Sparse $T1$ theorems

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1. Weighted inequalities and sparse domination
2. $T1$ theorems and their sparse version
3. Ideas from the proof

Weighted inequalities

Consider a (sub)linear operator $T : L^p(w) \to L^p(w)$,

$$\|T\|_{L^p(w)} \coloneqq \sup_{\substack{f \in C_c^\infty \\ \|f\|_{L^p(w)}=1}} \left(\int |Tf|^p w \, \mathrm{d}x \right)^{1/p}$$

Replace $\mathrm{d}x$ with $w(x)\,\mathrm{d}x$, for $w \in L^1_{\mathrm{loc}}$, $w(x) > 0$ a.e.

1. When is $\|T\|_{L^p(w)}$ finite?
2. How does $\|T\|_{L^p(w)}$ depend on w?

Definition (Calderón–Zygmund kernel)

A function $K(x,y)$ on $\mathbb{R}\times\mathbb{R}\setminus\{x=y\}$ such that

$$|K(x,y)| \le C|x-y|^{-1}$$

$$|K(x+h,y)-K(x,y)|+|K(x,y+h)-K(x,y)| \le \frac{C|h|^{\alpha}}{|x-y|^{1+\alpha}}$$

for some $\alpha\in(0,1]$ and all $|h|<\frac{1}{2}|x-y|$.

Definition (Singular Integral Operator)

A map $T\colon C_c^{\infty}\to(C_c^{\infty})'$ associated with a Calderón–Zygmund kernel K such that

$$\langle Tf,g\rangle=\iint K(x,y)f(y)g(x)\,\mathrm{d}y\,\mathrm{d}x$$

for $f,g\in C_c^{\infty}$ with disjoint supports.

Definition (Littlewood–Paley kernels)

A family of functions $\{\Psi_t(x,y)\}_{t>0}$ such that

$$|\Psi_t(x,y)| \le \frac{C}{t}\left(1+\frac{|x-y|}{t}\right)^{-(\alpha+1)}$$

$$|\Psi_t(x+h,y)-\Psi_t(x,y)|+|\Psi_t(x,y+h)-\Psi_t(x,y)| \le \frac{C|h|^\alpha}{(t+|x-y|)^{1+\alpha}}$$

for some $\alpha \in (0,1]$ and all $|h|<t$.

Consider the Square function

$$Sf(x) := \left(\int_0^\infty \left|\int \Psi_t(x,y)f(y)\,\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}$$

Example (Littlewood–Paley square function)

Let $\psi \in \mathscr{S}(\mathbb{R})$, $\int \psi = 0$ and consider $f * \psi_t$, $\psi_t(\cdot) = t^{-1}\psi(\cdot/t)$.

1. For which w is $\|T\|_{L^p(w)}$ finite?

Let M be the maximal function

$$Mf(x) := \sup_{B \ni x} \fint_B |f(y)| \, \mathrm{d}y$$

Definition (A_p weight)

A weight w is a A_p Muckenhoupt weight if

$$[w]_{A_p} := \sup_{I \subseteq \mathbb{R}} \left(\fint_I w \right) \left(\fint_I 1/w^{\frac{1}{p-1}} \right)^{p-1} \eqsim \|M\|^p_{L^p(w) \to L^{p,\infty}(w)}$$

is finite.

Theorem (Hunt, Muckenhoupt, Wheeden et al. ~ 1973)

Let $p > 1$ and $T \in \{$S. I. O., Square function, Maximal function$\}$, then

$$\|T\|_{L^p(w)} < \infty \iff w \in A_p$$

2. How does $\|T\|_{L^p(w)}$ depends on $[w]_{A_p}$?

Theorem (Hytönen 2012)

If T is Singular Integral Operator bounded on L^2, then

$$\|T\|_{L^2(w)} \leq c[w]_{A_2}$$

Singular Integral

$$\|T\|_{L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p}\cdot\max\{p,p'\}}$$

Square function

$$\|S\|_{L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p}\cdot\max\{\frac{p}{2},p'\}}$$

