Non elementary methods in combinatorial number theory: Roth's and Sarkozy's theorems

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Introduction

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Asymptotic density

Let $A \subseteq \mathbb{N}$, the asymptotic (upper) *density* of A is defined as

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N}$$

Roth's theorem

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Then $x, x + r, x + 2r \in A$ for some $x, r \in \mathbb{N}$.

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We will then use these approaches in a nonstandard setting.

Fourier analysis

Character

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$$e_\xi(n) = e^{2\pi i \frac{\xi n}{N}}$$

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Let $\xi \in \mathbb{Z}_N$ and $f : \mathbb{Z}_N \to \mathbb{C}$.
We define

$$\hat{f}(\xi) = \mathbf{E}_{n \in \mathbb{Z}_N} f(n)\overline{e_\xi(n)} = \frac{1}{N} \sum_{n \in \mathbb{Z}_N} f(n)\overline{e_\xi(n)}$$

Density increment

The *density increment* approach goes through two main steps

- no arithmetic progression $\Rightarrow$ correlation with a character e_ξ
- correlation with a character e_ξ $\Rightarrow$ density increment

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By iterating this process enough times we reach a contradiction.

Roth's theorem

It is convenient to define

$$\Lambda_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} \mathbb{1}_A(n)\mathbb{1}_A(n+r)\mathbb{1}_A(n+2r)$$

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Roth's theorem

For any $N \in \mathbb{N}$ and for any $A \subseteq [1, N]$ such that $|A| = \delta N > 0$ we have

$$\Lambda_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = \Omega_\delta(1)$$

i.e. $\Lambda_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) \geq C_\delta$ for some positive constant C_δ depending only on δ.

Step 1. No AP implies correlation

We want to show that if A does not contain an arithmetic progression of length 3 then A is correlated with some character e_ξ.

Proposition

Let $A \subseteq [1, N]$ with $|A| = \delta N$ for some $0 < \delta \leq 1$.
Assume $N \geq \frac{100}{\delta^2}$ and that A does not contain any arithmetic progression of length 3.
Then there exists ξ such that

$$\left| \mathbf{E}_{n \in [1,N]} (\mathbb{1}_A(n) - \delta) e_\xi(n) \right| = \Omega(\delta^2)$$

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$$[1, N] = \bigsqcup_{j=1}^{m} P_j \sqcup E$$

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$$|e_\xi(x) - e_\xi(y)| \leq \epsilon \text{ for } x, y \in P_j$$

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- E is small

To find the spacing h we use the

Kronecker approximation theorem

For any $M > 0$ and $\xi \in \mathbb{R}$ there exists an integer $0 < h < M$ such that

$$\|h \cdot \xi\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{M}$$

Correlation implies density increment

Let $A \subseteq [1, N]$ with density $\delta > 0$. If

$$\mathbf{E}_{n \in [1,N]}(\mathbb{1}_A(n) - \delta)e_\xi(n) \geq \sigma$$

for some ξ and $\sigma > 0$ then there exist $P \subseteq [1, N]$ arithmetic progression such that

$$|P| = \Omega(\sigma^2 N^{\frac{1}{2}}) \quad \text{and} \quad \frac{|A \cap P|}{|P|} \geq \delta + \frac{\sigma}{4}$$

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we can repeat the process until we obtain an absurdum since the density cannot exceed 1.

Nonstandard analysis

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- Is an ordered field and contains $\mathbb{R}$
- Contains both infinite and infinitesimal numbers
- Every finite hyperreal r is infinitely close to exactly one real number called *standard part* of r and denoted with $st(r)$
- Has the same "elementary" properties of $\mathbb{R}$ if we consider only **internal** sets and functions (**Transfer principle**)

Nonstandard setting

Roth's theorem

Let $N \in {}^*\mathbb{N}$ infinite and let $A \subseteq [1, N]$ be an internal subset such that $\frac{|A|}{N} \not\approx 0$. Then A contains an arithmetic progression of length 3.

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With slight alterations to the standard proof we obtain the result in nonstandard setting.
For instance, in the fragmentation step, we can take the length ν of the subprogression to be **infinite but infinitely smaller** than N.

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Let $N \in {}^*\mathbb{N}$ infinite and let $A \subseteq [1, N]$ be an internal subset such that $\frac{|A|}{N} \not\approx 0$. Then A contains two elements whose difference is a perfect square.

Similarly to what we have done for Roth's theorem we define

$$\Lambda_2(\mathbb{1}_A, \mathbb{1}_A) = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \mathbb{1}_A(n)\mathbb{1}_A(m)\mathbb{1}_S(n-m)$$

where $S = \{d^2 \colon 1 \leq d \leq \sqrt{N}\}$.

