Suslin's Problem and Martin Axiom

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In this seminar we will prove the coherence of the non-existence of Suslin's Tree building a model of ZFC where the Martin Axiom holds and $2^{\aleph_0} = \aleph_2$.

1 Suslin's Problem

Suslin's Problem. *Is there a linearly ordered set which satisfies the countable chain condition (ccc) and is not separable?*

Such a set is called a Suslin line. The existence of a Suslin line is equivalent to the existence of a normal Suslin tree.

Definition 1. A *tree* is a poset $(P, <)$ such that $\forall x \in T$ $\{y \colon y < x\}$ is well ordered by $<$.

$o(x) =$ order type of $\{y \colon y < x\}$
α^{th} -level $= \{x \colon o(x) = \alpha\}$
$height(B) = \sup_{x \in B}\{o(x) + 1\}$
A *branch* is a maximal linearly ordered subset of T.
An *antichain* is a set of pairwise incompatible elements of T.

Definition 2. A tree is called a *Suslin tree* if:

1. $height(T) = \omega_1$
2. every branch in T is at most countable
3. every antichain in T is at most countable

A Suslin tree is called *normal* if:

1. T has a unique least point
2. each level of T is at most countable
3. x not maximal $\Rightarrow \{z\colon z > x\}$ is infinite
4. $\forall x \in T$ there is some $z > x$ at each greater level
5. if $o(x) = o(y) = \beta$ with β limit and $\{z\colon z < x\} = \{z\colon z < y\}$ then $x = y$

2 Martin Axiom

Let k be an infinite cardinal.

Martin Axiom k (MA-k). *If a poset $(P, <)$ satisfies ccc and $\mathcal{D}$ is a collection of at most k dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

Martin Axiom (MA). *MA_k holds for every $k < 2^{\aleph_0}$.*

$MA_{\aleph_0}$ is always true while $MA_{2^{\aleph_0}}$ is always false.

Lemma 1. *If $MA_{\aleph_1}$ holds then there is no Suslin tree.*

Proof. Let $(T, <)$ be a normal Suslin tree, then $P_T = (T, >)$ is a poset that satisfies ccc. $\forall \alpha < \omega_1$ I define $D_\alpha = \{x \in T\colon o(x) > \alpha\}$ which is dense in P_T. Let $\mathcal{D} = \{D_\alpha\colon \alpha < \omega_1\}$, then there exists $\mathcal{G}$ $\mathcal{D}$-generic filter on P_T. But $\mathcal{G}$ is a branch of T and $|\mathcal{G}| = \omega_1$ which is absurd. □

Theorem (Solovay - Tennenbaum) . *There is a model $\mathcal{M}$ of ZFC such that $\mathcal{M} \models MA + 2^{\aleph_0} > \aleph_1$.*

3 Iterated Forcing

Let P be a forcing notion in $\mathcal{M}$ and $\mathcal{G}_1 \subseteq P$ a $\mathcal{M}$-generic filter.
Let Q be a poset in $\mathcal{M}[\mathcal{G}_1]$ and $\mathcal{G}_2 \subseteq Q$ a $\mathcal{M}[\mathcal{G}_1]$-generic filter.
I want to show that there exists a $\mathcal{G}$ $\mathcal{M}$-generic filter on R such that:

$$\mathcal{M}[\mathcal{G}_1][\mathcal{G}_2] = \mathcal{M}[\mathcal{G}]$$

We will define this filter using Boolean algebras.