Maximal function

$$\|M\|_{L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p}\cdot p'}$$

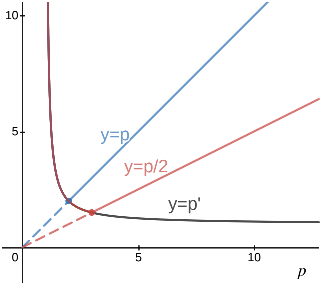


Figure: Dependence on $[w]_{A_p}^{1/p}$

Sparse domination

For $f, g \in C_c^\infty$

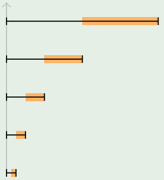
$$\int_{\mathbb{R}} Tf \cdot g \,\mathrm{d}x \;\lesssim\; \sum_{I \in \mathscr{S}} \int_I \langle |f| \rangle_I \langle |g| \rangle_I \,\mathrm{d}x \;\lesssim\; \int_{\mathbb{R}} Mf \cdot Mg \,\mathrm{d}x$$

Remark: $Tf(x) \not\lesssim Mf(x)$.
Flexibility: exists $E_I \subseteq I : |I| \leq 2|E_I|$ and $\{E_I\}_I$ are disjoint. ($\mathscr{S}$ is $\frac{1}{2}$-sparse)

$$\sum_{I \in \mathscr{S}} \langle |f| \rangle_I \langle |g| \rangle_I |I| \leq 2 \sum_{I \in \mathscr{S}} \int_{E_I} \langle |f| \rangle_I \langle |g| \rangle_I \,\mathrm{d}x \;\lesssim\; \int_{\mathbb{R}} Mf \cdot Mg \,\mathrm{d}x$$

Example

- Disjoint families
- Finite overlapping: $\sum_{I \in \mathscr{S}} \mathbb{1}_I \in L^\infty$
- $\mathscr{S} = \{[0, 2^{-n})\}_{n \in \mathbb{N}}$ is sparse.
  Take $E_{[0,2^{-n})} = [2^{-n-1}, 2^{-n})$



Sparse domination

Given f and g in C_c^∞ find $\mathscr{S}$ such that

$$|\langle Tf, g\rangle| \leq c \sum_{I \in \mathscr{S}} \langle |f|\rangle_I \langle |g|\rangle_I |I| \tag{1}$$

Remark 1. If T satisfies (1) then $\|T\|_{L^2(w)} \leq c[w]_{A_2}$.

$$\langle (Sf)^2, g\rangle \leq c \sum_{I \in \mathscr{S}} \langle |f|\rangle_I^2 \langle |g|\rangle_I |I| \tag{2}$$

Remark 2. If S satisfies (2) then $\|S\|_{L^3(w)}^2 \leq c[w]_{A_3}$.

$$\|Sf\|_{L^3(w)}^2 = \||Sf|^2\|_{L^{3/2}(w)} = \sup_{g \in L^3} \langle (Sf)^2, gw\rangle$$

A_2 theorem via sparse domination

Aim: show that if $T : L^2 \to L^2$ and $w \in A_2$, then $\|T\|_{L^2(w)} \le c[w]_{A_2}$.

$$\|T\|_{L^2(w)} = \sup_{\substack{f,g \in C_c^\infty \\ f \in L^2(w) \\ g \in L^2(w^{-1})}} \int_{\mathbb{R}} Tf \cdot g \,\mathrm{d}x \lesssim \sup_{f,g \in C_c^\infty} \sum_{I \in \mathscr{S}} \int_I \langle |f| \rangle_I \langle |g| \rangle_I \,\mathrm{d}x$$

Let $\sigma = w^{-1}$ and $M_w f(x) \coloneqq \sup_{\substack{I \in \mathscr{D} \\ I \ni x}} \left(\frac{1}{w(I)} \int_I |f| w \right)$.

$$\begin{aligned}
\sum_{I \in \mathscr{S}} \int_{E_I} \langle |f| \rangle_I \langle |g| \rangle_I \,\mathrm{d}x &= \sum_{I \in \mathscr{S}} \int_{E_I} \langle \sigma \rangle_I \langle w \rangle_I \langle |fw| \rangle_I^\sigma \langle |g\sigma| \rangle_I^w \\
&\le [w]_{A_2} \int_{\mathbb{R}} M_\sigma(fw)\, M_w(g\sigma) \,\mathrm{d}x \\
&\le [w]_{A_2} \|M_\sigma(fw)\|_{L^2(\sigma)} \, \|M_w(g\sigma)\|_{L^2(w)} \\
&\lesssim [w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(\sigma)}.
\end{aligned}$$

□

Sparse $T1$ theorem

"Conditions on $T \implies \|T\|_{L^2}$ finite."

