Sparse T1 theorems

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Weighted inequalities

Consider a (sub)linear operator $T : L^p \to L^p$

$$\|Tf\|_{L^p} \leqslant c(T)\|f\|_{L^p}$$

where

$$c(T) \coloneqq \sup_{\substack{f \in L^p \\ f \neq 0}} \frac{\|Tf\|_{L^p}}{\|f\|_{L^p}} \eqqcolon \|T\|_{\mathscr{B}(L^p)}$$

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2. Can we characterise them, given T and p?
3. How does $c(T, w)$ depend on w?

Definition (Calderón–Zygmund kernel)

A function $K(x, y)$ on $\mathbb{R} \times \mathbb{R} \setminus \{x = y\}$ such that $|K(x, y)| \leqslant C|x - y|^{-1}$ and

$$|K(x + h, y) - K(x, y)| + |K(x, y + h) - K(x, y)| \leqslant \frac{C|h|^{\alpha}}{|x - y|^{1+\alpha}}$$

for some $\alpha \in (0, 1]$ and all $|x - y| > 2|h|$.

Definition (Singular Integral Operator)

A map $T\colon \mathscr{C}_c^{\infty} \to (\mathscr{C}_c^{\infty})'$ associated with a Calderón–Zygmund kernel K such that

$$\langle Tf, g \rangle = \iint_{\mathbb{R}\times\mathbb{R}} K(x, y) f(y) g(x) \, dy \, dx$$

for $f, g \in \mathscr{C}_c^{\infty}$ with disjoint supports.

Definition (Littlewood–Paley kernels)

A family of functions $\{\Psi_t(x, y)\}_{t>0}$ such that

$$|\Psi_t(x, y)| \leqslant C \frac{t^\alpha}{(t + |x - y|)^{\alpha+1}}$$

$$|\Psi_t(x + h, y) - \Psi_t(x, y)| + |\Psi_t(x, y + h) - \Psi_t(x, y)| \leqslant \frac{C|h|^\alpha}{(t + |x - y|)^{1+\alpha}}$$

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Consider the Square function

$$\mathcal{S}f(x) \coloneqq \left(\int_0^\infty |\theta_t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}, \qquad \theta_t f(x) = \int \Psi_t(x, y) f(y)\, dy$$

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Example (Littlewood–Paley square function)

Let $\psi \in \mathscr{S}(\mathbb{R})$, $\int \psi = 0$ and consider $\theta_t f = f * \psi_t$, $\psi_t(\cdot) = t^{-1}\psi(\cdot/t)$.

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Theorem (Hunt, Muckenhoupt, Wheeden et al. $\sim$ 1973)

Let $T \in \{$*Singular Integral Operator, Square function, Maximal function*$\}$ *bounded on* L^p *for* $p > 1$, *then*

$$c(T, p, w) = \|T\|_{L^p(w)} < \infty \iff w \in A_p$$

Definition

A weight w is a A_p Muckenhoupt weight if

$$[w]_{A_p} \coloneqq \sup_{I \subseteq \mathbb{R}} \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I \sigma \right)^{p-1} < \infty$$

where $\sigma \coloneqq w^{1-p'}$ is the dual weight.

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Theorem (Hytönen 2012)

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Singular Integral

$$\|T\|_{L^p(w)} \leqslant c[w]_{A_p}^{\frac{1}{p}\cdot\max\{p,p'\}}$$

Square function

$$\|S\|_{L^p(w)} \leqslant c[w]_{A_p}^{\frac{1}{p}\cdot\max\{\frac{p}{2},p'\}}$$

Maximal function

$$\|M\|_{L^p(w)} \leqslant c[w]_{A_p}^{\frac{1}{p}\cdot p'}$$

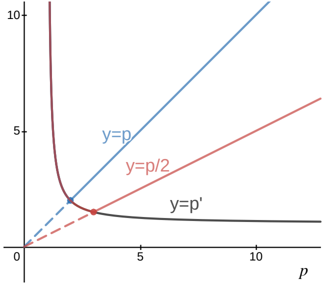

Figure: Dependence on $[w]_{A_p}^{1/p}$

Sparse domination

Definition (Sparse family)

Let $\eta \in (0,1)$. A collection $\mathscr{S} \subseteq \mathscr{D}$ is η-sparse if for every $I \in \mathscr{S}$ there exists $E_I \subseteq I$ such that $\{E_I\}_I$ are disjoint and $|E_I| > \eta |I|$.

