What does maximize a Strichartz estimate?

ANALYSIS GRADUATE SEMINAR

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joint work with

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§0. Sharp inequalities

Consider an operator $T : X \to Y$

$$\|Tf\|_Y \leq C(T)\,\|f\|_X \quad \forall f \in X$$

$$\|T\| := \sup_{f \neq 0} \frac{\|Tf\|_Y}{\|f\|_X} = C(T)$$

- When is this sup achieved?
- For which f?

Today
- $X, Y \simeq L^p$, $1 \leq p \leq \infty$.
- $T \in \{$ Fourier transform ; Solution map of some PDE $\}$

$$f \mapsto \int f(x)\, e^{-ix\xi}\, dx \qquad u_0 \longmapsto u(t)$$

§0. Sharp inequalities

$$\| Tf \|_Y \leq C(T) \| f \|_X \quad \forall f \in X$$

__Def__ A maximiser is a function f such that $\| Tf \|_Y = C \| f \|_X$

__Example__ (Fourier transform)

- (Plancherel) $\quad \| \hat{f} \|_{L^2} = \| f \|_{L^2}$

- (Hausdorff-Young) $\quad \| \hat{f} \|_{L^{p'}} \leq \| f \|_{L^p}$, $1 \leq p \leq 2$, $\frac{1}{p'} + \frac{1}{p} = 1$

▷ (Sharp Hausdorff-Young) $\quad \| \hat{f} \|_{L^{p'}} \leq C_p \| f \|_{L^p}$, $1 \leq p \leq 2$

$$C_p = \left(\frac{p^{1/p}}{p'^{1/p'}} \right)^{d/2} \leq 1$$

Thm (Babenko, Beckner, Lieb): $\{ c\, e^{-\langle Ax, x \rangle} \}$ are the only maximisers.

§0. Sharp inequalities

How do we find maximisers?

Def A maximiser is a function f such that $\|Tf\|_Y = C\|f\|_X$

Def A maximising sequence is a sequence $\{f_n\}_n$ of functions with $\|f_n\|_X = 1$ such that $\lim_{n\to\infty} \|Tf_n\|_Y = C$

If $f_n \to f$ then f is a maximiser.

Maximising sequences may NOT converge!

Example

$$c_0 := \left\{ \{x_n\}_{n\in\mathbb{N}} : \lim_n x_n = 0 \right\}$$

$$\ell^1 := \left\{ \{x_n\}_{n\in\mathbb{N}} : \sum_n |x_n| < +\infty \right\}$$

$$\ell^\infty := \left\{ \{x_n\}_{n\in\mathbb{N}} : \sup_n |x_n| < +\infty \right\}$$

$$(c_0)^* \simeq \ell^1, \quad (\ell^1)^* \simeq \ell^\infty$$

Consider $u = \{1-\frac{1}{n}\}_n \in \ell^\infty$

$\varphi : \ell^1 \to \mathbb{R}$

$$v \mapsto \varphi(v) = \langle u, v\rangle = \sum_{i=1}^\infty v_i\left(1-\frac{1}{n}\right), \quad \|\varphi\| = \sup_{v\in\ell^1} \frac{|\varphi(v)|}{\|v\|} \le 1$$

$$e_n = (0, \dots, \overset{\overset{n}{\downarrow}}{1}, 0, \dots) \in \ell^1, \quad \|e_n\| = 1. \qquad \varphi(e_n) = 1 - \frac{1}{n}$$

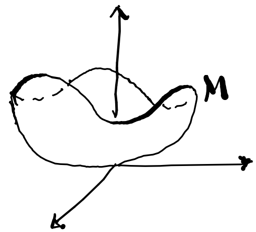
$$\langle e_n, a\rangle \xrightarrow[n\to\infty]{} 0 \quad \forall a \in c_0 \qquad \text{weak-* conv.}$$

if weak limit $\exists$, it must be 0, but

$$|\varphi(e_n)| = 1 - \frac{1}{n} \xrightarrow{n\to\infty} 1 \qquad \Rightarrow \text{NO strong limit.}$$

§1. Restriction theory

Let $M \subseteq \mathbb{R}^3$ a surface.

