Exam of the course “Dynamical Systems” - Scuola Normale Superiore

The Grobman-Hartman theorem in Banach spaces

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4 aprile 2017

Abstract

I will present a version of the Grobman-Hartman theorem in Banach spaces, generalizing the statement given during the course, which was substantially focused on $\mathbb{R}^n$.

First of all I will recall the main definitions and preliminary results about hyperbolic endomorphisms of a Banach space, then I will state the Grobman-Hartman theorem, showing the relation with the classical version seen in the course, and finally I will give a proof of this result.

Now let us go into the details of the presentation.

Definitions and preliminary results

Let $(E, \|\cdot\|)$ be a Banach space and $T : E \to E$ be a continuous linear map.

Definition. The spectrum of T is defined as

$$\mathrm{Sp}(T) \coloneqq \{\lambda \in \mathbb{C} \ : \ T_{\mathbb{C}} - \lambda \mathrm{id}_{\mathbb{C}} \text{ is not an automorphism of } E_{\mathbb{C}}\} ,$$

where $E_{\mathbb{C}}$, $T_{\mathbb{C}}$, $\mathrm{id}_{\mathbb{C}}$ are the complexifications of E, T, id.

I take the complexifications since in this way the spectum turns out to be nonempty, besides being compact.

Definition. We say that $T : E \to E$ is hyperbolic if

$$\mathrm{Sp}(T) \cap \{|z| = 1\} = \emptyset .$$

Notice that, if T is hyperbolic, since $\mathrm{Sp}(T)$ is compact, there exist $0 < \kappa_s < 1 < \kappa_u$ such that $\mathrm{Sp}(T) \cap \{\kappa_s \le |z| \le \kappa_u\} = \emptyset$. In this case, T is said to be (κ_s, κ_u)-hyperbolic.

We now quote some properties of hyperbolic maps, which are well-known in the finite-dimensional case, but are far from being obvious in this general setting.

Proposition. *Let $T : E \to E$ be a (κ_s, κ_u)-hyperbolic continuous linear map, then*

(i) there exist subspaces E_s and E_u of E such that $E = E_s \oplus E_u$, $T(E_s) \subseteq E_s$, $T(E_u) = E_u$ and $\mathrm{Sp}(T|_{E_s}) = \mathrm{Sp}(T) \cap \{|z| < 1\}$, $\mathrm{Sp}(T|_{E_u}) = \mathrm{Sp}(T) \cap \{|z| > 1\}$.

(ii) there exists on E an adapted norm $\|\cdot\|_T$ for T, that is a norm equivalent to $\|\cdot\|$ such that $\|x_s + x_u\|_T = \max\{\|x_s\|_T, \|x_u\|_T\}$ for all $x_s \in E_s$, $x_u \in E_u$ and $\|T|_{E_s}\|_T \le \kappa_s$, $\|(T|_{E_u})^{-1}\|_T \le \kappa_u^{-1}$.

Statement of the Grobman-Hartman theorem

From now on, I suppose that E is a Banach space, $T : E \to E$ is a (κ_s, κ_u)-hyperbolic endomorphism of E, with $\kappa_s < 1 < \kappa_u$, and I assume that $\|\cdot\|$ is an adapted norm for T.

Theorem (Grobman-Hartman)**.** *Suppose that T is also an automorphism of E. Let $f : E \to E$ be a map such that $\Delta f \coloneqq f - T$ is bounded and Lipschitz with*

$$\mathrm{Lip}(\Delta f) < \varepsilon_1 = \min(\|T^{-1}\|^{-1}, 1 - \kappa_s, 1 - \kappa_u^{-1}) .$$

Then there exists a unique homeomorphism $h = \mathrm{id}_E + \Delta h$ with Δh bounded such that

$$f = h \circ T \circ h^{-1} .$$

Let us investigate the connection with the classical statement seen during the course. In that case, we have a smooth vector field X defined in a neighborhood U of 0 in $\mathbb{R}^n$, which has a hyperbolic equilibrium point in 0. Call X^{lin} the linear part of X and Φ^t_X, $\Phi^t_{X^{\mathrm{lin}}}$ the flows of X and X^{lin} respectively. We know that $\Phi^t_{X^{\mathrm{lin}}}$ coincides with the linear part of the flow Φ^t_X and consequently it holds

$$\Phi^t_X(x) = \Phi^t_{X^{\mathrm{lin}}}(x) + \mathcal{O}(|x^2|) .$$

Then there exists a smaller neighborhood $V \subseteq U$ of 0 such that $\Phi^t_X(x) - \Phi^t_{X^{\mathrm{lin}}}(x)$ is Lipschitz in V with constant strictly less then ε_1. Consequently, we can take $T = \Phi^t_{X^{\mathrm{lin}}}$ and f a Lipschitz extension of Φ^t_X to all $\mathbb{R}^n$, with the same constant, and apply this version of the Grobman-Hartman theorem to obtain a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\Phi^t_X = h \circ \Phi^t_{X^{\mathrm{lin}}} \circ h^{-1} .$$

Notice that during the course we said that there exists a local homeomorphism $\psi : W \to \mathbb{R}^n$ defined in a neighborhood W of 0, such that in W it holds

$$\Phi^t_X \circ \psi = \psi \circ \Phi^t_{X^{\mathrm{lin}}}$$

for every t in a neighborhood of zero small enough. However, with the version of the Grobman-Hartman theorem presented here, we can obtain the conjugacy only at fixed time.

Proof of the Grobman-Hartman theorem

In order to prove the Grobman-Hartman theorem, we show the following slightly more general result, which directly implies the theorem.

Theorem. *Take E, T and ε_1 as in the Grobman-Hartman theorem. Take $f = T + \Delta f$ and $g = T + \Delta g$ such that Δf and Δg are bounded and Lipschitz with $\mathrm{Lip}(\Delta f), \mathrm{Lip}(\Delta g) < \varepsilon_1$. Then there exists a unique homeomorphism $h = \mathrm{id}_E + \Delta h$ with Δh bounded such that*

$$f \circ h = h \circ g .$$

This theorem will be proven using a fixed point lemma in Banach spaces, which will produce the sought homeomorphism h.

Lemma. *Let E be a Banach space, $T : E \to E$ a (κ_s, κ_u)-hyperbolic endomorphism and $f = T + \Delta f$ such that Δf is Lipschitz with $\mathrm{Lip}(\Delta f) < \varepsilon_0 = \min(1 - \kappa_s, 1 - \kappa_u^{-1})$. Then f has a unique fixed point x in E and moreover it holds*

$$\|x\| < (\varepsilon_0 - \mathrm{Lip}(\Delta f))^{-1} \|f(0)\| .$$

References

[1] J.-C. Yoccoz, *Introduction to hyperbolic dynamics*, Real and Complex Dynamical Systems, NATO ASI Series, **464** (1995), pp. 265-291.