Theorem (David & Journé 1984)

Let T be a S.I.O. If there exists $C > 0$ such that

$$\langle |T\mathbb{1}_I|, \mathbb{1}_I\rangle + \langle |T^\star \mathbb{1}_I|, \mathbb{1}_I\rangle \leq C|I| \tag{T1}$$

holds for all intervals $I \subseteq \mathbb{R}$, then $\|T\|_{L^2} < \infty$.

Can we derive sparse bounds from minimal conditions?

Theorem (Lacey & Mena 2016)

Let T be a S.I.O. satisfying (T1). Then for any $f, g \in C_c^\infty$

$$|\langle Tf, g\rangle| \leq c \sum_{I \in \mathscr{S}} \langle |f|\rangle_I \langle |g|\rangle_I |I|$$

The domination

$$\int_{\mathbb{R}} Tf \cdot g \,\mathrm{d}x \lesssim \sum_{I \in \mathscr{S}} \int_I \langle |f| \rangle_I \langle |g| \rangle_I \,\mathrm{d}x$$

implies

- strong L^p bounds for all $p > 1$: $\|T\|_{L^p \to L^p} < \infty$
- weighted L^p estimates for all $p > 1$ and $w \in A_p$

$$\|T\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\{p',p\}\frac{1}{p}}$$

- weak $(1,1)$ estimate [Benea–Bernicot 2018]
- weak weighted estimate for $w \in A_1$ [Frey–Nieraeth 2019]

$$\|T\|_{L^1(w) \to L^{1,\infty}(w)} \lesssim [w]_{A_1} \log(e + [w]_{A_1})$$

- upper bound on the asymptotic behaviour at the endpoints

$$\lim_{p \to 1^+} \|T\|_{L^p \to L^p} \simeq (p-1)^{-\gamma_1}, \qquad \lim_{p \to \infty} \|T\|_{L^p \to L^p} \simeq p^{\gamma_2}$$

- sufficient conditions on (w, v) for $\|T\|_{L^p(w) \to L^p(v)} < \infty$ [Lacey–Spancer 2015, K. Li 2017, Lerner 2020]
- vector value estimates [Lorist–Nieraeth 2020]

Sparse $T1$ theorem for square functions

"Conditions on $S \implies \|S\|_{L^2}$ finite."

Theorem (Christ & Journé 1987, Auscher, Hofmann, Lacey, et al. 2002)

Let S be a square function. If there exists $C > 0$ such that

$$\langle S(\mathbb{1}_I)^2, \mathbb{1}_I \rangle \leq C|I| \tag{S1}$$

holds for all intervals $I \subseteq \mathbb{R}$, then $\|S\|_{L^2} < \infty$.

Can we derive sparse bounds from minimal conditions?

Theorem (B. 2020)

Let S be a square function satisfying (S1)*. Then for any $f, g \in C_c^\infty$*

$$\langle (Sf)^2, g \rangle \leq c \sum_{I \in \mathscr{S}} \langle |f| \rangle_I^2 \langle |g| \rangle_I |I|$$

Difficulties: non-linear form, extra scale parameter

Ideas in the proof

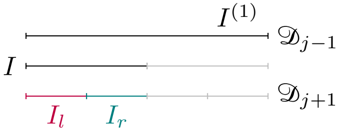
Remark

If f and g are constant and supported on I, then $\mathscr{S} = \{I\}$

$$\int_I Tf \cdot g \, \mathrm{d}x = \langle f \rangle_I \langle g \rangle_I \langle T\mathbb{1}_I, \mathbb{1}_I \rangle \leq C \langle |f| \rangle_I \langle |g| \rangle_I |I|$$

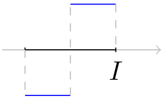
Dyadic intervals. For $j \in \mathbb{Z}$, consider $\mathscr{D}_j \coloneqq \{2^{-j}([0,1) + m), m \in \mathbb{Z}\}$

$$\mathscr{D} \coloneqq \bigcup_{j \in \mathbb{Z}} \mathscr{D}_j$$



Haar functions. The set $\{h_I\}_{I \in \mathscr{D}}$ is a basis of $L^2(\mathbb{R})$

$$h_I \coloneqq (\mathbb{1}_{I_r} - \mathbb{1}_{I_l})|I|^{-\frac{1}{2}}$$



Let $\Delta_I f \coloneqq \langle f, h_I \rangle h_I$, then

$$f = \sum_{I \in \mathscr{D}} \Delta_I f = \sum_{I \in \mathscr{D}} \sum_{i \in \{r,l\}} (\langle f \rangle_{I_i} - \langle f \rangle_I) \mathbb{1}_{I_i}$$