3.1 Definition of $B * \mathbf{C}$

Let B be a complete Boolean algebra in $\mathcal{M}$.
Let $\mathbf{C} \in \mathcal{M}^B$ such that $||\mathbf{C}$ is a complete Boolean algebra $|| = 1$.
I consider the class of all $\mathbf{c} \in \mathcal{M}^B$ such that $||\mathbf{c} \in \mathbf{C}|| = 1$ and I define the equivalence relationship $\mathbf{c_1} \sim \mathbf{c_2} \iff ||\mathbf{c_1} = \mathbf{c_2}|| = 1$.
Then there is a set D which contains exactly one element for each $\sim$-equivalence class.
D is a maximal subset in $\mathcal{M}^B$ such that:

1. $||c \in \mathbf{C}|| = 1 \ \forall c \in D$
2. $c_1, c_2 \in D, c_1 \neq c_2 \Rightarrow ||c_1 = c_2|| < 1$

I define $+_D$:

$$\forall c_1, c_2 \in D \ \exists c \in D \text{ such that } ||c = c_1 +_C c_2|| = 1$$

this c is unique and I define $c = c_1 +_D c_2$.
The operations $\cdot_D$ and $-_D$ are defined similarly.
With this operations D is a complete Boolean algebra (in $\mathcal{M}$).
I define $B * \mathbf{C} = D$.

Observation 1. *There exists an embedding $B \hookrightarrow B * \mathbf{C}$ given by:*

$$b \mapsto c_b \colon ||c_b = 0_C|| = -b$$
$$||c_b = 1_C|| = b$$

So we can assume that B is a complete subalgebra of $B * \mathbf{C}$.

Lemma 2. *Let B be a complete Boolean algebra in $\mathcal{M}$, let $\mathbf{C} \in \mathcal{M}^B$ be such that $||\mathbf{C}$ is a complete Boolean algebra$|| = 1$ and let $D = B * \mathbf{C}$ such that B is a complete subalgebra of D. Then*

1. *If $\mathcal{G}_1$ is an $\mathcal{M}$-generic ultrafilter on B, $C = i_{\mathcal{G}_1}(\mathbf{C})$ and $\mathcal{G}_2$ is an $\mathcal{M}[\mathcal{G}_1]$-generic ultrafilter on C then there is an $\mathcal{M}$-generic ultrafilter $\mathcal{G}$ on $B * \mathbf{C}$ such that:*
$$\mathcal{M}[\mathcal{G}_1][\mathcal{G}_2] = \mathcal{M}[\mathcal{G}]$$
2. *If $\mathcal{G}$ is an $\mathcal{M}$-generic ultrafilter on $B * \mathbf{C}$. $\mathcal{G}_1 = \mathcal{G} \cap B$ and $C = i_{\mathcal{G}_1}(\mathbf{C})$ then there is an $\mathcal{M}[\mathcal{G}_1]$-generic ultrafilter $\mathcal{G}_2$ on C such that:*
$$\mathcal{M}[\mathcal{G}_1][\mathcal{G}_2] = \mathcal{M}[\mathcal{G}]$$

Proof. (Idea) I define $\forall c \in B * \mathbf{C}\ c \in \mathcal{G} \iff i_{\mathcal{G}_1}(c) \in \mathcal{G}_2$ and prove that the resulting set (either $\mathcal{G}$ or $\mathcal{G}_2$) is the wanted generic set. □

Lemma 3. *B satisfies ccc and $||\mathbf{C}$ satisfies ccc $|| = 1$ iff $B * \mathbf{C}$ satisfies ccc.*

4 Direct Limit

Let α be a limit ordinal and $\forall i \in \alpha$ let B_i be a complete Boolean algebra such that $\forall j > i$ B_i is a complete subalgebra of B_j. Let $C = \bigcup_{i<\alpha} B_i$, then we call the completion B of C the *direct limit* of $\{B_i\}$, $B = limdir_{i\leq\alpha} B_i$.

Lemma 4. *Let k be a regular cardinal, $k > \aleph_0$.*
Let α be a limit ordinal and $\{B_i\}$ a sequence of complete Boolean algebras such that $\forall j > i$ B_i is a complete subalgebra of B_j and for each limit $\gamma < \alpha$ we have $B_\gamma = limdir_{i\leq\gamma} B_i$. Let $B = limdir_{i\leq\alpha} B_i$.
Then if each B_i is k-saturated then B is k-saturated.
In particular if each B_i satisfies ccc then B satisfies ccc.

