3. LIPSCHITZ FUNCTIONS

Definition 3.1 (Open sets)**.** A subset $D \subseteq \mathbb{R}$ is *open* if for every $x_0 \in \Omega$ there exists $r > 0$ such that

$$(x_0 - r, x_0 + r) \subseteq \Omega.$$

A subset $D \subseteq \mathbb{R}^2$ is open if for every $(x_0, y_0) \in \Omega$ there exists $r > 0$ such that

$$(x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r) \subseteq \Omega.$$

Definition 3.2 (Bounded functions)**.** A function of one or more variables g defined on $\Omega \subseteq \mathbb{R}^n$ is *bounded* if there exists $M \in \mathbb{R}$ such that

$$|g(x)| \leq M$$

for every $x \in \Omega$.

If g is not bounded, for every a in $\mathbb{R}$ there exists x_a such that

$$|g(x_a)| \geq a.$$

Actually, by the Archimedean property of the set of real numbers, to prove that a function is not bounded, it is enough to show that there exists x_n as above, only when n is a natural number.

Definition 3.3 (Lipschitz functions)**.** A one-variable function $y\colon I \to \mathbb{R}$ is Lipschitz if and only if there exists a constant L such that

$$|y(x_1) - y(x_2)| \leq L|x_1 - x_2|$$

for every $x_1, x_2 \in I$.

Unless otherwise stated, in the next proposition and theorems I will be an interval containing at least two elements.

Proposition 3.1. *If $y\colon I \to \mathbb{R}$ is Lipschitz (with constant L) and derivable at $x \in I$, then $|y'(x)| \leq L$.*

Proof. Since y is Lipschitz, for $h > 0$, we have

$$\left| \frac{y(x+h) - y(x)}{h} \right| \leq L.$$

By taking the limit, we obtain $|y'(x)| \leq L$. □

Proposition 3.2. *If $y\colon I \to \mathbb{R}$ is derivable on I and y' is bounded (by a constant M), then y is Lipschitz (with constant M).*

Proof. Given x_1, x_2 by the Mean Value Theorem, there exists $x_1 < x_* < x_2$ such that

$$y(x_1) - y(x_2) = y'(x_*)(x_1 - x_2).$$

By taking the absolute value, we obtain

$$|y(x_1) - y(x_2)| \leq |y'(x_*)||x_1 - x_2| \leq L|x_1 - x_2|.$$

□

Proposition 3.3. *A Lipschitz function is continuous.*

Proof. Let us call g this function and K the Lipschitz constant. We fix x_0 in I and ε. We have to show that there exists $\delta > 0$ such that

$$|x_0 - x| < \delta \implies |g(x_0) - g(x)| < \varepsilon. \tag{6}$$

We have

$$|g(x_0) - g(x)| \leq L|x_0 - x| < L\delta.$$

So, if we choose $\delta < \varepsilon / L$ we obtain the inequality (6). $\square$

Continuous functions can be non-Lipschitz. The function

$$y_0 \colon (0,1) \to \mathbb{R}, \quad y_0(x) = \frac{1}{x}$$

is continuous on the interval $(0,1)$ but not Lipschitz. In order to prove this, we argue by contradiction. Let L be a Lipschitz constant for y_0. Since it is derivable, if y_0 is Lipschitz, y_0' must be bounded on $(0,1)$ from L, according to Proposition 3.1. That is,

$$\frac{1}{x^2} \leq L$$

for every $x \in (0,1)$. However, this is not true: take the sequence (x_n) where $x_n = n^{-1/2}$. Then

$$|y'(x_n)| = n$$

which is bigger than L is n is large enough.

Continuous and bounded functions can be non-Lipschitz. The function

$$y_1 \colon (0,1) \to \mathbb{R}, \quad y_1(x) = \sin(1/x) \tag{7}$$

is derivable and bounded. But it is not Lipschitz. In fact, its derivative

$$y_1'(x) = -\frac{1}{x^2} \cos \frac{1}{x}.$$

is not bounded: if we evaluate y_1' on the sequence

$$x_n = \frac{1}{2\pi n}$$

we obtain

$$y_1'(x_n) = -4\pi^2 n^2$$

which diverges to $-\infty$ as $n \to +\infty$.

