The (real) Kempf–Ness theorem

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after Böhm and Lafuente

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Summer School on
Brascamp–Lieb inequalities

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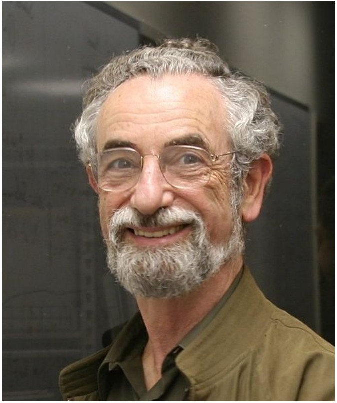

Figure: Elliot Lieb (2010)

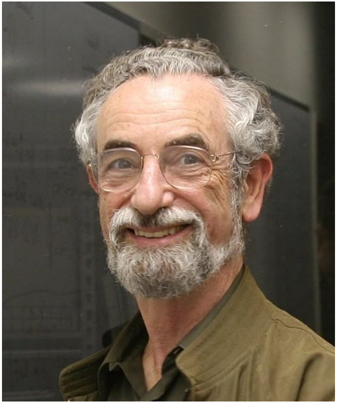

Figure: Elliot Lieb (2010)
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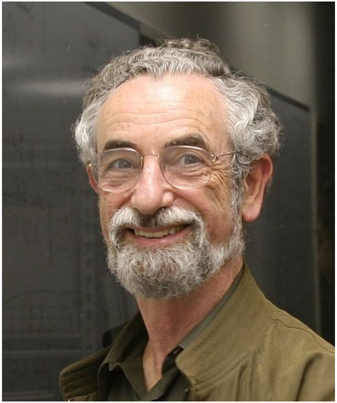

Figure: Elliot Lieb (2010)
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Figure: Avi Wigderson (2012)
Abel prize 2021

Together with László Lovász *"for their foundational contributions to theoretical computer science and discrete mathematics, and their leading role in shaping them into central fields of modern mathematics."*

Brascamp–Lieb inequalities

Let $B_j \colon \mathbb{R}^d \to \mathbb{R}^{d_j}$ surjective linear maps.

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} |f_j(B_j x)|^{p_j} \, dx \leqslant \mathrm{BL}(\{B_j, p_j\}) \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} |f_j(y)| dy \right)^{p_j}$$

Feasibility ($\mathrm{BL}(\{B_j, p_j\}) < \infty$):

- $\sum_j p_j d_j = d$
- $\dim V \leqslant \sum_j p_j \dim B_j V \quad$ for all subspaces $V \subset \mathbb{R}^d$

Geometric data:

- (projection) $B_j B_j^t = I_{d_j}$
- (isotropy) $\sum_j p_j B_j^t B_j = I_d$

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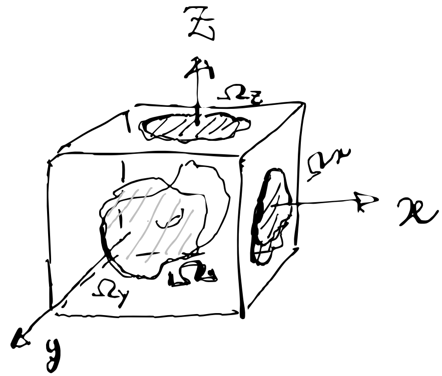
Loomis–Whitney in $d = 3$:

$$\int_{\mathbb{R}^3} \prod_{j=1}^{3} |f_j(B_j x)|^{\frac{1}{2}}\, dx \leqslant \mathrm{BL}(\{B_j, (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\}) \prod_{j=1}^{3} \left(\int_{\mathbb{R}^2} |f_j(y)| dy \right)^{\frac{1}{2}}$$

Loomis - Whitney example



$$\mathrm{Vol}(\Omega) \leq |\Omega_x|^{1/2} \cdot |\Omega_y|^{1/2} \, |\Omega_z|^{1/2}$$

equality for 

From Gaussians to matrices

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} |f_j(B_j x)|^{p_j} \, dx \leqslant \mathrm{BL}(\{B_j, p_j\}) \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} |f_j(y)| dy \right)^{p_j}$$