Equivalently: if there exists $\Lambda \geqslant 1$ such that for any $J \in \mathscr{D}$

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Example

- Finite families are sparse ($\Lambda = \#\mathscr{S}$)
- Finite overlapping: $\sum_{I \in \mathscr{S}} \mathbb{1}_I \in L^\infty$
- $\mathscr{S} = \{[0, 2^{-n})\}_{n \in \mathbb{N}}$ is 1/2-sparse

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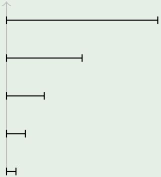

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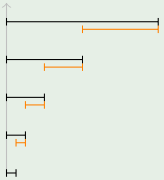
  $E_{[0,2^{-n})} = [2^{-n-1}, 2^{-n})$



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Given f and g find $\mathscr{S} \subseteq \mathscr{D}$ such that

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$$\|Sf\|^2_{L^3(w)} = \||Sf|^2\|_{L^{3/2}(w)} = \sup_{g \in L^3} \langle (Sf)^2, g\rangle$$

Theorem (A_2 theorem for sparse operators)

$$\sum_{I \in \mathscr{S}} \left(\frac{1}{|I|} \int_I |f| \right) \left(\frac{1}{|I|} \int_I |g| \right) |I| \lesssim [w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(\sigma)}$$

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Let $w(I) \coloneqq \int_I w$ and consider

$$M_w^{\mathscr{D}} f(x) \coloneqq \sup_{I\in\mathscr{D}} \left(\frac{1}{w(I)}\int_I |f| w\right)\mathbb{1}_I(x)$$

Lemma

For $1 < p < \infty$

$$\|M_w^{\mathscr{D}} f\|_{L^p(w)} \leqslant c_p \|f\|_{L^p(w)}$$

Let $\sigma \coloneqq w^{-1}$, then

$$[w]_{A_2} = \sup_{I\subseteq\mathbb{R}} \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I \sigma\right) = \sup_{I\subseteq\mathbb{R}} \frac{w(I)}{|I|}\frac{\sigma(I)}{|I|}$$

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\sum_{I \in \mathscr{S}} \left(\fint_I |f| \right) \left(\fint_I |g| \right) |E_I| &= \sum_{I \in \mathscr{S}} \frac{\sigma(I)}{|I|} \frac{w(I)}{|I|} \langle |fw| \rangle_I^\sigma \langle |g\sigma| \rangle_I^w |E_I| \\
&\leqslant [w]_{A_2} \sum_{I \in \mathscr{S}} \int_{E_I} \langle |fw| \rangle_I^\sigma \langle |g\sigma| \rangle_I^w \, dx \\
&\leqslant [w]_{A_2} \int_{\mathbb{R}} M_\sigma^{\mathscr{D}}(fw)\sigma \quad w M_w^{\mathscr{D}}(g\sigma) \, dx \\
&\lesssim [w]_{A_2} \|fw\|_{L^2(\sigma)} \|g\sigma\|_{L^2(w)} \\
&= [w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(\sigma)}
\end{aligned}$$

□

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Can we derive sparse bounds from minimal conditions?

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Theorem (David & Journé 1984)

Let T *be a Singular Integral Operator (SIO) with Calderón–Zygmund kernel. If there exists* $\mathrm{C} > 0$ *such that*

$$\langle |\mathsf{T}\mathbb{1}_{\mathrm{I}}|, \mathbb{1}_{\mathrm{I}} \rangle + \langle |\mathsf{T}^{\star}\mathbb{1}_{\mathrm{I}}|, \mathbb{1}_{\mathrm{I}} \rangle \leqslant \mathrm{C}|\mathrm{I}| \tag{3}$$

holds for all intervals $\mathrm{I} \subseteq \mathbb{R}$*, then* $\|\mathsf{T}\|_{\mathrm{L}^2} < \infty$*.*

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Theorem (Lacey & Mena 2016)

Let T *be a SIO with Calderón–Zygmund kernel that satisfies* (3)*. Then for any* $f, g \in \mathscr{C}_c^\infty$ *there exists a sparse collection* $\mathscr{S}$ *such that*

$$|\langle Tf, g\rangle| \leqslant c \sum_{I \in \mathscr{S}} \left(\fint_I |f| \right) \left(\fint_I |g| \right) |I|$$