Decomposition

In [Lacey–Mena 2016]:

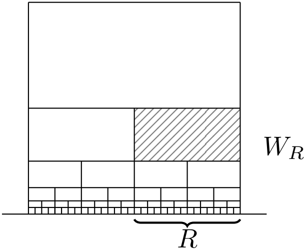
$$\langle Tf, g\rangle = \sum_{Q\in\mathscr{D}} \sum_{P\in\mathscr{D}} \langle T\Delta_P f, \Delta_Q g\rangle$$

In [B. 2020]:

Decompose $\mathbb{R}^2_+$ using $W_R := R \times \left[\frac{\ell R}{2}, \ell R\right)$

$$\iint_{\mathbb{R}^2_+} |\theta_t f|^2 g \frac{\mathrm{d}t}{t}\,\mathrm{d}x = \sum_{R\in\mathscr{D}} \iint_{W_R} \left|\theta_t f\right|^2 g \frac{\mathrm{d}t}{t}\,\mathrm{d}x$$

where $\theta_t f(x) = \int \Psi_t(x,y) f(y)\,\mathrm{d}y$.



Then decompose f and g

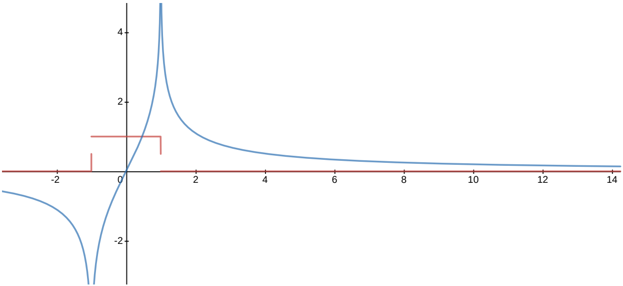
$$\sum_{R\in\mathscr{D}} \iint_{W_R} \left|\theta_t \sum_{P\in\mathscr{D}} \Delta_P f\right|^2 \sum_{\substack{Q\in\mathscr{D}\\ Q\subseteq R}} \Delta_Q g \frac{\mathrm{d}t}{t}\,\mathrm{d}x$$

Interactions

$$\langle Tf, g\rangle = \sum_{J\in\mathscr{D}} \sum_{I\in\mathscr{D}} \langle f, h_I\rangle \langle g, h_J\rangle \langle Th_I, h_J\rangle$$

Different cases given J:

I far from J | I close to J | $I \supset J$ or $I \subseteq J$



When $I \cap J = \emptyset$

$$\langle Th_I, h_J\rangle \lesssim \iint \frac{|h_I(y) h_J(x)|}{|x-y|} \, \mathrm{d}y \, \mathrm{d}x$$

Stopping intervals

Definition (Stopping intervals)

Given $I_0 \in \mathscr{D}$ let $\mathscr{A}(I_0)$ be the collection of maximal $S \in \mathscr{D}$ in I_0 :

$$\text{either} \quad \langle |f| \rangle_S > 2\langle |f| \rangle_{I_0} \quad \text{or} \quad \langle |g| \rangle_S > 2\langle |g| \rangle_{I_0}.$$

Define $\mathscr{S}$ iteratively:

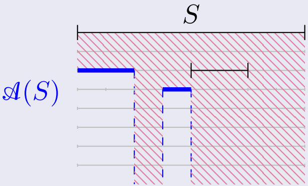
$$\mathscr{S}_0 \coloneqq \{I_0\}, \qquad \mathscr{S}_{n+1} \coloneqq \bigcup_{S \in \mathscr{S}_n} \mathscr{A}(S), \qquad \mathscr{S} \coloneqq \bigcup_{n \in \mathbb{N}} \mathscr{S}_n.$$

Definition (Stopping tree)

Given $J \subset I_0$, let

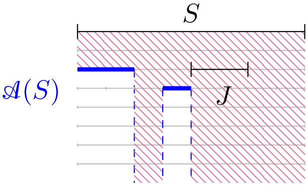
$$\widehat{J} \coloneqq \min\{S \in \mathscr{S},\, S \supseteq J\}$$

Let $\mathrm{Tree}(S) \coloneqq \{J \in \mathscr{D} : \widehat{J} = S\}$.