Let $g_j(x) \coloneqq \exp(-\frac{1}{2}\langle A_j x, x \rangle)$ on $\mathbb{R}^{d_j}$, then

$$\mathrm{BL}(\{B_j, p_j\}) \geqslant \frac{\int_{\mathbb{R}^d} \prod_j |g_j(B_j x)|^{p_j} \, dx}{\prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} g_j(y) dy \right)^{p_j}} = \left(\frac{\det(\sum_j p_j B_j^* A_j^* A_j B_j)}{\prod_{j=1}^{m} \det(A_j^* A_j)^{p_j}} \right)^{-\frac{1}{2}} .$$

$$\mathrm{BL}(\{B_j, p_j\}) = \sup_{\substack{A_j \in GL(d_j) \\ A \in GL(d)}} \left(\frac{\det(\sum_j p_j B_j^* A_j^* A_j B_j)}{\prod_{j=1}^{m} \det(A_j^* A_j)^{p_j}} \right)^{-\frac{1}{2}}$$

- When $\mathrm{BL}(\{B_j, p_j\}) < \infty$, how can we approximate it?

- How to approximate $\mathrm{BL}(\{B_j, p_j\})$?

$$\mathrm{BL}(\{B_j, p_j\}) = \sup_{\substack{A_j \in \mathrm{GL}(d_j) \\ A \in \mathrm{GL}(d)}} \left(\frac{\det(\sum_j p_j B_j^* A_j^* A_j B_j)}{\prod_{j=1}^m \det(A_j^* A_j)^{p_j}} \right)^{-\frac{1}{2}}$$

Gressman rewrote it as

$$\mathrm{BL}(\{B_j, p_j\})^{-1} = \inf_{\substack{A_j \in \mathrm{SL}(d_j) \\ A \in \mathrm{SL}(d)}} \prod_{j=1}^{m} \left(d_j^{-\frac{1}{2}} \|A_j B_j A^*\|_{\mathrm{HS}} \right)^{p_j d_j}.$$

Let $\mathbf{B} = (B_1, \ldots, B_m) \in V = \mathbb{R}^{d \times d_1} \times \cdots \times \mathbb{R}^{d \times d_m}$.
Let $G = \mathrm{SL}(d_1) \times \cdots \times \mathrm{SL}(d_m) \times \mathrm{SL}(d)$ acting on V as

$$\begin{aligned} \mathrm{SL}(d_1) \times \cdots \times \mathrm{SL}(d_m) \times \mathrm{SL}(d) \times V &\to V \\ ((A_1, \ldots, A_m, A), \mathbf{B}) &\mapsto (A_1 B_1 A^*, \ldots, A_m B_m A^*) \end{aligned}$$

Then

$$\mathrm{BL}(\{\mathbf{B}, p_j\})^{-1} \approx \inf_{g \in G} \|g \cdot \mathbf{B}\|$$

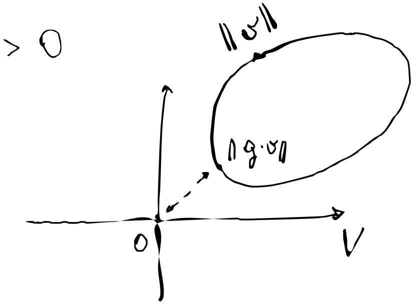
is the length of a minimal vector w.r.t $\|\cdot\|$ in a given orbit.

QUESTIONS:

- Can we find other way to approximate $BL(\vec{B},\vec{p})$?

- Is there an "oracle" which tells when $BL(\vec{B},\vec{p}) < \infty$?

$$BL(\vec{B},\vec{p})^{-1} = \inf_{g\in G} |||g\cdot\vec{B}||| = |||g_0\cdot\vec{B}||| > 0$$



<u>Null cone</u>: $\mathcal{N}_G := \{ v\in V : 0 \in \overline{G\cdot v}\}$

∘ Finding f such that $f(\vec{B}) = \begin{cases} 1 & \text{if } \vec{B}\notin \mathcal{N}_G \\ 0 & \text{otherwise}\end{cases}$

<u>Remark</u>: $f(g\cdot v) = f(v) \quad \forall g\in G.$

Examples of group actions

Example (Non closed orbits) : $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$

$(\lambda \ , \ a) \mapsto \lambda a$

two orbits, $0 \in \overline{\mathbb{R}_+ \cdot a}$

Example : $ST(2) := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \lambda > 0 \right\} \subseteq SL(2)$.