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Theorem (Christ & Journé 1987, Auscher, Hofmann, Lacey, et al. 2002)

Let S be a square function associated to $\{\theta_t\}_{t>0}$*. If there exists* $C > 0$ *such that*

$$\int_I \int_0^{\ell I} |\theta_t \mathbb{1}_I|^2 \frac{dt}{t} \, dx \leqslant C|I| \tag{4}$$

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holds for all intervals $I \subseteq \mathbb{R}$, then $\|S\|_{L^2} < \infty$.

Theorem (B. 2020)

Let S be a square function associated to $\{\theta_t\}_{t>0}$ satisfying (4). Then for any $f, g \in \mathscr{C}_c^\infty$ there exists a sparse collection $\mathscr{S}$ such that

$$|\langle (Sf)^2, g\rangle| \leqslant c \sum_{I \in \mathscr{S}} \left(⨍_I |f| \right)^2 \left(⨍_I |g| \right) |I|$$

Remark

If f and g are constant and supported on I, then

$$\int_I (Tf)\, g \, dx = \langle f \rangle_I \langle g \rangle_I \langle T\mathbb{1}_I, \mathbb{1}_I \rangle \leqslant C \langle |f| \rangle_I \langle |g| \rangle_I |I|$$

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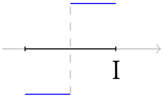
Decompose f, g on Haar functions $\{h_I\}_{I\in\mathscr{D}}$

$$h_I := (\mathbb{1}_{I_r} - \mathbb{1}_{I_l})|I|^{-\frac{1}{2}}$$



Let $\Delta_I f := \langle f, h_I\rangle h_I$, then

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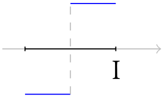

Let $\Delta_I f \coloneqq \langle f, h_I\rangle h_I$, then

$$f = \sum_{I\in\mathscr{D}} \Delta_I f \coloneqq \sum_{I\in\mathscr{D}} \sum_{i\in\{r,l\}} (\langle f\rangle_{I_i} - \langle f\rangle_I)\mathbb{1}_{I_i}$$

Decomposition

In [Lacey–Mena 2016]:

$$\langle Tf, g\rangle = \sum_{Q\in\mathscr{D}} \sum_{P\in\mathscr{D}} \langle T\Delta_P f, \Delta_Q g\rangle$$

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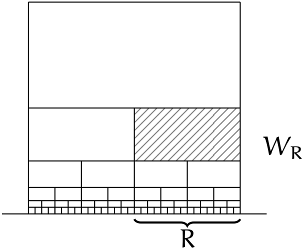

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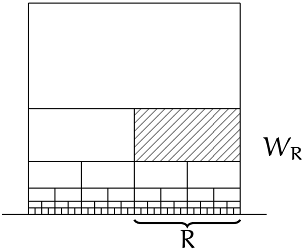

and decompose f and g

$$\iint_{\mathbb{R}^2_+} |\theta_t f|^2 g \frac{dt}{t}\, dx = \sum_{R\in\mathscr{D}} \iint_{W_R} \left|\theta_t \sum_{P\in\mathscr{D}} \Delta_P f\right|^2 \sum_{\substack{Q\in\mathscr{D}\\ Q\subseteq R}} \Delta_Q g \frac{dt}{t}\, dx$$

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Different cases given R:

P far from R | P close to R | $P \supset R$ or $P \subseteq R$

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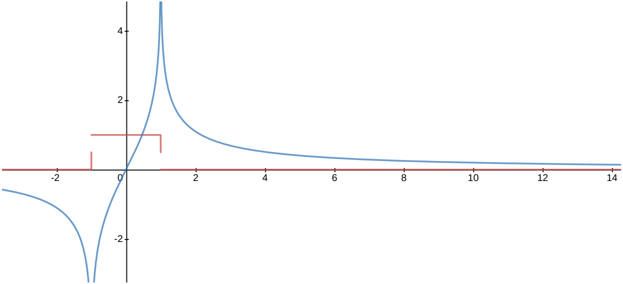

Figure: Hilbert transform of $\mathbb{1}_{[-1,1)}$

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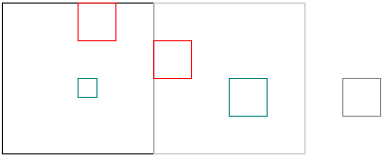
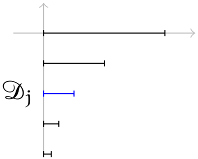


Shifted dyadic grids $\mathscr{D}^{\omega}$

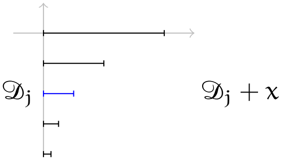
Let $x \in (0, 1)$.