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- An estimate on $\|\hat{\mathbb{1}}_S\|_{L^2}$
- A "quadratic fragmentation" of $[1, N]$, i.e. $P_j = \{s_j + h^2 n\}_{n \le \nu}$

To obtain the estimate we use a bound on

Weyl sum

We define *Weyl sum* the quantity

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Quadratic recurrence

For all $N \in \mathbb{Z}$ sufficiently large and $\xi \in \mathbb{R}$ there exists an integer $1 \leq h \leq N$ such that

$$\left\| h^2 \cdot \xi \right\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{N^{\frac{1}{10}}}$$

Energy increment

The energy increment approach aims to find a decomposition of $\mathbb{1}_A$ in:

- A "periodic" component
- A "pseudo-random" component

Ergodic proof

Using **Furstenberg correspondence principle** we have that Roth's theorem is equivalent to

Ergodic Roth

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. For any $E \in \mathcal{B}$ with $\mu(E) > 0$ there exists some $n > 0$ such that

$$\mu(E \cap T^n E \cap T^{2n} E) > 0$$

To prove this theorem we use the decomposition

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$$f \in WM(X) \quad \text{if } \mathcal{D}\text{-lim } \langle f, T^n f\rangle = \mathbf{E}(f)^2$$

The almost periodic component represents the structured part of our set and the weak mixing component represents the pseudo-random factor.

Roth's theorem

To use this idea in the discrete setting $[1, N]$ we aim to decompose a function in:

$$f = f_U + f_{U^\perp} \text{ with } \begin{cases} f_{U^\perp} \text{ almost periodic} \\ f_U \text{ negligible} \end{cases}$$

Almost periodicity

Almost periodicity

Let $k > 1$ be an integer and $\sigma > 0$. A function $f : [1, N] \to \mathbb{C}$ is (k, σ)-*almost periodic* if there exist frequencies $\xi_1, \dots, \xi_k$ and $c_1, \dots, c_k \in \mathbb{C}$, $|c_1|, \dots, |c_k| \le 1$ such that

$$\|f - \sum_{j=1}^{k} c_j e_{\xi_j}\|_{L^2} \le \sigma$$

Almost periodic functions are recurrent

Let $f\colon [1,N] \to \mathbb{R}^+$, $0 \leq f \leq 1$ and $\mathbf{E}(f) \geq \delta$. If f is (k,σ)-almost periodic for some $k \geq 1$ and $0 < \sigma < \frac{\delta^3}{8}$ then

$$\Lambda_3(f,f,f) = \Omega\left(\left(\frac{\delta}{k}\right)^k \delta^3\right)$$

Negligible component

Let $f : \mathbb{Z}_N \to \mathbb{C}$, we define

$$\|f\|_{u^2} = \max_{\xi \in \mathbb{Z}_N} \left|\hat{f}(\xi)\right|$$

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Estimate on Λ_3

Let $f, g, h \colon \mathbb{Z}_N \to \mathbb{C}$. Then we have the estimate

$$|\Lambda_3(f, g, h)| \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{u^2}$$

Energy Increment

We define the *energy* of an algebra $\mathcal{B}$ on $[1, N]$ with respect to f to be

$$\mathcal{E}_f(\mathcal{B}) = \| \mathbf{E}(f|\mathcal{B})\|_{L^2}^2 = \mathbf{E}_{x \in Z} \left|\mathbf{E}(f|\mathcal{B})(x)\right|^2$$

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with

$$\mathbf{E}(f|\mathcal{B})(x) = \frac{1}{|\mathcal{B}(x)|} \sum_{y\in\mathcal{B}(x)} f(y)$$

where $\mathcal{B}(x)$ is the unique atom of $\mathcal{B}$ which contains x.

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Energy increment

If $f - \mathbf{E}(f|\mathcal{B})$ has one large Fourier coefficient then we can find a new σ-algebra $\mathcal{B}'$ with more energy with respect to f.

Koopman-von Neumann decomposition

Let $\sigma > 0$ and let $F\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary function. Then there exists k such that for any $f\colon [1, N] \to [0, 1]$ there exists a decomposition $f = f_{U^\perp} + f_U$ with:

- $f_{U^\perp}$ is (k, σ)-almost periodic
- $\|f_U\|_{u^2} \leq \dfrac{1}{F(\sigma, k)}$

Proof

Decomposing

$$\mathbb{1}_A = f_U + f_{U^\perp} \text{ with } \begin{cases} f_{U^\perp} \text{ almost periodic} \\ \|f_U\|_{u^2} \text{ small} \end{cases}$$

Proof

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using the inequality

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using the inequality

$$|\Lambda_3(f,g,h)| \leq \|f\|_{L^2} \|g\|_{L^2} \|h\|_{u^2}$$

we obtain

$$\begin{aligned} \Lambda_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) &= \Lambda_3(f_U, f_U, f_U) + \Lambda_3(f_U, f_U, f_{U^\perp}) + \\ &+ \cdots + \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_U) + \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_{U^\perp}) = \\ &= \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_{U^\perp}) + \epsilon = \Omega_\delta(1) \end{aligned}$$

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Loeb measure

There is a unique σ-additive extension of μ to the σ-algebra $\mathcal{L}$ generated by the internal sets. The completion of this measure is called *Loeb measure* and is denoted with μ_L.

