EXERCISES OF WEEK SIX

Exercise 1 (Check problems, 1,2, 3 and 4 at page 59 of "Advanced Engineering Mathematics")**.** For each of the following equations, check whether they are exact or not. If they are exact, find an implicit solution

$$(2x-1)dx+(3y-7)dy=0 \tag{1}$$

$$(2x+y)dx-(x+6y)dy=0 \tag{2}$$

$$(5x+4y)dx+(4x-8y^3)dy=0 \tag{3}$$

$$(\sin y-y\sin x)dx+(\cos x+x\cos y-y)dy=0. \tag{4}$$

Moreover,

1. in (1), find all the solutions y defined on $(-\infty,+\infty)$ such that

$$y(x)>\frac{7}{3}$$

for every x in $(-\infty,+\infty)$

2. in (3): is there a solution (y,I) such that $0\in I$ and $y(0)=-1/2$ (you do not have to find this solution explicitly, just give a reason why this solution exists or not)

3. in (4), is there a solution such that $y(\pi)=\pi$?

Solution.

(1) the equation is exact and an implicit solution is given by

$$G(x,y)=x^2-x+\frac{3}{2}y^2-7y+c.$$

(2) $\partial_y M=1\neq -1=\partial_x N$, so it is not exact

(3) it is exact and an implicit solution is

$$G(x,y)=\frac{5x^2}{2}+4xy-2y^4+c.$$

(4) it is exact and an implicit solution is

$$G(x,y)=x\sin y+y\cos x-\frac{y^2}{2}+c.$$

Now, let us address questions 1,2 and 3

1. We are looking for explicit solutions. A solution to $G(x,y(x))=0$ exists if the discriminant of the equation is non-negative, that is

$$49-6(x^2-x+c)\geq 0$$

for every x real number. That is,

$$6x^2-6x+6c-49\neq 0$$

for every x. However, for any choice of c, the function attains positive values. So, a solution on $(-\infty,+\infty)$ does not exist.

Date: 2014, October 1.

2. A solution exists by the Implicit Function Theorem:

$$\partial_y G(0,-1/2) = N(0,-1/2) = 4 \cdot 0 - 8(-1/2)^3 = 1 \neq 0.$$

3. A solution exists by the Implicit Function Theorem:

$$\partial_y G(\pi,\pi) = N(\pi,\pi) = \cos\pi + \pi\cos\pi - \pi = -1 - \pi + \pi = -1 \neq 0.$$

□

Exercise 2. Check whether the following differential equation

$$(1-y)\cos x + (2y - 1 - \sin x)y' = 0$$

is exact. Moreover, for every $0 \leq k \leq 3$ find a solution y_k such that $y_k(0) = 0$ and

$$\int_{\pi/2}^{13\pi/2} y_k(x)dx = 2k\pi.$$

Solution. We compare the partial derivatives of

$$M = (1-y)\cos x, \quad N = 2y - 1 - \sin x.$$

We have

$$\partial_y M = -\cos x = \partial_x N.$$

So, there is the chance to find G such that $\partial_x G = M$ and $\partial_y G = N$. From

$$(1-y)\cos x = \partial_x G$$

we obtain

$$G(x,y) = (1-y)\sin x + c(y).$$

Then, from

$$\partial_y G(x,y) = -\sin x + c'(y) = 2y - 1 - \sin x$$

we obtain $c'(y) = 2y - 1$, whence $c(y) = y^2 - y + c$. Then

$$G(x,y) = (1-y)\sin x + y^2 - y + c.$$

Now, we consider explicit solutions to the differential equation. If $c = 0$, we see that such solution should satisfy

$$(1 - y(x))\sin x + y(x)^2 - y(x) = 0$$

which can be written as

$$(y(x) - 1)(y(x) - \sin x) = 0.$$

So, we can point out at least two solutions to the differential equation:

$$(z_1 = 1, (-\infty, +\infty)), \quad (z_2 = \sin x, (-\infty, +\infty)).$$

Clearly,

$$\int_{\pi/2}^{13\pi/2} z_2 = \Big[-\cos x\Big]_{\pi/2}^{13\pi/2} = \cos\pi/2 - \cos 13\pi/2 = 0.$$

Since $z_2(0) = 0$, we can choose

$$y_0(x) = \sin x.$$

Clearly, z_1 satisfies

$$\int_{\pi/2}^{13\pi/2} z_2 = 6\pi = 3 \cdot 2\pi$$

which makes him a perfect candidate for y_3. Unfortunately, $y_3(0)$ is 1 and not 0. However,

$$z_1(\pi/2) = z_2(\pi/2) = 1, \quad z_1'(\pi/2) = z_2'(\pi/2) = 0.$$

So, through

$$y_3(x) := \sin(x)\#_{\pi/2}1$$

we obtain a solution such that $y_3(0) = 0$, and on the interval $(\pi/2, 13\pi/2)$ is equal to 1. Then

$$\int_{\pi/2}^{13\pi/2} y_3 = 6\pi.$$

The other solutions are

$$y_1 = (\sin x)\#_{9\pi/2}1$$

and

$$y_2 = (\sin x)\#_{5\pi/2}1$$

□