Serre's Uniformity Question and proper subgroups of $C_{ns}^{+}(p)$

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Joint work with Davide Lombardo

Open Image Theorem

Definition

Let K be a number field and ${}^{E}\!/_{K}$ an elliptic curve. Setting $\mathbf{G}_K := \mathrm{Gal}\left({}^{\overline{K}}\!/_{K}\right)$, we define the Galois representation

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Does there exist an integer N such that for every prime $p > N$ and for every elliptic curve ${}^{E}\!/_{\mathbb{Q}}$ without CM the representation $\rho_{E,p}$ is surjective?

Serre's Uniformity Question

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If $E/\mathbb{Q}$ is an elliptic curve without CM, then there exists an integer N_E such that for every prime $p > N_E$ the representation $\rho_{E,p}$ is surjective.

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current strategy $\rightarrow$ trying to exclude that $\operatorname{Im}\rho_{E,p}$ is contained in maximal proper subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$.

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Notation

We call $C_{ns}(p)$ a non-split Cartan subgroup in $\mathrm{GL}_2(\mathbb{F}_p)$ and $C_{ns}^+(p)$ its normaliser.

What do we know about the non-split Cartan case?

Let $p > 37$ be a prime and let ${}^{E}\!/_{\mathbb{Q}}$ be an elliptic curve without CM such that $\operatorname{Im}\rho_{E,p} \subseteq C_{ns}^{+}(p)$.

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We call $G(p)$ the subgroup of $C_{ns}^{+}(p)$ of index 3 such that $G(p) \cap C_{ns}(p) = C_{ns}(p)^3$ (unique up to conjugation).

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Theorem (Zywina, 2015)

There are just two possibilities:

- $\operatorname{Im}\rho_{E,p} = C_{ns}^{+}(p)$;
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Theorem (Le Fourn–Lemos, 2021)

If $\operatorname{Im} \rho_{E,p} = G(p)$, *then* $p < 1.4 \cdot 10^7$ *and* $j(E) \in \mathbb{Z}$.

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Theorem (F.–Lombardo, 2023)

In this setting, we always have $\operatorname{Im} \rho_{E,p} = C_{ns}^{+}(p)$.

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- Adapting Gaudron and Rémond's effective results on the degrees of minimal isogenies, one can show an 'effective surjectivity' theorem, obtaining

$$p < c \cdot \log |j|,$$

for some explicit constant c.

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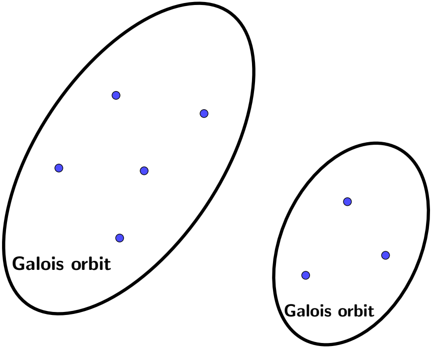
- The effective surjectivity theorem can be slightly improved, keeping the effective constant not too large and making it work for elliptic curves with small heights.
- A detailed study of the image of the inertia subgroups and the canonical subgroup of $E[p]$ allows one to show that the j-invariant must be of the form

$$j = c^3 \cdot p^k,$$

with $k \geq 4$. This allows us to filter the remaining cases and perform a feasible computation.



The modular units defined over $\mathbb{Q}$ of the curve $X_{G(p)}$ have zeros and poles on the cusps of the modular curve, and all the cusps in a same Galois orbit over $\mathbb{Q}$ are of the same type (zero or pole).
The rank of the group of modular units up to constants is equal to the number of Galois orbits of cusps minus 1, hence we need at least 2 orbits for the existence of a non-trivial modular unit.
We need to find a modular unit U integral over $\mathbb{Z}[j]$, which is integer when valued in $j \in \mathbb{Z}$. This holds also for $p^3 U^{-1}$.

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However the best improvement is achieved on the estimates on $\log |j|$, in particular we have

$$\log |j| \leq 40,$$

while the estimates by Le Fourn and Lemos give only $\log |j| \leq 27000$.

How can we further filter the j-invariants?

One can observe that j must be of the form

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hence almost a cube. This allows us to divide by 3 on a 'logarithmic scale' the number of cases to consider.

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Moreover, one can show that j is 'large enough' by proving that $k \geq 4$. This can be achieved by studying the canonical subgroup of the corresponding elliptic curve.

The canonical subgroup

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We know that the geometric p-torsion of the reduction modulo p of E is either ${}^{\mathbb{Z}}\!/_{p\mathbb{Z}}$ or 0.

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$$0 \longrightarrow E_1[p] \longrightarrow E[p] \longrightarrow \widetilde{E}[p] \longrightarrow 0$$

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Remark

$E_1[p]$ is the subgroup of points of p-adic valuation greater than 0.

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Theorem (Lubin, 1979)

Let A be the Hasse invariant of E. The group $E[p]$ admits a canonical subgroup if and only if

$$v_p(A) < \frac{p}{p+1}.$$

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- If E admitted a canonical subgroup, there would be a subgroup of order p stable for the Galois action, hence the image of $\operatorname{Gal}\left(\overline{\mathbb{Q}}_p/K\right)$ would be contained in a Borel subgroup.
- However, the image is contained in $C_{ns}^{+}(p)$, hence is diagonal. This cannot happen, because there must be an element of order $\frac{p^2-1}{6}$ (as shown by Serre).

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If we take an integral model in $\mathcal{O}_K$

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With some calculations, one can show that $v_p(A) = v_p(a)$.
Moreover, since the curve has good reduction $v_p(\Delta) = 0$. We have

$$v_p(j(E)) = v_p\left(12^3 \cdot \frac{(64a)^3}{\Delta}\right) = 3v_p(a) = 3v_p(A)$$

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Finally, studying the image of the inertia one can show that $3 \nmid v_p(j)$, so $p^4 \mid j$.

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To conclude, we test all primes $p < 22000$ and all (isom. classes of) elliptic curves with integral $|j| = |p^k \cdot c^3| \leq e^{40}$ by searching an element of $\operatorname{Im} \rho_{E,p}$ which is not in $G(p)$.

Thank you for your attention