Nonstandard Roth

We define

$$\Lambda_3(f,g,h) = \int_{[1,N]} \int_{[-N,N]} f(n)g(n+r)h(n+2r)\, d\mu_L(r) d\mu_L(n)$$

for any $f, g, h\colon [1,N] \to \mathbb{C}$ Loeb measurable.

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for any $f, g, h\colon [1, N] \to \mathbb{C}$ Loeb measurable.

Roth's theorem

Let $N \in {}^*\mathbb{N}$ be infinite and let $f \in L^\infty(\mu)$, $f\colon [1, N] \to \mathbb{R}$ bounded, non negative, with $\mathbf{E}_{n\in[1,N]} f(n) > 0$. Then

$$\Lambda_3(f,f,f) > 0$$

Loeb integral

Let $F, G, H\colon [1, N] \to {}^*\mathbb{C}$ be internal functions bounded by a finite number, then

$$\Lambda_3(st\,(F), st\,(G), st\,(H)) = st\left(\frac{1}{N(2N+1)}\sum_{n=1}^{N}\sum_{r=-N}^{N} F(n)G(n+r)H(n+2r)\right)$$

Almost periodicity

Using the properties of the Loeb integral we have that

Almost periodic functions are recurrent

Let $f \in L^{\infty}(\mu)$, $0 \leq f \leq 1$, $\mathbf{E}_{[N]}(f) = \delta > 0$.
If f is (k, σ)-almost periodic with $\sigma \leq \frac{\delta^3}{8}$ then

$$\Lambda_3(f, f, f) > 0$$

Compact factor

We define $\mathcal{Z}^1$ to be the σ-algebra generated by the characters $\{e_\xi : \xi \in [1, N]\}$.

Theorem

Let $f : [1, N] \to \mathbb{C}$ be $\mathcal{Z}^1$-measurable and $f \in L^\infty(\mu)$. Then for any $\sigma > 0$ there exists k such that f is (k, σ)-almost periodic.

Roth's theorem

We can then decompose

$$\mathbb{1}_A = f_U + f_{U^\perp} \text{ with} \begin{cases} f_{U^\perp} = \mathbf{E}(\mathbb{1}_A|\mathcal{Z}^1) \\ f_U = \mathbb{1}_A - f_U \end{cases}$$

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Lemma

Let $f, g, h \colon [1, N] \to \mathbb{C}$, $f, g, h \in L^\infty(\mu)$.
If $\mathbf{E}(f|\mathcal{Z}^1) = 0$ then $\Lambda_3(f, g, h) = 0$.

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Lemma

Let $f, g, h \colon [1, N] \to \mathbb{C}$, $f, g, h \in L^\infty(\mu)$.
If $\mathbf{E}(f|\mathcal{Z}^1) = 0$ then $\Lambda_3(f, g, h) = 0$.

Thus

$$\begin{aligned} \Lambda_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) &= \Lambda_3(f_U, f_U, f_U) + \Lambda_3(f_U, f_U, f_{U^\perp}) + \\ &+ \cdots + \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_U) + \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_{U^\perp}) = \\ &= \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_{U^\perp}) > 0 \end{aligned}$$

Final remarks

- The density increment proof of Roth's theorem in a nonstandard settings is easier to obtain, reduces a bit the length of computations and is easily adapted to prove Sarkozy's theorem but does not give any estimate.

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- The density increment proof of Roth's theorem in a nonstandard settings is easier to obtain, reduces a bit the length of computations and is easily adapted to prove Sarkozy's theorem but does not give any estimate.
- The energy increment proof of Roth's theorem in a nonstandard settings provides an easy way to obtain the decomposition by using both hyperfinite (discrete) and continuous techniques.

Possible developments

- Extend the density increment proof of Sarkozy's theorem to patterns of the form $x, x + P(n)$ with $P(n)$ polynomial, $P(0) = 0$.

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- Extend the density increment proof of Sarkozy's theorem to patterns of the form $x, x + P(n)$ with $P(n)$ polynomial, $P(0) = 0$.
- Find a "pure" nonstandard proof of Roth's theorem via energy increment by replacing the characters e_ξ with a suitable subspace of $\mathbb{C}^N$ with $N \in {}^*\mathbb{N}$ infinite.