$$ST(2) \times M(2) \longrightarrow M(2) \simeq \mathbb{R}^{2\times 2}$$
$$(S \ , \ M) \mapsto SMS^{-1}$$

Example : $ST(n) := \{ \text{diag. metrices} \subseteq SL(n) . \}$

$$ST(n) \times M(n) \longrightarrow M(n)$$
$$(S \ , \ M) \mapsto SM$$

INVARIANT POLYNOMIALS

Given G acting linearly on V, consider

$$IP := IP_G[V] := \left\{ p \text{ polynomial in } \mathbb{C}[V] \;\middle|\; p(g.v) = p(v) \;\; \forall g \in G, \forall v \in V \right\}$$

<u>Example</u> $\quad G = SL(n) \times SL(n)$. $\qquad G \times M(n, \mathbb{C}) \longrightarrow M(n, \mathbb{C})$

$$((A,B), M) \longmapsto AMB^t$$

What does a element in IP look like?

The determinant is an invariant, homogeneous polynomial.

- $\det(AMB^t) = \det(A)\det(M)\det(B^t) = \det(M)$
- $\det(\lambda M) = \lambda^n \det(M)$

Any other invariant polynomial is $p(\det(\cdot))$, for $p(x)$ univariate polynom.

BACK TO THE NULL CONE

For all groups we consider in this talk:

$$\mathcal{N}_G = \{ v \in V : p(v) = 0 \ \forall \text{ homogeneous } p \in \mathbb{P}[V] \}$$

$\Rightarrow$ if $v \notin \mathcal{N}_G$ there exists a homog. $p \in \mathbb{P}$ such that $p(v) \neq 0$.

<u>Example</u> $G = SL(n) \times SL(n)$. $\quad G \times M(n,\mathbb{C}) \to M(n,\mathbb{C})$

$$((A,B), M) \mapsto AMB^t$$

then $\mathcal{N}_G = \{ M \in M(n,\mathbb{C}) : \det M = 0 \}$.

- **How do we find such homogeneous $p \in \mathbb{P}$ in general?**
- How to show that $v \in \mathcal{N}_G$?

How to show that $v \in N_G$?

(<u>Hilbert-Mumford criteria</u>)

If $v \in N_G$ then there exists $g(t) \subset G$ such that

$$\lim_{t \to \infty} g(t) \cdot v = 0 .$$

<u>**Example**</u> consider the action

$$SL(n) \times M(n) \longrightarrow M(n)$$
$$(S, M) \longmapsto SM$$

What does one-parameter subgroup of $SL(n)$ look like?

$$g(t) = S \begin{pmatrix} e^{-t} & & & \\ & \ddots & & \\ & & e^{-t} & \\ & & & e^{(n-1)t} \end{pmatrix} S^{-1}, \quad \text{with } S \in SL(n).$$

Sanity check for $n=2$. Let $M \in N_G$, find $g(t)$.

There exists $S \in SL(2)$: $S^{-1}M = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, then

$$S \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = S \begin{pmatrix} e^{-t}a & e^{-t}b \\ 0 & 0 \end{pmatrix} \xrightarrow[t \to \infty]{} \mathbf{0}$$

MORAL: one-parameter subgroup are powerful!

How to show that $v \notin N_G$?

Given $v \in V$ consider the map $g \mapsto \|g \cdot v\|^2$.

Then $v \notin N_G \iff \inf_{g \in G} \|g \cdot v\|^2 > 0$

▷ understand minimal vectors and critical points.

Def: The Lie algebra $\mathfrak{g}$ of a Lie group $G \subseteq GL(n)$

$$\mathfrak{g} := \{ A \in M(n) \mid e^A \in G \}$$

Example $SL(n) = \{ S \in M(n) \mid \det S = 1 \}$

$$\mathfrak{sl}(n) = \{ A \in M(n) \mid \operatorname{tr} A = 0 \} \qquad \det(e^A) = e^{\operatorname{tr}(A)}$$

Consider $t \mapsto \| e^{tA} \cdot v \|^2 = \langle e^{tA} \cdot v, e^{tA} v \rangle$ for $A = A^t$.

Then $\frac{d}{dt} \left(\langle e^{tA} \cdot v, e^{tA} v \rangle \right) \Big|_{t=0} = 2 \langle A \cdot v, v \rangle =: m(v)$

moment map

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Theorem (**Kempf-Ness**)

$v \notin N_G \iff m(w) = 0$ for some $w \in \overline{G \cdot v} \setminus \{0\}$

Proof. Recall $m(w) = \langle A \cdot w, w \rangle$ with $A \in \mathfrak{g}$, $A = A^t$.