Let $J \subset I$ and

$$\widehat{J} := \min\{S \in \mathscr{S},\ S \supseteq J\}$$



Let $\mathsf{Tree}(S) := \{J \subseteq S \,:\, \widehat{R} = S\}$.

$$\sum_I \sum_{J \subset I} \langle T\Delta_I f \mathbb{1}_S, \Delta_J g\rangle = \sum_{S \in \mathscr{S}} \sum_{J \in \mathsf{Tree}(S)} \sum_{I \supset J} \langle T\Delta_I f \mathbb{1}_S, \Delta_J g\rangle$$

$$\sum_{I \supset J} (\langle f\rangle_{I_J} - \langle f\rangle_I)\mathbb{1}_S \approx \langle f\rangle_J \mathbb{1}_S$$

$$\lesssim \sum_{S \in \mathscr{S}} \langle |f|\rangle_S \int T\mathbb{1}_S\, P_S(g)\,\mathrm{d}x$$

where

$$P_S(g) := \sum_{J \in \mathsf{Tree}(S)} \langle g, h_J\rangle h_J.$$

$$\int T\mathbb{1}_S\, P_S g\,\mathrm{d}x = \int T\mathbb{1}_S (P_S g - \langle P_S g\rangle) + \int T\mathbb{1}_S \langle P_S g\rangle$$

$$= \int \big(T\mathbb{1}_S - T\mathbb{1}_S(x_S)\big)\widetilde{P_S g} + \int T\mathbb{1}_S \langle P_S g\rangle$$

Enough to control $P_S(g) \coloneqq \sum_{J \in \mathsf{Tree}(S)} \langle g, h_J \rangle h_J$ so that

$$\int_S T\mathbb{1}_S \, \langle P_S g \rangle_S \lesssim \langle |g| \rangle_S \int_S |T\mathbb{1}_S| \, \mathrm{d}x \leq \langle |g| \rangle_S \, C|S|$$

Fact 1 : $\|P_S(g)\|_{L^2} \leq \|g\|_{L^2}$
Proof.

$$\|P_S(g)\|_{L^2}^2 = \sum_{J \in \mathsf{Tree}(S)} \langle g, h_J \rangle^2 \leq \sum_{J \subseteq S} \langle g, h_J \rangle^2 = \|g \mathbb{1}_S\|_{L^2}^2$$

Fact 2 : P_S is "astigmatic".
Proof.
Recall $\mathscr{A}(S) = \{S' \subseteq S$ maximal such that $\langle g \rangle_{S'} > 2\langle g \rangle_S\}$. Define

$$[g]_S(x) \coloneqq \begin{cases} g(x) & \text{if } x \in S \setminus \mathscr{A}(S) \\ \langle g \rangle_{S'} & \text{if } x \in S' \text{ for some } S' \in \mathscr{A}(S). \end{cases}$$

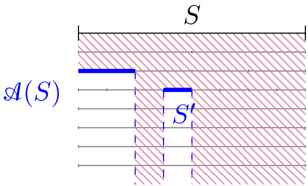
Then

$$P_S(g - [g]_S) = 0$$

Proof of $\langle P_S(g)\rangle_S \lesssim \langle |g|\rangle_S$

Recall

$$[g]_S(x) := \begin{cases} g(x) & \text{if } x \in S \setminus \mathscr{A}(S) \\ \langle g\rangle_{S'} & \text{if } x \in S',\ S' \in \mathscr{A}(S). \end{cases}$$



Then $\|[g]_S\|_{L^\infty(S)} \lesssim \langle |g|\rangle_S$. We have

$$\begin{aligned} \int_S |P_S g| = \int_S |P_S[g]_S| &\leq \|\mathbb{1}_S\|_{L^2}\|P_S[g]_S\|_{L^2} \\ &\leq \|\mathbb{1}_S\|_{L^2}\|[g]_S\|_{L^2(S)} \\ &\leq |S|^{1/2}|S|^{1/2}\|[g]_S\|_{L^\infty} \\ &\lesssim |S|\ \langle |g|\rangle_S. \end{aligned}$$

□

Thank you for listening