5 Construction of the model

Let $\mathcal{M}$ be a transitive model of ZFC + GCH. We will construct a complete Boolean algebra B such that if $\mathcal{G}$ is an $\mathcal{M}$-generic filter on B then

$$\mathcal{M}[\mathcal{G}] \models MA + 2^{\aleph_0} \leq \aleph_2$$

We will have $|B| = \aleph_2$ so $[2^{\aleph_0}]^{\mathcal{M}[\mathcal{G}]} \leq [|B|^{\aleph_0}]^{\mathcal{M}} = [{\aleph_2}^{\aleph_0}]^{\mathcal{M}} = [\aleph_2]^{\mathcal{M}}$ ([1], lemma 19.4) and B will satisfy ccc and so cardinals will be preserved.

5.1 Definition of B

Let $\{B_\alpha\}$ be a sequence such that:

1. $\alpha < \beta \Rightarrow B_\alpha$ is a complete subalgebra of B_β
2. γ limit $\Rightarrow B_\gamma = limdir_{i\leq\gamma} B_i$
3. each B_α satisfies ccc
4. $|B_\alpha| \leq \aleph_2$

I define $B = limdir_{i<\omega_2} B_i$.
Using lemma 3 we have that B satisfies ccc and, since $C = \bigcap_{\alpha<\omega_2} B_\alpha$ is dense in B and $|C| = \aleph_2$, we have $|B| \leq {\aleph_2}^{\aleph_0} = \aleph_2$.

5.2 Construction of B_α

Observation 2. *If D is a complete Boolean algebra such that $|D| \leq \aleph_2$ that satisfies ccc then the number of D-valued binary relationships on $\check{\omega}_1$ is $\aleph_2$.*
So $\mathcal{R} = \{\mathbf{R}^D_\alpha \ D-valued\ relationship\ on\ \check{\omega}_1\}$ can be indexed with ω_2.

Let $\alpha \mapsto (\beta_\alpha, \gamma_\alpha)$ be a mapping of ω_2 onto $\omega_2 \times \omega_2$ such that $\alpha \leq \beta_\alpha$.
I define B_α by induction.
$B_0 = \{0, 1\}$ and $B_\gamma = limdir_{i<\gamma} B_i$ for γ limit.
I construct $B_{\alpha+1}$ given $\{B_i\}_{i\leq\alpha}$.
Let $D = B_{\beta_\alpha}$ and $\mathbf{R} = \mathbf{R}^D_{\gamma_\alpha}$ γ_α-th relationship on $\check{\omega}_1$, $\mathbf{R} \in \mathcal{M}^{B_\alpha}$.
Let $b = ||\mathbf{R}$ is a partial ordering of $\check{\omega}_1$ and $(\check{\omega}_1, \mathbf{R})$ satisfies ccc$||$.
Let $\mathbf{C} \in \mathcal{M}^{B_\alpha}$ be the complete Boolean algebra such that:

- $||\mathbf{C}$ is the trivial algebra $|| = -b$
- $||\mathbf{C} = r.o.(\check{\omega}_1, \mathbf{R})|| = b$

I define $B_{\alpha+1} = B_\alpha * \mathbf{C}$.
I show $B_{\alpha+1}$ has the required properties.
By definition

$||\mathbf{C}$ is a complete Boolean algebra, satisfies ccc
and has a dense subset of size $\leq \aleph_1|| = 1$

So by lemma 3 $B_{\alpha+1}$ satisfies ccc.
To show that $|B_{\alpha+1}| \leq \aleph_2$ we take a subset $\mathbf{Q} \subseteq \mathbf{C}$ of size $\leq \aleph_1$ such that $|B_\alpha * \mathbf{Q}| \leq \aleph_2$ and $B_\alpha * \mathbf{Q}$ is dense in $B_{\alpha+1}$.

