Random shifted dyadic grids

and what they can do for you

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UNIVERSITY OF BIRMINGHAM

Analysis & PDE Working Seminar

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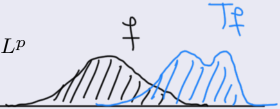

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Remark

If $Tf = f * K$, for some kernel K, then

$$\tau_h(Tf) = T(\tau_h f)$$

Weighted estimates

$$\left(\int_{\mathbb{R}} |Tf|^p w \,\mathrm{d}x\right)^{1/p} \leq c(T,w) \left(\int_{\mathbb{R}} |f|^p w \,\mathrm{d}x\right)^{1/p} \tag{1}$$

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$$c(T,w) \coloneqq \sup_{\substack{f \in L^p(w) \\ f \neq 0}} \frac{\|Tf\|_{L^p(w)}}{\|f\|_{L^p(w)}}$$

Example: the Hilbert transform

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Properties: $\|Hf\|_{L^2} = \|f\|_{L^2}$ and H commutes with

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Remark: If $T \in \mathscr{L}(L^2)$ satisfying 1. 2. 3., there exists $c \in \mathbb{R}$:

$$Tf = c\,Hf$$

$$\left(\int_{\mathbb{R}} |Hf|^2 w \,\mathrm{d}x\right)^{\frac{1}{2}} \leq c(H,w) \left(\int_{\mathbb{R}} |f|^2 w \,\mathrm{d}x\right)^{\frac{1}{2}}$$

Theorem (Hunt, Muckenhoupt, Wheeden 1973)

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holds for all $f \in L^2(w)$ *if and only if*

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Theorem (Petermichl 2007)

φ is linear: $c(H,w) \leq c[w]_{A_2}$.

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$$Tf(x) = \int_{\mathbb{R}} K(x,y) f(y) \, \mathrm{d}y \qquad \text{for } x \notin \operatorname{supp} f$$

where $|K(x,y)| \leq C|x-y|^{-1}$ on $\mathbb{R} \times \mathbb{R} \setminus \{x = y\}$ and

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Remark: By extrapolation, (2) implies the sharp bound for $1 < p < \infty$

$$\|Tf\|_{L^p(w)} \leq c[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}$$

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It is enough to show that [Pérez, Treil, Volberg 2010]

$$\|T(w\mathbb{1}_I)\|_{L^2(w^{-1})} \leq c(T)[w]_{A_2}\|\mathbb{1}_I\|_{L^2(w)}$$

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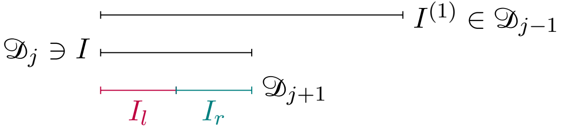

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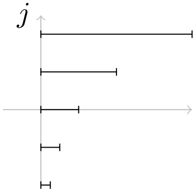

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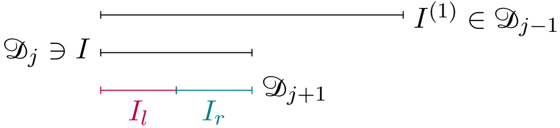

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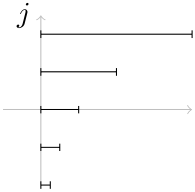

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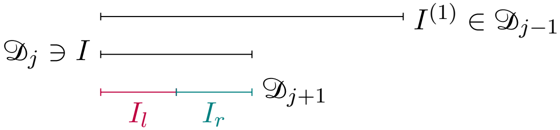

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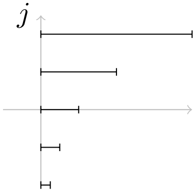

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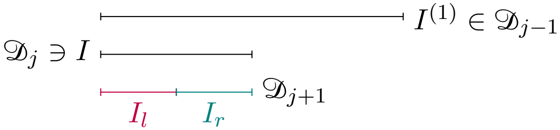

Drawback of $\mathscr{D}$: two quadrants.
There is **no** $I \in \mathscr{D}$, such that $I \supseteq [-1,0) \cup [0,1)$.

