Corollary 3.1. *Let f a locally Lip_y function. And let (y_1, I_1) and (y_2, I_2) are two solutions of the differential equation*

$$y'(x) = f(x, y(x)).$$

Suppose that there exists x_* in $I_1 \cap I_2$ such that $y_1(x_*) = y_2(x_*)$, then

$$y_1(x) = y_2(x) \text{ for every } x \in I_1 \cap I_2.$$

4. Linear differential equations

An ordinary differential equation

$$F(x, y(x), y'(x), \dots, y^{(n)}) = 0$$

is said *linear* of order n if

$$F(x, z, p_1, \dots, p_n) = a_n(x)p_n + \dots + a_1(x)p_1 + a_0(x)z - g(x)$$

for some functions

$$a_n, a_{n-1}, \dots, a_0, g \colon J \to \mathbb{R}$$

defined on a given open interval $J \subset \mathbb{R}$ and

$$a_n \neq 0.$$

The functions a_i are called *coefficients* and g is called *non-homogeneous term*.

Definition 4.1. A linear differential equation (d.e.) is said *homogeneous* if $g \equiv 0$

Definition 4.2. A linear d.e. is called *constant coefficients d.e.* if a_i are constant functions for every $0 \leq i \leq n$.

We will assume that the coefficients are constant functions and that the equation is homogeneous. Then, the equation can be written as

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_0 y = 0$$

and $a_n \neq 0$. Up to divide by a_n, we can suppose that the equation is

$$y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_0 y = 0.$$

We introduce the following notation for the derivative

$$Dy := y'.$$

Given a function $a \colon J \to \mathbb{R}$, we define

$$(D - a)y(x) := y'(x) - a(x)y(x).$$

Moreover,

$$D^0 y := y$$
$$D^k y := D(D^{k-1} y) \quad k \geq 1.$$

for every $\alpha \in \mathbb{R}$. From the notation above it follows the relation

$$(D - \alpha)(D - \beta)y = D^2 y - (\alpha + \beta)Dy + \alpha\beta y$$

for every α, β real numbers. In fact,

$$\begin{aligned}(D - \alpha)(D - \beta)y &= (D - \alpha)(y' - \beta y) = D(y' - \beta y) - \alpha(y' - \beta y)\\ &= y'' - \beta y' - \alpha y' + \alpha\beta y = y'' - (\alpha + \beta)y' + \alpha\beta y.\end{aligned}$$

For the sake of simplicity, we will denote a linear differential equation with $Ly = g$.

Proposition 4.1 (The Superposition Principle). *Given a n^{th} homogeneous order linear d.e. $Ly = 0$, if y and z are solutions, then $cy + dz$ is a solution for every real numbers c, d.*

Proof. We have

$$L(cy + dz) = \sum_{k=0}^{n} a_k(x)(cy + dz)^{(k)} = \sum_{k=0}^{n} a_k(x)(cy^{(k)} + dz^{(k)})$$
$$= c\sum_{k=0}^{n} a_k(x)y^{(k)}(x) + d\sum_{k=0}^{n} a_k(x)z^{(k)}(x) = cLy + dLz.$$

So, if y and z are solutions, then $Ly = Lz = 0$. Then $L(cy + dz) = 0$. □

Definition 4.3. To a linear homogeneous ODE with constant coefficients, we can associate its *characteristic polynomial* given by

$$p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0.$$

Theorem 4.1. *The solutions to the differential equation $(D - \alpha)(D - \beta)y = 0$ are*

$$(\alpha \neq \beta) \qquad y = c_1e^{\alpha x} + c_2e^{\beta x}$$
$$(\alpha = \beta) \qquad y = c_1e^{\alpha x} + c_2xe^{\alpha x}.$$

Proof. We use the substitution $z = (D - \beta)y$. Then

$$(D - \alpha)z = 0 \implies z = ce^{\alpha x}.$$

Then

$$(D - \beta)y = ce^{\alpha x}$$

whence

$$e^{-\beta x}(D - \beta)y = ce^{(\alpha-\beta)x}$$
$$D(e^{-\beta x}y) = ce^{(\alpha-\beta)x}.$$