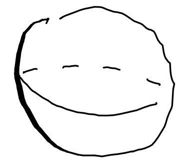


M

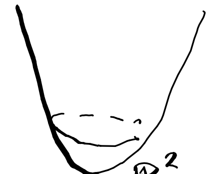
$f \in C^0(\mathbb{R}^3)$

$f \longmapsto f|_M \in C^0(M)$

Examples:



$\mathbb{S}^2$



$\mathbb{P}^2$

$f \in L^1(\mathbb{R}^3) \Rightarrow \hat{f} \in C^0(\mathbb{R}^3)$

$\hat{f}|_M \in C^0(M)$ makes sense

$\lightning$

$f \in L^2(\mathbb{R}^3) \Rightarrow \hat{f} \in L^2(\mathbb{R}^3)$

$\hat{f}|_M$ is not defined!

What if $f \in L^p(\mathbb{R}^3)$, $1 < p < 2$?

$\hat{f} \in L^{p'}(\mathbb{R}^3) \longmapsto \hat{f}|_M \in L^q(M)$?

If so, $\|\hat{f}\|_{L^q(M)} \leq C \|f\|_{L^p(\mathbb{R}^3)}$ Fourier Restriction Estimate

§ 1.1 Dual formulation: Fourier Extension

Given f :  $\mathbb{S}^2, \sigma \longrightarrow \mathbb{R}$ or $\mathbb{C}$

$d\sigma(x, x') = \delta(|x|^2 - 1)$ measure on $\mathbb{R}^3$ supp on $\mathbb{S}^2$.

$$f\, d\sigma \longmapsto (f\, d\sigma)^{\vee}(x) \in L^r(\mathbb{R}^3) \;?$$

$$\|(f\, d\sigma)^{\vee}\|_{L^r(\mathbb{R}^3)} \leq C \|f\|_{L^q(\mathbb{S}^2, \sigma)}$$

Fourier Extension inequality

Thm (Stein-Tomas):

$$\|(f\, d\sigma)^{\vee}\|_{L^4(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{S}^2, \sigma)}$$

$M \subseteq \mathbb{R}^{d+1}$ d-dimensional manifold

$$\|(f\, d\sigma)^{\vee}\|_{L^r(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(M^d, \sigma)} \qquad r = 2 + \frac{4}{d}$$

§ 2. Schrödinger equation

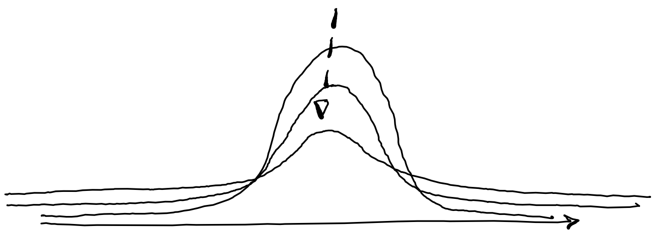
$$\begin{cases} i\partial_t u(t,x) = \partial_x^2 u(t,x) \\ u(0,x) = f(x) \qquad \in \mathcal{S}(\mathbb{R}) \end{cases}$$

$$u(t,x) = e^{-it\partial_x^2} f(x) := \left(e^{+it\xi^2}\hat{f}(\xi)\right)^{\vee}(x) = c\left(f * \frac{e^{-i|\cdot|^2/t}}{\sqrt{4\pi t}}\right)(x)$$

Estimates:

▷ $\|u(t,\cdot)\|_{L^2_x} = \|e^{it|\cdot|^2}\hat{f}\|_{L^2_\xi} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}$ (constant!)

▷ $\|u(t,\cdot)\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{4\pi t}}\|f\|_{L^1}$



How to measure dispersion?

Does $\left(t \mapsto \|u(t,\cdot)\|_{L^p_x}\right)$ belong to some L^q?

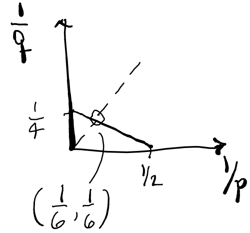
§ 3. Strichartz estimates

$2 \le q, p \le \infty$

$$\Big\| \, \| e^{-it\partial^2} f \|_{L^p_x} \Big\|_{L^q_t} \le C_{p,q} \| f \|_{L^2}$$

optimal constant

scaling: $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$



- What is the value of $C_{p,q}$?
- Are there functions which maximise the inequality?
- Can we characterise them?
- Stability of maximisers?

Remark Similar estimates hold in higher dimension and for other dispersive equations.