Haar functions

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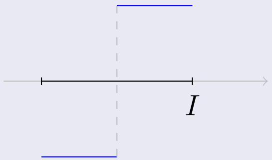
$$h_I := (\mathbb{1}_{I_r} - \mathbb{1}_{I_l})|I|^{-\frac{1}{2}}$$



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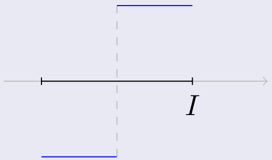

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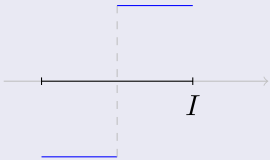

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Given $f \in L^2(\mathbb{R})$, decompose

$$f = \sum_{I \in \mathscr{D}} \Delta_I f$$

where $\Delta_I f := \langle f, h_I \rangle h_I$.

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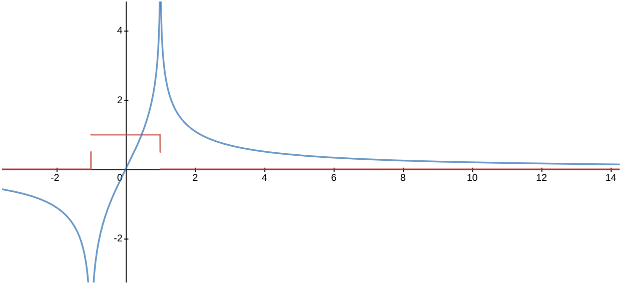

Figure: Hilbert transform of $\mathbb{1}_{[-1,1)}$

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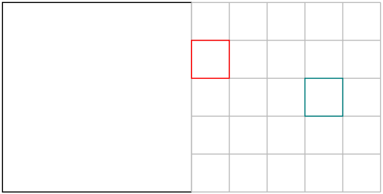

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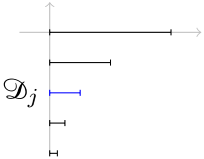

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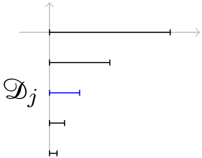

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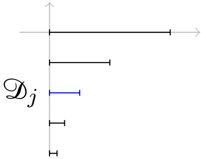

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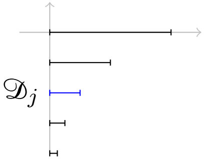

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For $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ define $\mathscr{D}_j^\omega := \{I \dot{+} \omega, I \in \mathscr{D}_j\}$ and

$$\mathscr{D}^\omega := \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j^\omega$$

Independence

Definition

Fix $r \in \mathbb{N}$. An interval $I \in \mathscr{D}^{\omega}$ is good if $d(I, J) > \ell(I)$ for any $J \in \mathscr{D}^{\omega}$ with $\ell(J) > 2^r \ell(I)$, otherwise I is bad.

Position of I depends on

$$I = I_0 + \sum_{2^{-n} < \ell(I)} \omega_n 2^{-n}$$

Goodness of I depends on the position of J

$$\begin{aligned} J &= J_0 + \sum_{2^{-n} < \ell(J)} \omega_n 2^{-n} \\ &= J_0 + \sum_{2^{-n} < \ell(I)} \omega_n 2^{-n} + \sum_{\ell(I) \leq 2^{-n} < \ell(J)} \omega_n 2^{-n} \end{aligned}$$

Independence

Let $\mathbb{P}$ be a probability measure on $\Omega := \{0,1\}^{\mathbb{Z}}$.
Let $\mathscr{D}^\omega = \mathscr{D}^\omega_{\text{good}} \cup \mathscr{D}^\omega_{\text{bad}}$. Then

$$\mathbb{1}_I \quad \text{and} \quad \mathbb{1}_{\{I \in \mathscr{D}^\omega_{\text{good}}\}}$$

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If $\mathbb{P}\big(\{I \in \mathscr{D}^\omega_{\text{good}}\}\big) > 0$, then

$$\mathbb{E}\left[\mathbb{1}_I\right] = \frac{1}{\mathbb{E}\left[\mathbb{1}_{\{I \in \mathscr{D}^\omega_{\text{good}}\}}\right]} \mathbb{E}\left[\mathbb{1}_{I_{\text{good}}}\right]$$

Definition (Dilated grids)

For $s \in [1,2)$ define

$$s\mathscr{D}^{\omega} := \{sI := [sa, sb), I = [a,b) \in \mathscr{D}^{\omega}\}$$