Now, we need to integrate both sides of the equation. If $\alpha \neq \beta$, then

$$\int e^{(\alpha-\beta)x} = \frac{1}{\alpha - \beta}e^{(\alpha-\beta)x}.$$

Then

$$e^{-\beta x}y = \frac{c}{\alpha - \beta}e^{(\alpha-\beta)x} + d$$

whence

$$y(x) = \frac{c}{\alpha - \beta}e^{\alpha x} + de^{\beta x}.$$

If we set $c_1 = c/(\alpha - \beta)$ and $c_2 = d$, then we obtain the solutions in the first case. Now, suppose that $\alpha = \beta$. Then (13) becomes

$$D(e^{-\beta x}y) = c.$$

Then

$$e^{-\beta x}y = cx + d$$

and

$$y(x) = cxe^{\beta x} + de^{\beta x}.$$

We set $c_1 = d$ and $c_2 = c$. Since $\alpha = \beta$, we obtain the solutions in (13). □

4.1. **Non factorizable characteristic polynomial.** We start by considering the case where

$$y'' + y = 0, \quad p(X) = X^2 + 1. \tag{10}$$

We can see that there are at least two solutions

$$y_1(x) = \sin x \quad y_2(x) = \cos x$$

and, by the Superposition Principle, all the linear combinations

$$y = c_1 \sin x + c_2 \cos x \tag{11}$$

are also solutions. In the next proposition, we show that all the solutions are as in (11).

Proposition 4.2. *Let $(y, (a, b))$ be a solution to*

$$y'' + y = 0.$$

Then, there are two (unique) constants c_1 and c_2 such that

$$y(x) = c_1 \cos x + c_2 \sin x.$$

Therefore, the solution can be defined on $(-\infty, +\infty)$.

Proof. Let us fix a point x_0 in (a, b). We define

$$z(x) = y(x + x_0).$$

Clearly $(z, (a - x_0, b - x_0))$ is a solution to (10).
We prove that

$$z(x) = z(0) \cos x + z'(0) \sin x.$$

We define

$$w(x) := z(x) - z(0) \cos x - z'(0) \sin x.$$

By the Superposition principle, w satisfies

$$w'' + w = 0.$$

Moreover,

$$w(0) = 0, \quad w'(0) = 0. \tag{12}$$

We claim that $w = 0$ on $(a - x_0, b - x_0)$. In fact, if we multiply by $2w'$ and obtain

$$2w''w + 2ww' = 0 \implies D\big((w')^2 + w^2\big) = 0.$$

Then, there exists a constant c such that

$$(w'(x))^2 + w^2(x) = c.$$

From (12), this constant is equal to zero. Then

$$(w'(x))^2 + w^2(x) = 0$$

for every x in $(a - x_0, b - x_0)$, which implies $w = 0$. Hence

$$\begin{aligned} y(x) &= z(x + x_0) = z(0) \cos(x + x_0) + z'(0) \sin(x + x_0) \\ &= (z(0) \cos x_0 - z'(0) \sin x_0) \cos x + (z'(0) \cos x_0 - z(0) \sin x_0) \sin x. \end{aligned}$$

Then we can choose

$$c_1 = z(0) \cos x_0 - z'(0) \sin x_0, \quad c_2 = z'(0) \cos x_0 - z(0) \sin x_0.$$

We show that, if the equality (11) holds for another pair of constants (d_1, d_2), then $c_1 = d_1$ and $c_2 = d_2$. In fact, since

$$(c_1 - d_1)\cos x + (c_2 - d_2)\sin x = 0 \tag{13}$$

for every x in (a, b), there holds

$$\begin{aligned}(c_1 - d_1)\cos x_0 + (c_2 - d_2)\sin x_0 = 0\\ -(c_1 - d_1)\sin x_0 + (c_2 - d_2)\cos x_0 = 0.\end{aligned}$$