Definition 3.4 (Locally Lipschitz functions)**.** A function $y \colon I \to \mathbb{R}$ is *locally Lipschitz* if for every $x_0 \in I$ there exists $r > 0$ such that y is Lipschitz on

$$(x_0 - r, x_0 + r) \cap I.$$

Proposition 3.4. *A Lipschitz function is locally Lipschitz.*

Proof. Suppose that $y \colon I \to \mathbb{R}$ is Lipschitz with constant L. Let $x_0 \in I$ be a point and let $r = 1$. Then y is Lipschitz on $(x - 1, x + 1) \cap I$: given

$$x_1, x_2 \in (x - 1, x + 1) \cap I$$

we have $x_1, x_2 \in I$. Since y is Lipschitz,

$$|y(x_1) - y(x_2)| \leq L|x_1 - x_2|.$$

Then y is Lipschitz on $(x - 1, x + 1) \cap I$. $\square$

Example 3.1 (Locally Lipschitz does not imply Lipschitz)**.** In general, the converse is not true. For instance y_1 in (7) is locally Lipschitz: given $0 < x_0 < 1$, if we take $r = x_0/2$, then

$$|y_1'(x)| = |1/x^2| \le \frac{4}{x_0^2}, \quad \text{for every } x \in (x_0/2, 3x_0/2).$$

Notation 3.1. Given x_0, y_0 in $\mathbb{R}$ and $r > 0$, we define

$$I_r(x_0) := (x_0 - r, x_0 + r) \subset \mathbb{R}$$
$$Q_r(x_0, y_0) := I_r(x_0) \times I_r(y_0) \subset \mathbb{R}^2.$$

This definition generalizes in $\mathbb{R}^n$ as

$$Q_r(x_0) := \prod_{i=1}^{n} I_r(x_0^i).$$

Definition 3.5 (Locally bounded functions)**.** A function $g\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *locally bounded* if and only if, for every $x_0 \in \Omega$, there exists $r > 0$ such that g is bounded on $Q_r(x_0) \cap \Omega$.

Local properties. In general, when we define a property P of a function g over Ω, we can also define the corresponding local property: g is locally P if and only for every x_0 in Ω there exists $r > 0$ such that g satisfies P in $Q_r(x_0) \cap \Omega$. In most of the cases we will see that the global property implies the local property, while usually the converse will not be true.

Example 3.2 (Lipschitz functions may not be derivable)**.** A function can be Lipschitz but not derivable. A simple example is given by $y_2(x) = |x|$. It follows from the inequality

$$||x_1| - |x_2|| \le |x_1 - x_2|$$

for every x_1, x_2 in $\mathbb{R}$. So, we can take $L = 1$.

Proposition 3.5. *Suppose that y is a continuous function on an interval I and there exists x_* in I such that y is Lipschitz on $I_1 := (-\infty, x_*) \cap I$ and is Lipschitz on $I_2 := (x_*, +\infty) \cap I$. Then y is Lipschitz on I.*

Proof. We claim the y is a Lipschitz function with constant $L := \max\{L_1, L_2\}$. Since y is Lipschitz on I_1, there exists L_1 such that

$$|y(x_1) - y(x_2)| \le L_1|x_1 - x_2| \le L|x_1 - x_2| \tag{8}$$

for all x_1, x_2 in I_2. First, we show that y is Lipschitz on $(-\infty, x_*] \cap I$. Let x_+ be an element of I_1. Then, there exists $\varepsilon > 0$ such that

$$x_* + \varepsilon < x_+.$$

So, $x_* + \varepsilon$ is in I_2. Then

$$\begin{aligned} y(x_+) - y(x_*) &= y(x_+) - y(x_* + \varepsilon) + y(x_* + \varepsilon) - y(x_*) \\ &= y(x_+) - y(x_* + \varepsilon) + \alpha(\varepsilon). \end{aligned}$$

From (8), we have

$$\begin{aligned} |y(x_+) - y(x_*)| &\le |y(x_+) - y(x_* + \varepsilon)| + |\alpha(\varepsilon)| \\ &\le L|x_+ - x_* - \varepsilon| + |\alpha(\varepsilon)|. \end{aligned}$$

Now, we take the limit as $\varepsilon \to 0$. The function $\alpha(\varepsilon)$ converges to zero because y is continuous at the point x_*. In a similar way, we can show that y is Lipschitz on $(-\infty, x_*] \cap I$ with Lipschitz constant L.

