Sometimes, given a differential equation, we can show that there are two solutions on two different intervals

$$(y, [x_0, b)), \ (z, (a, x_0])$$

and both of them satisfy some differential equation

(1) $$F(x, y(x), y'(x)) = F(x, z(x), z'(x)) = 0.$$

A natural question one can ask is whether it is possible to find a solution w on the interval (a, c) such that

$$w(x) = y(x), \ x \in (a, b)$$
$$w(x) = z(x), \ x \in (b, c).$$

This is possible when the following conditions hold:

1. $y(b) = z(b) =: L$
2. y and z are derivable in b and

$$y'(b) = z'(b) =: L_1.$$

If conditions 1 and 2 hold, we can define

$$w(x) = y\#z(x) := \begin{cases} z(x) & \text{if } a < x \leq x_0 \\ y(x) & \text{if } x_0 < x < b \end{cases}$$

We show that w is derivable at x_0. We define

$$\alpha(h) := \frac{w(x_0 + h) - w(x_0)}{h}$$

for $h \neq 0$. We prove that the limit of α exists when α converges to zero. In fact, if $h > 0$, then

$$\alpha(h) = \frac{w(x_0 + h) - w(x_0)}{h} = \frac{w(x_0 + h) - y(x_0)}{h} = \frac{y(x_0 + h) - y(x_0)}{h}.$$

As $h \to 0^+$, $\alpha(h)$ converges to $y'(x_0)$. If $h < 0$,

$$\alpha(h) = \frac{w(x_0 + h) - w(x_0)}{h} = \frac{z(x_0 + h) - z(x_0)}{h} = \frac{z(x_0 + h) - z(x_0)}{h}.$$

As $h \to 0^-$, $\alpha(h)$ converges to $z'(x_0)$. Since $z'(x_0) = y'(x_0)$ the two limits (with h negative and h positive) are equal, so the limit of α exists. Hence

$$w'(x_0) = L_1.$$

Finally, we show that $(w, (a, b))$ is a solution to the differential equation (1). In fact, when $x \geq x_0$

$$F(x, w(x), w'(x)) = F(x, y(x), y'(x)) = 0,$$

while for $x > x_0$ we have

$$F(x, w(x), w'(x)) = F(x, z(x), z'(x)) = 0.$$

In the next example, we use this method to find all the solutions to the differential equation

(2) $$xy'(x) = 2y(x).$$

This equation can be integrated by using the separable variables method: we divide by x

(3) $$y'(x) = \frac{2y(x)}{x}$$

we divide by y

$$\frac{y'(x)}{y(x)} = \frac{2}{x} \tag{4}$$

and obtain

$$\ln|y(x)| = \ln|x|^2 + C$$

for every $C \in \mathbb{R}$. Then

$$|y(x)| = c|x|^2$$

for every $c > 0$. Clearly, there is a one-parameter family of solutions

$$(y_d(x) = dx^2, (-\infty, +\infty)), \quad d \neq 0. \tag{5}$$

But we also notice, that $z = 0$ is a solution to (2) and it does not appear in (5). It seems that when we divide by y, we lose the solution $y = 0$. Also, we should notice that (5) are solutions to (2), but not solutions to (4). In conclusion, dividing by x and y triggers a lost of solutions.

Now, we expose an argument whose purpose is to find all the solutions to (2): suppose that (y, I) is a solution to (2). Then

$$(y, I \cap (0, +\infty)), \quad (y, I \cap (-\infty, 0))$$

are solutions to (2). We use the notations

$$I_+ := I \cap (0, +\infty), \quad I_- := I \cap (-\infty, 0).$$

We use the notations y_+ and y_- for the function on I_+ and I_-. Then

$$(y_+, I_+), \quad (y_-, I_-)$$

are solutions to (2). Since I_- and I_+ do not containt $x = 0$, these are also solutions to (3). We show that on I_+ the function y does not have zeroes, unless $y = 0$ on I_+. In fact, suppose that there exists $x_* \in I_+$ such that $y(x_*) = 0$. On the domain

$$(0, +\infty) \times \mathbb{R}$$

the function

$$g(x, y) = \frac{2y}{x}$$

has partial derivative $\partial_y g = 2/x$, locally bounded. Then g is locally Lip_y. Hence, the solution to the initial value problem

$$\begin{cases} y'(x) = g(x, y(x)) \\ y(x_*) = 0 \end{cases}$$

is unique. Since $(y = 0, I_+)$ is a solution to the initial value problem, we have $y_+ = 0$.

Suppose that y_+ is different from zero at every point. Then (y_+, I_+) is a solution to (5). Then, there exists $c \neq 0$ such that

$$y_+ = cx^2$$

Similarly, there exists $d \neq 0$ such that

$$y_- = dx^2.$$

Now, we try to paste the two solutions $(cx^2, [0, +\infty))$ and $(dx^2, (-\infty, 0])$ together. Clearly,

$$y_+(0) = 0 = y_-(0)$$

and

$$y'_+(0) = y'_-(0) = 0.$$

Then

$$y = y_- \#_0 y_+.$$

In conclusion, if (y, I) is a solution to (2), then there are c and d such that

$$y = cx^2 \#_0 dx^2.$$

Then, we are able to list all the solutions to (2)

$$(cx^2 \#_0 dx^2, (-\infty, +\infty)), \quad c, d \in \mathbb{R}.$$