Remark when $p = q = 6$

$$\| e^{it\partial^2} f \|_{L^6_{t,x}(\mathbb{R}^2_+)} \le C_0 \| f \|_{L^2(\mathbb{R})}$$

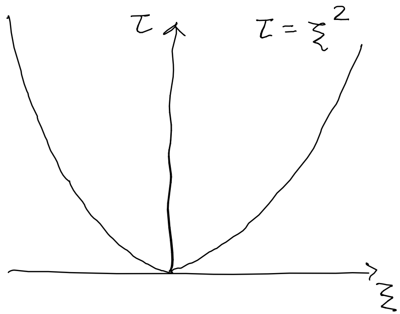
- Kunze (2003) proved existence of extremisers
- Foschi (2007) maximisers are $\{ g(e^{-x^2}) \}_{g \in G}$

§3. Strichartz estimates via Fourier Extension

$$\|e^{-it\partial^2} f\|_{L^6_{t,x}} \leq C_0 \|f\|_{L^2}$$

$$e^{-it\partial^2} f(x) = \left(e^{it|\cdot|^2}\hat{f}\right)^\vee(x) = \int_{-\infty}^{\infty} e^{ix\xi + it|\xi|^2}\hat{f}(\xi)\,d\xi$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi + t\tau)}\,\delta(\tau - |\xi|^2)\,\hat{f}(\xi)\,d\xi\,d\tau$$



$$= \mathcal{F}^{-1}(\hat{f}\,d\sigma)(x,t)$$

$$f \longmapsto e^{-it\partial^2} f$$

$$f \mapsto \hat{f} \mapsto \hat{f}\,d\sigma \mapsto \mathcal{F}^{-1}(\hat{f}\,d\sigma)$$

Approach by Foschi (2007)

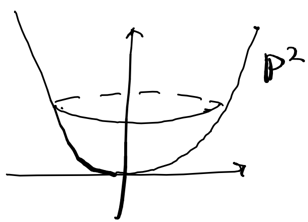
$$\| (f\sigma)^\vee \|_{L^4(\mathbb{R}^3)}^4 = \| (f\sigma)^\vee \cdot (f\sigma)^\vee \|_{L^2}^2 = \| f\sigma * f\sigma \|_{L^2}^2$$

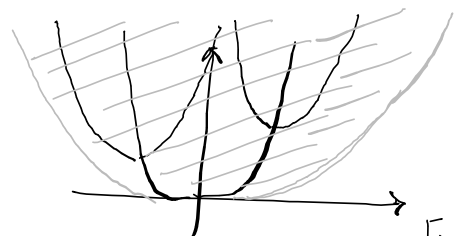
$$\boxed{\widehat{fg} = c \; \hat{f} * \hat{g}}$$

Studying convolution of singular measures!

$$\sigma(\tau, \xi) = \delta(\tau - |\xi|^2)$$

$$\operatorname{supp}(\sigma * \sigma) \subseteq \operatorname{supp}(\sigma) + \operatorname{supp}(\sigma)$$





$$\| f\sigma * f\sigma \|_{L^2}^2 \leq \| \sigma * \sigma \|_{L^\infty}^2 \; \| f\sigma * f\sigma \|_{L^2}^2 = c \, \|f\|_{L^2}^4$$

[in the notes]

- Maximisers are Gaussians in dim 1 and 2.
- Characterise maximisers in $d \geq 3$ is open.

§4. 4th order Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) + \partial_x^4 u(t,x) = 0 \\ u(0,x) = f(x) \in L^2(\mathbb{R}) \end{cases}$$

solution: $u(t,x) = e^{it\partial^4} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x\xi + t\xi^4)} \hat{f}(\xi)\, d\xi =: \mathcal{E}\hat{f}(x,t)$

Fourier extension from $\{\tau = \xi^4\}$

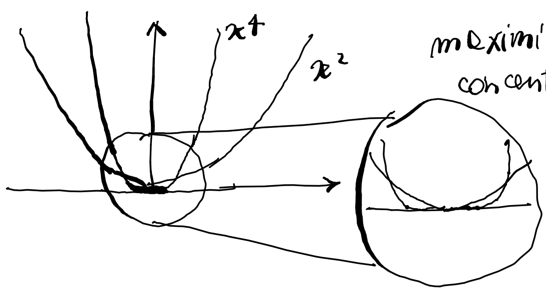
▷ Refined Strichartz estimate

$$\| \partial^{1/3} e^{it\partial^4} f \|_{L^6_{t,x}(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R})}$$

Do maximisers exist?

Thm (Jiang, Pausader, Shao) (2010): $C_0 \leq C$. If $C_0 < C$, then maximisers exist.