5.3 $\mathcal{M}[\mathcal{G}] \models MA_{\aleph_1}$

Let $\mathcal{G}$ be a generic ultrafilter on B (see Observation 3).
We define $\mathcal{G}_\alpha = \mathcal{G} \cap B_\alpha$.

Lemma 5. *If $X \in \mathcal{M}[\mathcal{G}]$ is a subset of ω_1 then exists $\alpha < \omega_2$ such that $X \in \mathcal{M}[\mathcal{G}_\alpha]$.*

Let $(P,<)$ be a poset in $\mathcal{M}[\mathcal{G}]$ that satisfies ccc. We can assume $|P| \leq \aleph_1$ (see [1], lemma 23.2), so there is a binary relationship $\mathcal{R}$ on $\check{\omega}_1$ such that $(P,<) \cong (\omega_1, \mathcal{R})$.

So we can assume $(P,<) = (\omega_1, \mathcal{R})$.

Let $\mathcal{D} \in \mathcal{M}[\mathcal{G}]$ be a collection of at most $\aleph_1$ dense subsets of ω_1.

Using the previous lemma, since $\mathcal{D}$ can be encoded in $\omega_1 \times \omega_1$, there is $\beta < \omega_2$ such that $\mathcal{D}, \mathcal{R} \in \mathcal{M}[\mathcal{G}_\beta]$.

Let $\mathbf{R} \in \mathcal{M}^{B_\beta}$ be a name for $\mathcal{R}$, then $\mathbf{R} = \mathbf{R}_\gamma^{B_\beta}$ for some $\gamma < \omega_2$.

Let $\alpha < \omega_2$ be such that $\alpha \mapsto (\beta_\alpha, \gamma_\alpha) = (\beta, \gamma)$, $\alpha \geq \beta$.

Now, since $\mathcal{M}[\mathcal{G}_\alpha]$ is a submodel of $\mathcal{M}[\mathcal{G}]$, we have

$$\mathcal{M}[\mathcal{G}] \models (\omega_1, \mathcal{R}) \text{ satisfies ccc } \Rightarrow \mathcal{M}[\mathcal{G}_\alpha] \models (\omega_1, \mathcal{R}) \text{ satisfies ccc}$$

So we have $b = ||(\check{\omega}_1, \mathbf{R}) \text{ satisfies ccc } || \in \mathcal{G}_\alpha$.

By construction $B_{\alpha+1} = B_\alpha * \mathbf{C}$ and $||\mathbf{C} = r.o.(\check{\omega}_1, \mathbf{R})|| = b$ so

$$\mathcal{M}[\mathcal{G}_\alpha] \models C = r.o.(\omega_1, R)$$

Using lemma 2 exists $\mathcal{H}$ $\mathcal{M}[\mathcal{G}_\alpha]$-generic ultrafilter on C and filter on (ω_1, R) such that

$$\mathcal{M}[\mathcal{G}_{\alpha+1}] = \mathcal{M}[\mathcal{G}_\alpha][\mathcal{H}]$$

Since $\mathcal{D} \in \mathcal{M}[\mathcal{G}_\alpha]$ and $\forall D \in \mathcal{D}$ D is dense in (ω_1, R) we have $\mathcal{H} \cap D \neq \emptyset$ so $\mathcal{H}$ is $\mathcal{D}$-generic on (ω_1, R).

We can now conclude that $\mathcal{M}[\mathcal{G}] \models MA_{\aleph_1}$.

Observation 3. *We can assume an $\mathcal{M}$-generic set over B exists because the following sentences are absolute between transitive models:*

- *the definition of ω_2*
- *the definition of $B * \mathbf{C}$*
- *the definition of $limdir B_i$*

So the definition of B is absolute between transitive models and we can use the same argument as in [1], pag. 175.

References

[1] Thomas Jech. *Set Theory.* Springer Berlin Heidelberg, 1997.