MIDTERM EXAM

Exercise 1. Integrate the differential equation

$$2xy''(x) = y'(x)^2 - 1$$

with a substitution. Then, find
(a) a solution defined on $(-\infty, +\infty)$
(b) a solution which cannot be defined on $(-\infty, +\infty)$.

Solution. We use the substitution $y' = z$. Then

$$2xz' = z^2 - 1.$$

This is a separable variables equation. The constant $z = 1$ is a solution; then

$$(y_c(x) := x + c, (-\infty, +\infty)) \tag{1}$$

is a solution to the differential equation. Since we are not trying to find all the solutions, we consider the case where $x > 0$ and $z \neq 1, -1$. Then

$$\frac{2}{z^2 - 1} = \frac{1}{x}$$

and

$$\ln\left|\frac{z(x) - 1}{z(x) + 1}\right| = \ln|x| + c,$$

$$z(x) = \frac{1 + Cx}{1 - Cx}, \quad C \neq 0$$

which is defined, for instance, when $x > C^{-1}$. Then

$$y_{D,C}(x) = D + \int z(x)dx = D + \int \left(\frac{1 + Cx}{1 - Cx} + \frac{1 + Cx - 2 + 2}{1 - Cx}\right) dx$$
$$= D + \int \left(\frac{Cx - 1}{1 - Cx} + \frac{2}{1 - Cx}\right) dx = D - x - \frac{2}{C}\ln(Cx - 1)$$

If $C = 1$ and $D = 0$, we obtain the solution

$$(y(x) = -x - 2\ln(x - 1), (1, +\infty)). \tag{2}$$

Clearly, there is no way to extend y to $(-\infty, +\infty)$, not even as a continuous function. Then
(a) (1)
(b) (2)

□

Exercise 2. By integrating the Bernouilli equation

$$-2y'(x) = y(x) + e^x y^5(x)$$

find a positive solution (y, I) such that $y(0) = 1$.

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Solution. Since we are looking for positive solution, we are allowed to make the substitution $y := z^{-1/4}$ with $z > 0$. Then z satisfies:

$$z'(x) = 2z(x) + 2e^x.$$

$$(e^{-2x}z(x))' = 2e^{-x}$$

whence

$$e^{-2x}z(x) = -2e^{-x} + c$$

and

$$z_c(x) = ce^{2x} - 2e^x$$

The domain of z_c must be restricted to an interval where z_c is positive. Then, we consider the solutions

$$(y_c(x) = (ce^{2x} - 2e^x)^{-1/4}, (\ln(2/c), +\infty)), \quad c > 0.$$

Since we need to satisfy the requirement $y(0) = 1$, we should consider the solutions y_c such that

$$0 \in (\ln(2/c), +\infty))$$

that is

$$\ln(2/c) < 0 \Rightarrow c > 2.$$

Finally, the condition $y_c(0) = 1$ gives

$$c = 3.$$

Then, the desired solution is

$$(y_3(x) := (ce^{2x} - 2e^x)^{-1/4}, (\ln(2/3), +\infty)).$$

□

Exercise 3. Consider the following differential equation

$$y(x) + y'(x)(1 - y(x)e^{-x}) = 0.$$

(a) is it exact?
(b) is there an integrating factor which does not depend on x?
(c) is there an integrating factor which does not depend on y?
(d) integrate the equation.
(e) find one solution defined on $(-\infty, +\infty)$.

Solution. We have $M = y$ and $N = (1 - ye^{-x})$.

(a) no. Because

$$\partial_y M = 1 \neq \partial_x N = ye^{-x}.$$

(b) since

$$\frac{\partial_x N - \partial_y M}{M} = \frac{ye^{-x} - 1}{y}$$

does not depend only on y, the answer is no.

(c) since

$$\frac{\partial_y M - \partial_x N}{N} = 1$$

does not depend on y, the answer is yes and $\mu(x) = e^x$.

(d) we use the integrating factor e^x

$$Me^x = ye^x = \partial_x G \Rightarrow G(x,y) = ye^x + c(y)$$
$$\partial_y G = e^x + c'(y).$$

From

$$e^x + c'(y) = e^x N = e^x - y$$

we obtain $c(y) = -y^2 + c$. Then

$$G(x,y) = ye^x - y^2 + c.$$

(e) For example, $(0, (-\infty, +\infty))$.

□

Exercise 4. Make an example of a first order differential equation which is linear, homogeneous and not exact.

Solution. The most simple first order, linear and homogeneous differential equation is

$$1 + y' = 0.$$

However, this is also exact. Then, we multiply by x

$$x + xy' = 0.$$

This is still homogeneous (of degree 1), but it is not exact. □