Why so?



maximisers for Schrödinger concentrating at the origin do as well of a job for the quartic

§ 4. 4^{th} order Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) + \partial_x^4 u(t,x) = 0 \\ u(0,x) = f(x) \in L^2(\mathbb{R}) \end{cases}$$

solution: $u(t,x) = e^{it\partial^4} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^4)} \hat{f}(\xi)\, d\xi =: \mathcal{E}\hat{f}(x,t)$

Fourier extension from $\{\tau = \xi^4\}$

▷ Refined Strichartz estimate

$$\| \partial^{1/3} e^{it\partial^4} f \|_{L^6_{t,x}(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R})}$$

Do maximisers exist ?

Thm (B., Oliveira e Silva, Quilodran, 2020): Yes.

There exists $f \in \mathcal{S}(\mathbb{R})$ such that

$$\| \partial^{1/3} e^{it\partial^4} f \|_{L^6_{t,x}(\mathbb{R}^2)} = C \|f\|_{L^2(\mathbb{R})}$$

<u>Thm</u> (B., Oliveira e Silva, Quilodran, 2020): Maximisers do exist.

<u>GOAL</u> : show that $C > C_0$.

$$\sup_{f\neq 0} \frac{\| \partial^{1/3} e^{it\partial^4} f \|_{L^6_{t,x}}}{\|f\|_{L^2}} > \sup_{g\neq 0} \frac{\| e^{-it\partial^2} g \|_{L^6_{t,x}}}{\|g\|_{L^2}} = \left(\frac{1}{12}\right)^{\frac{1}{12}}$$

(Foschi '07)

▷ <u>Convolution form</u>:

$$\| \partial^{1/3} e^{it\partial^4} f \|^6_{L^6_{t,x}} = c \, \| \mathcal{F}^{-1} (\hat{f} \, |\cdot|^{1/3} \nu) \|^6_{L^6} \qquad \nu(\tau,\xi) = \delta(\tau - \xi^4)\, d\tau\, d\xi$$

$$\| \, | \mathcal{F}^{-1} (\hat{f} \, |\cdot|^{1/3} \nu) |^3 \, \|^2_{L^2} = \| \hat{f} |\cdot|^{\frac{1}{3}} \nu * \hat{f} |\cdot|^{\frac{1}{3}} \nu * \hat{f} |\cdot|^{1/3} \nu \|^2_{L^2}$$

▷ reduce to understand $\nu * \nu * \nu (\tau, \xi)$.

§ 4.1 Study the measure $\upsilon(\xi,\tau) = \delta(\tau - \xi^4)\, d\tau\, d\xi$

$$w(\xi) = |\xi|^{2/3}, \qquad \sigma(\tau,\xi) = w(\xi)\,\upsilon(\xi,\tau)$$

$$\mu(\xi,\tau) = (w\upsilon * w\upsilon * w\upsilon)(\xi,\tau)$$

Proposition

(a) The measure $\mu \ll$ Lebesgue measure

(b) $\operatorname{supp}\mu = \{(\tau,\xi) \in \mathbb{R}^2 : \tau \geq \frac{1}{3^3}|\xi|^4\}$

(c) $\mu(\lambda\xi, \lambda^4\tau) = \mu(\xi,\tau) \quad \forall \lambda > 0$ (0-homogeneous)

$\mu(-\xi,\tau) = \mu(\xi,\tau) \quad \forall \xi \in \mathbb{R}.$ ("radial")

Idea: exploit symmetries.

<u>Remark</u> : enough to show $\dfrac{\| \partial^{1/3} e^{it\partial^4} f \|_{L^6_{tx}}}{\|f\|_{L^2}} > \left(\dfrac{1}{12}\right)^{\frac{1}{12}}$ for <u>one</u> f.

Consider $f(x) = e^{-x^4} |x|^{1/3}$.

$$\| f |\cdot|^{\frac{1}{3}} \nu * f |\cdot|^{\frac{1}{3}} \nu * f |\cdot|^{1/3} \nu \|_{L^2}^2 = \int_{\mathbb{R}^2} |e^{-\tau} (\sigma * \sigma * \sigma)(\tau, \xi)|^2 \, d\tau \, d\xi$$

We have :

$$\frac{\| f\sigma * f\sigma * f\sigma \|_{L^2}^2}{\|f\|_{L^2}^6} = c \int_{-1}^{1} \mu\left(1, \frac{1}{3^3} \frac{1}{t^4}\right)^2 dt =: \|g\|^2_{L^2([-1,1])}$$

Expand g in the Legendre basis of $L^2([-1,1])$: $c_n = \langle g, \mathcal{L}_n \rangle$

$$\|g\|_{L^2}^2 = \sum_{n \geq 0} c_n^2 \geq c_0^2 + c_2^2 + c_4^2 > \text{Schr const.}$$

$\square$