We conclude by showing that y is Lipschitz on I with constant L: the only case that we did not discuss is the one where x_1 and x_2 are such that

$$x_1 \le x_* \le x_2.$$

Then

$$\begin{aligned} |y(x_2) - y(x_1)| &\le |y(x_2) - y(x_*)| + |y(x_*) - y(x_1)| \\ &\le L(x_2 - x_*) + L(x_* - x_1) = L(x_2 - x_1). \end{aligned}$$

□

3.1. **Two-variables Lipschitz functions.**

Definition 3.6 (Two variables Lipschitz funtions)**.** A function $g\colon \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz if there exists a constant L such that

$$|g(x_1, y_1) - g(x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|).$$

Most of the propositions we proved for one-variables Lipschitz functions apply to two-variable Lipschitz functions. For instance, if g is Lipschitz and $\partial_x g$ exists at a given point (x_0, y_0), then

$$|\partial_x g(x_0, y_0)| \le L. \tag{9}$$

This follows from the definition of partial derivative: for every $h \neq 0$,

$$\left| \frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h} \right| \le L.$$

Taking the limit as $h \to 0$, we obtain (9). For our purposes, it is interesting to look at two-variables functions which are Lipschitz only on one variable:

Definition 3.7. A function $g\colon \Omega \to \mathbb{R}$ is Lip_y if there exists L such that for every (x, y_1) and (x, y_2), there holds

$$|g(x, y_1) - g(x, y_2)| \le L|y_1 - y_2|.$$

If g is Lip_y, and $\partial_y g$ exists, then $|\partial_y g| \le L$. In one-variable functions, Proposition 3.2 ensures that if y' exists on I (interval) and it is bounded, then y is Lipschitz.

Unfortunately, in two variables, it is not true that if $\partial_x g$ and $\partial_y g$ exist on Ω and are bounded, then g is Lipschitz on Ω (or that if $\partial_y g$ is bounded, then g is Lip_y on Ω). It is true if Ω satisfies some special requirements, [1]. as in the next proposition.

Proposition 3.6. *Let g be function on $Q_r(x_0, y_0)$ such that $\partial_y g$ is bounded. Then g is Lip_y.*

Proof. Let (x, y_1) and (x, y_2) be two distinct points of Q. For every $0 \le t \le 1$, the segment

$$(x, y_1 + t(y_2 - y_1)) \in Q.$$

We can check this directly. Since $(x, y_1) \in Q$, we have

$$x \in (x_0 - r, x_0 + r).$$

[1] For example, if Ω is a convex subset of $\mathbb{R}^2$

Now, y_1 and y_2 belong to $I_r(y_0)$. Then

$$\begin{aligned}|y_1 + t(y_2 - y_1) - y_0| &= |y_1 + t(y_2 - y_1) - ty_0 - (1-t)y_0| \\ &= |(1-t)(y_1 - y_0) + t(y_2 - y_0)| \\ &\le (1-t)|y_1 - y_0| + |t(y_2 - y_0)| \le (1-t)r_0 + tr_0 = r_0.\end{aligned}$$

We define

$$h\colon [0,1] \to \mathbb{R}, \quad h(t) := g(x, y_1 + t(y_2 - y_1)).$$

Since

$$h'(t) = \partial_y g(x, y_1 + t(y_2 - y_1))(y_2 - y_1)$$

by the Mean Value Theorem,

$$g(x, y_2) - g(x, y_1) = h(1) - h(0) = h'(t_*) = \partial_y g(x, y_1 + t(y_2 - y_1))(y_2 - y_1).$$

Then

$$|g(x, y_2) - g(x, y_1)| \le |h'(t_*)| = |\partial_y g(x, y_1 + t_*(y_2 - y_1))||y_2 - y_1| \le L|y_2 - y_1|.$$

□

Definition 3.8 (Locally Lip_y functions)**.** The function g is locally Lip_y on Ω if and only if for every (x_0, y_0) in Ω, there exists $r > 0$ such that g is bounded on $Q_r(x_0, y_0)$.

Theorem 3.1 (Picard-Lindelöf)**.** *Let f be a continuous and locally Lip_y function on a open subset of $\mathbb{R}^2$, Ω. Let (x_0, y_0) be a point of Ω. Then,*

(i) there exists $r > 0$ and a function y on $I_r(x_0)$ such that $(x, y(x))$ is in Ω for every x in $I_r(x_0)$. Moreover,

(IVP) $$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

(ii) if (y_1, I_1) and (y_2, I_2) solve (IVP), then

$$y_1(x) = y_2(x) \text{ for every } x \in I_1 \cap I_2.$$

Corollary 3.1. *Let f a locally Lip_y function. And let (y_1, I_1) and (y_2, I_2) are two solutions of the differential equation*

$$y'(x) = f(x, y(x)).$$

Suppose that there exists x_* in $I_1 \cap I_2$ such that $y_1(x_*) = y_2(x_*)$, then

$$y_1(x) = y_2(x) \text{ for every } x \in I_1 \cap I_2.$$