$\mathscr{D}_j + x$

Shifted dyadic grids $\mathscr{D}^{\omega}$

Let $x \in (0,1)$. Then $[x]_2 = 01001001000101\ldots = \{\omega_n\}_{n\in\mathbb{N}}$



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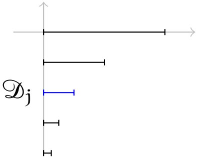

$$\mathscr{D}_j + x = \mathscr{D}_j + \sum_{n=1}^{\infty} \omega_n 2^{-n}$$

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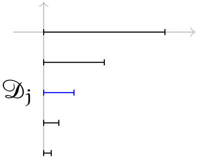

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if $2^{-n} \geqslant 2^{-j}$ then $\mathscr{D}_j + 2^{-n} = \mathscr{D}_j$

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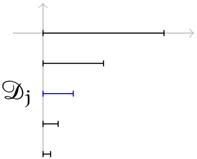

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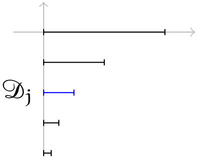

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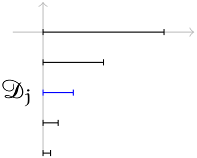
$\mathscr{D}_j$ 

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Let $\mathscr{D}_j^{\omega} := \{R \dotplus \omega, R \in \mathscr{D}_j\}$ and

$$\mathscr{D}^{\omega} := \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j^{\omega}$$

Independence

Definition

Fix $r \in \mathbb{N}$. An interval $R \in \mathscr{D}^{\omega}$ is good if $d(R, \partial P) > \ell R$ for any $P \in \mathscr{D}^{\omega}$ with $\ell P > 2^r \ell R$, otherwise R is bad.

Position of R depends on

$$R = R_0 + \sum_{2^{-n} < \ell R} \omega_n 2^{-n}$$

Goodness of R depends on the position of P

$$P = P_0 + \sum_{2^{-n} < \ell P} \omega_n 2^{-n}$$

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Goodness of R depends on the position of P

$$\begin{aligned} P &= P_0 + \sum_{2^{-n} < \ell P} \omega_n 2^{-n} \\ &= P_0 + \sum_{2^{-n} < \ell R} \omega_n 2^{-n} + \sum_{\ell R \leqslant 2^{-n} < \ell P} \omega_n 2^{-n} \end{aligned}$$

Independence

Let $\mathbb{P}$ be a probability measure on $\Omega \coloneqq \{0,1\}^{\mathbb{Z}}$ and $\mathscr{D}^{\omega} = \mathscr{D}^{\omega}_{\text{good}} \cup \mathscr{D}^{\omega}_{\text{bad}}$.
If $R = R_0 \dot{+} \omega$ then

$$\begin{aligned} \mathbb{1}_{R_0 \dot{+} \omega} &: \Omega \to L^{\infty}(\mathbb{R}) \\ \mathbb{1}_{\mathscr{D}^{\omega}_{\text{good}}} &: \Omega \to L^{\infty}(\mathscr{D}^{\omega}) \end{aligned}$$

are independent random variables.