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Remark

- $s\mathscr{D}^\omega$ is a dyadic grid for all $s \in [1, 2)$ and $\omega \in \{0, 1\}^{\mathbb{Z}}$.
- Any dyadic grid on $\mathbb{R}$ equals $s\mathscr{D}^\omega$ for some s and ω
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Example (Infinitely many shifts)

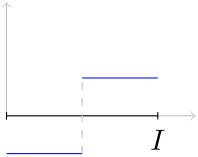
Since

$$\left[\frac{1}{3}\right]_2 = 01\overline{01}\ldots = \omega^1 \qquad \left[\frac{2}{3}\right]_2 = 10\overline{10}\ldots = \omega^2$$

the corresponding grids $\mathscr{D}^{\omega^1}, \mathscr{D}^{\omega^2}$ have one quadrant.

Petermichl's dyadic shift

$$h_I := (\mathbb{1}_{I_r} - \mathbb{1}_{I_l})|I|^{-\frac{1}{2}}$$



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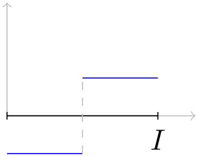

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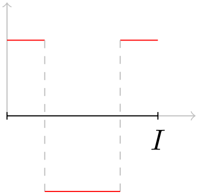

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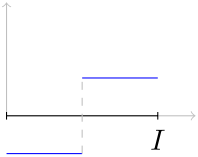

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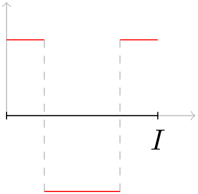

Definition (Petermichl's dyadic shift operator)

$$Ш f := \sum_{I \in \mathscr{D}} \langle f, h_I \rangle \hbar_I$$

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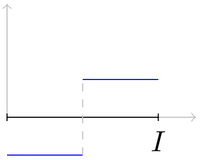

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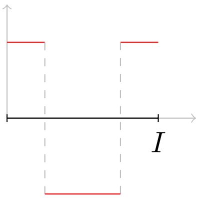

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$$\text{Ш} f := \sum_{I \in \mathscr{D}} \langle f, h_I \rangle \hbar_I$$

Note that $\text{Ш} h_I = \hbar_I$ and $\|\text{Ш} f\|_{L^2} = \|f\|_{L^2}$.

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Theorem (Hytönen 2008)

Let μ be a probability measure on $\Omega \times [1, 2)$, then

$$\int_\Omega \text{Ш}^{s,\omega} f \, \mathrm{d}\mu_{(s,\omega)} = c\, H f$$

Thank you for all the shifts

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Sparse domination

Definition (Sparse family)

Let $\eta \in (0,1)$. A collection $\mathscr{S} \subseteq \mathscr{D}$ is η-sparse if for every $I \in \mathscr{S}$ there exists $E_I \subseteq I$ such that $\{E_I\}_I$ are disjoint and $|E_I| > \eta|I|$.

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Given f and g find $\mathscr{S} \subseteq \mathscr{D}^\omega$ such that

$$|\langle Tf, g\rangle| \leq c \sum_{I \in \mathscr{S}} \left(\fint_I |f| \right) \left(\fint_I |g| \right) |I| \tag{3}$$

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Remark. If T satisfies (3) then $\|Tf\|_{L^2(w)} \leq c[w]_{A_2} \|f\|_{L^2(w)}$ for all $w \in A_2$, and so $T\colon L^p(w) \to L^p(w)$ for all $1 < p < \infty$ and all $w \in A_p$.

Sparse $T1$ theorems

Theorem (David & Journé 1984)

Let T be a singular integral operator (SIO) with Calderón–Zygmund kernel, then $T \in \mathscr{L}(L^2)$ if for all I

$$\langle |T(\mathbb{1}_I)| + |T^\star(\mathbb{1}_I)|, \mathbb{1}_I \rangle \leq C|I| \tag{4}$$

Theorem (Lacey & Mena 2016)

Let T be a SIO with Calderón–Zygmund kernel that satisfies (4). Then for any $f, g \in \mathscr{C}_c^\infty$ there exists a sparse collection $\mathscr{S}$ such that

$$|\langle Tf, g\rangle| \leq c \sum_{I \in \mathscr{S}} \left(\fint_I |f| \right) \left(\fint_I |g| \right) |I|$$