$

Then, by Proposition 1, h is a constant function. Then

$$g(x_1, y) = h(x_1) = h(x_2) = g(x_2, y).$$

So, if we fix x_0 in J_1,

$$g(x, y) = g(x_0, y) =: c(y).$$

For every (x, y). □

2.2. **Order of a differential equation.** Roughly, speaking the order of a differential equation is the order of the highest derivative appearing on the differential equation. When we are given explicitly a differential equation, it is not difficult to define its order. For instance, the order of

$$x''(t) = -\frac{GM}{|x(t)|^3} x(t)$$

is two, while the order of

$$y'(x) = y(x)(1 - y(x))$$

is one. But we need to take more care when we define the order of an equation given with a normal form. For example, in

$$F(x, y(x), y'(x), y''(x)) = 0. \tag{1}$$

we are tempted to infer that (1) is a second order differential equation. However, if

$$F(x, y, p_1, p_2) = x - y - p_1 \tag{2}$$

then we should say that the order is one. We notice that in (2) F does not depend on the variable p_2. Or, equivalently,

$$\partial_{p_2} F = 0,$$

according to Proposition 2.2. So, it seems reasonable to state that the order of a differential equation given in the form

$$F(x, y(x), \dots, y^{(n)}) = 0$$

is n if F depends on p_n. While, if it does not depend on p_n, but depends on p_{n-1}, then the order is $n-1$ and so on. The next definition gives a formalization of this process.

Definition 2.1 (Order of a differential equation)**.** Given a function F of $n+2$ variables, *the order of the differential equation*

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

is the highest natural number k such that $\partial_{p_k} F \neq 0$.

2.3. **Initial value problems.** A solution to the Initial Value Problem (IVP)

$$\text{(IVP)} \qquad \begin{cases} F(x, y(x), y'(x)) = 0 \\ y(x_0) = y_0. \end{cases}$$

is a derivable function $y\colon J \to \mathbb{R}$, where J is in an interval such that $x_0 \in J$.
A solution to an initial value problem does not necessarily exists and if exists, there can be more than one. For example,

$$\begin{cases} y'(x)^2 + y(x) = 0 \\ y(0) = 1. \end{cases}$$

does not have any solution because, if we substitute $x = 0$ and $y(0) = 1$ into the equation, we obtain $y'(0)^2 + 1 = 0$ which is not possible. The problem

$$\begin{cases} y'(x) = 2\sqrt{y(x)} \\ y(0) = 0. \end{cases}$$

has two solutions which can be checked directly: $(0, (-\infty, +\infty))$ and $(x^2, [0, +\infty))$.

2.4. **Separable variables differential equations.** Given two functions $h, g\colon \mathbb{R} \to \mathbb{R}$, a separable variable differential equation is given by

$$h(y(x))y'(x) = g(x) \tag{3}$$

and its normal form is

$$F(x, y, p) = h(y)p - g(x).$$

If g and h are continouous (or even just piece-wise continuous), then there are function H and G such that

$$H' = h, \quad G' = g.$$

We argue as follows: if there exists a solution (y, J) to (3), then this solution satisfies

$$\frac{d}{dx}\left(H(y(x)) - G(x)\right) = 0, \quad x \in J.$$

Since J is an interval, we can apply Proposition 2.1. There exists a constant $c \in \mathbb{R}$, such that

$$H(y(x)) - G(x) = c, \text{ for every } x \in J. \tag{4}$$

At this, point, if H is invertible, we have

$$y(x) = H^{-1}(c + G(x)).$$

Conversely, if (y, J) satisfies (4), it also satisfies (3). We can check this by taking the derivative of (4).

2.5. **Constant solutions.** As we will see during the course finding all the solutions (or even one solution) of a differential equation can be a hard task. Sometimes, however, it is possible to find solutions with prescribed features. A very common exercise is finding constant solutions to a given differential equation

$$y'(x) = y(x)(1 - y(x)). \tag{5}$$

It is convenient to argue as follows: if $(y = c, J)$ is a constant solution, then $y' = 0$. Thus,

$$0 = c(1 - c)$$

which implies that $c = 0$ or $c = 1$. Conversely,

$$(0, (-\infty, +\infty)), \quad (1, (-\infty, +\infty))$$

are constant solutions to (5). There are only two constant solutions and, as we will be able to check, there are a lot of solutions which are not constant.