We multiply the first equation by $\cos x_0$ and the second equation by $\sin x_0$ and take the difference. Then

$$\begin{aligned}(c_1 - d_1)\cos^2 x_0 + (c_2 - d_2)\sin x_0 \cos x_0 = 0\\ -(c_1 - d_1)\sin x_0 \sin x_0 + (c_2 - d_2)\cos x_0 \sin x_0 = 0.\end{aligned}$$

Now, we take the difference between the first and the second equation.

$$(c_1 - d_1)(\cos^2 x_0 + \sin^2 x_0) = 0$$

which implies $c_1 = d_1$. Together with (13) we obtain $c_2 = d_2$.

Finally, since the domain of $\sin x$ and $\cos x$ is $(-\infty, +\infty)$, then we can choose $(-\infty, +\infty)$ as existence interval for y. □

4.2. **Second case:** $p(X) = X^2 + \beta^2$ **with** $\beta \neq 0$**.** Now, we wish to solve the differential equation

$$y'' + \beta^2 y = 0 \tag{14}$$

with $\beta > 0$. Clearly, $\cos \beta x$ and $\sin \beta x$ are solutions to (14), and, by the Superposition Principle, for every c_1, c_2 real numbers

$$c_1 \cos \beta x + c_2 \cos \sin \beta x$$

is a solution to (14).

Proposition 4.3. *Let* $(y, (a, b))$ *be a solution to* (14)*. Then there exists a unique pair* (c_1, c_2) *such that*

$$y(x) = c_1 \cos \beta x + c_2 \sin \beta x.$$

and the existence interval can be extended to $(-\infty, +\infty)$*.*

Proof. We set

$$z(x) := y(\beta^{-1} x) \tag{15}$$

Then

$$z'(x) = \beta^{-1} y'(\beta^{-1} x)$$

and

$$z''(x) = \beta^{-2} y''(\beta^{-1} x) = \beta^{-2}(-\beta^2 y(\beta^{-1} x) = -y(\beta^{-1} x) = z.$$

Then

$$z'' + z = 0.$$

By Proposition 4.2, there are two constants c_1 and c_2 such that

$$z(x) = c_1 \sin x + c_2 \cos x.$$

From (15),

$$y(x) = z(\beta x) = c_1 \sin \beta x + c_2 \cos \beta x.$$

Since this pair of constants is unique for z, it is also unique for y. □

4.3. **Last case: $p(X) = (X-\alpha)^2 + \beta^2$ with $\beta \neq 0$.** We wish to reduce to problem to the previous case where the polynomial is $X^2 + \beta^2$. We have

$$\begin{aligned}(D-\alpha)^2 y &= (D-\alpha)(D-\alpha)y \\ &= (D-\alpha)[e^{\alpha x}e^{-\alpha x}(D-\alpha)y] = (D-\alpha)[e^{\alpha x}D(e^{-\alpha x}y)].\end{aligned}$$

We use the substitution

$$z(x) = e^{-\alpha x}y. \tag{16}$$

Then the last term of the equality above can be written as

$$\begin{aligned}(D-\alpha)(e^{\alpha x}Dz) &= D(e^{\alpha x}Dz) - \alpha(e^{\alpha x}Dz) \\ &= \alpha e^{\alpha x}Dz + e^{\alpha x}D^2 z - \alpha e^{\alpha x}Dz = e^{\alpha x}D^2 z.\end{aligned}$$

Then

$$(D-\alpha)^2 y + \beta^2 y = e^{\alpha x}D^2 z + e^{\alpha x}\beta^2 z = e^{\alpha x}(D^2 z + \beta^2 z)$$

and

$$e^{\alpha x}(D^2 z + \beta^2 z) = 0 \implies (D^2 + \beta^2)z = 0.$$

From Proposition 4.3,

$$z(x) = c_1 \cos \beta x + c_2 \sin \beta x.$$

By (16),

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$