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$$\mathbb{E}_{\omega}\left[\mathbb{1}_R\right] \cdot \mathbb{E}_{\omega}\left[\mathbb{1}_{\mathscr{D}^{\omega}_{\text{good}}}(R)\right] = \mathbb{E}_{\omega}\left[\mathbb{1}_R \cdot \mathbb{1}_{\mathscr{D}^{\omega}_{\text{good}}}(R)\right]$$

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$$\mathbb{1}_{R_0 \dot{+} \omega} : \Omega \to L^{\infty}(\mathbb{R})$$
$$\mathbb{1}_{\mathscr{D}^{\omega}_{\text{good}}} : \Omega \to L^{\infty}(\mathscr{D}^{\omega})$$

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If $\mathbb{P}\big(\{R \in \mathscr{D}^{\omega}_{\text{good}}\}\big) > 0$, then

$$\mathbb{E}\left[\mathbb{1}_R\right] = \frac{1}{\mathbb{E}\left[\mathbb{1}_{\mathscr{D}^{\omega}_{\text{good}}}\right]} \mathbb{E}\left[\mathbb{1}_{R_{\text{good}}}\right]$$

New decomposition

$$\sum_{R \in \mathscr{D}^\omega} \iint_{W_R} \left|\theta_t \sum_{P \in \mathscr{D}^\omega} \Delta_P f\right|^2 g \frac{dt}{t}\, dx$$

Different cases given good $R \in \mathscr{D}^\omega$:

P far from R	P close to R	$P \supset R$	$P \subseteq R$
$\ell P > 2^r \ell R$	$\ell R \leqslant \ell P \leqslant 2^r \ell R$	$\ell P > 2^r \ell R$	
$d(R, \partial P) > \ell R$	$d(R, \partial P) \leqslant \ell R$	$d(R, \partial P) > \ell R$	

New decomposition

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Definition (Stopping cubes)

Given $Q_0 \in \mathscr{D}$ let $\mathscr{A}(Q_0)$ be the collection of maximal $S \in \mathscr{D}$ in Q_0 :

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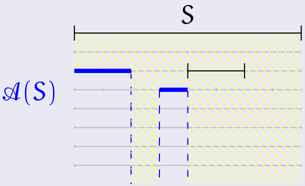

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Testing in action

Assume that $\mathbb{1}_R g(x) = \langle |g| \rangle_R$.

Lemma

Let $S = \widehat{R^{(r)}}$. *Then*

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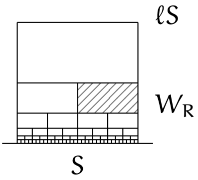

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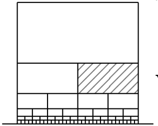

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For $(x,t) \in W_R$

$$\int \big(\Psi_t(x,y) - \Psi_t(x_Q,y)\big) f(y) \qquad \int \big(\Psi_t(x,y) + \Psi_t(x_Q,y)\big) f(y)$$
$$\lesssim \int \frac{(\ell Q)^\alpha}{(t+|x-y|)^{\alpha+1}} |f(y)| \, dy \qquad \int \frac{t^\alpha}{(t+|x-y|)^{\alpha+1}} |f(y)| \, dy$$
$$= \Big(\int \frac{(\ell Q \cdot t)^{\alpha/2}}{(t+|x-y|)^{\alpha+1}} |f(y)| \, dy \Big)^2 =: (\mathscr{K}_t^Q f)^2$$

Can we write

$$\mathbb{1}_R g \leqslant c\langle |g| \rangle_R \mathbb{1}_R + \sum_{Q \subset R} b_Q$$

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Let $r \in \mathbb{N}$ and $g \in L^1(\mathbb{R})$. For any $\lambda > 0$ there exists a maximal collection $\mathscr{L} \subseteq \mathscr{D}$ and functions a and b such that $g = a + b$, with $\|a\|_{L^\infty} \leqslant 2^{r+1}\lambda$ and

$$b \coloneqq \sum_{Q \in \mathscr{L}} \sum_{Q_r \in \mathrm{ch}_r(Q)} b_{Q_r}, \quad \text{where} \quad b_{Q_r} \coloneqq \Big(g - \langle g\rangle_{Q_r}\Big)\mathbb{1}_{Q_r}.$$

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Proof. Let $\{Q\}_{Q \in \mathscr{L}}$ be the maximal cubes from the Calderón–Zygmund decomposition g at $\lambda > 0$. Let Q_r be a r-grandchild of Q. Then

$$\|a\|_{L^\infty} \leqslant \sup_{Q_r} \fint_{Q_r} |g| \leqslant \sup_{Q_r} \frac{|Q|}{|Q_r|} \fint_Q |g| \leqslant 2^{r+1}\lambda$$

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Apply the decomposition:

$$\sum_{R\in\mathscr{D}} \iint_{W_R} |\theta_t f|^2 |g| \frac{dt}{t}\, dx$$

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