$(\Rightarrow)$ if $v \notin N_G$ then $\overline{G \cdot v}$ contains a minimal $w \neq 0$.

$(\Leftarrow)$ is equivalent to: $\underset{w \text{ minimal}}{w \in \overline{G \cdot v} \setminus \{0\}} \Rightarrow 0 \notin \overline{G \cdot v}$

<u>no</u> minimal <u>non-zero</u> $w \in \overline{G \cdot v}$ $\Leftarrow$ $0 \in \overline{G \cdot v}$

$0 \in \overline{G \cdot v} \implies$ no minimal non-zero $w \in \overline{G \cdot v}$

Recall: $\exists\, g(t) \subset G$: $\lim_{t \to \infty} g(t) \cdot v = 0$

Fact: The map $t \mapsto \|e^{tA} \cdot v\|^2 =: d(t)$ is CONVEX for $A = A^T$.

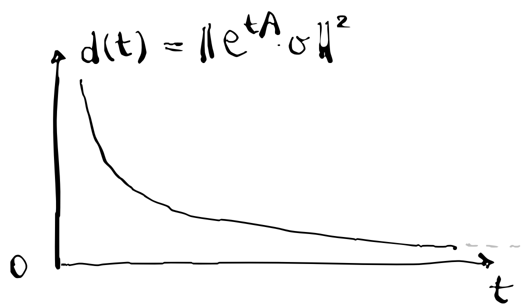
proof $d'(0) = 2 \langle A \cdot e^{tA} v, e^{tA} v \rangle \big|_{t=0} = 2 \langle A \cdot v, v \rangle$

$d''(t) = 2 \langle A^2 e^{tA} v, e^{tA} v \rangle + 2 \langle A e^{tA} v, A e^{tA} v \rangle = 4 \cdot \|A \cdot e^{tA} v\|^2 \geq 0$

Corollary: Assume that $\lim_{t \to \infty} g(t) \cdot v = 0 \in \overline{G \cdot v}$

Then $G \cdot v$ does NOT contain minimal vectors, since $\|e^{tA} \cdot v\| > 0$.

proof



A map to separate closed orbits

What are "weights" in representation theory?

Short answer: Weights are $A_1,\dots,A_n \in \mathfrak{g}$ so that given $A \in \mathfrak{g}$, the action on $(V, \langle\cdot,\cdot\rangle)$

$$e^{A} v = \left(e^{\langle A, A_1\rangle} v_1, \dots, e^{\langle A, A_n\rangle} v_n\right)$$

better answer: If G is abelian, there is $V = V_1 \oplus \dots \oplus V_n$ eigenspaces such that $S(V_i) \subseteq V_i \quad \forall i = 1,\dots,n \quad \forall S \in G$.

Take $e_j \in V_j$ consider

$$\phi_{e_j} : G \to \mathbb{C}^* \simeq Gl(1,\mathbb{C}) \qquad \text{GROUP HOMOM.}$$
$$S \mapsto \lambda_j \ , \quad S e_j = \lambda_j e_j$$

induces

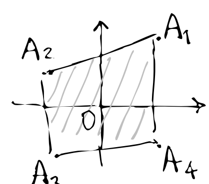
$$\tilde{\phi}_{e_j} : \mathfrak{g} \to \mathfrak{gl}(1,\mathbb{C}) \simeq \mathbb{C} \qquad \text{homom. of Lie algebras}$$

$$\Rightarrow \tilde{\phi}_{e_j} \in \mathfrak{g}^* \Rightarrow \tilde{\phi}_{e_j}(A) = \langle A, A_j \rangle \quad \forall A \in \mathfrak{g}$$

by Riesz representation

Proposition G abelian.

$G \cdot v$ is closed $\iff$



$$\sum_{i=1}^{n} \theta_i A_i = 0$$

where $\sum_{i=1}^{n} \theta_i = 1$, $\theta_i > 0$.

A map to seperate orbits

Let G be abelian with Lie algebra $\mathfrak{g}$.

$$\begin{aligned} \phi : \mathfrak{g} &\longrightarrow V \\ A &\longmapsto (\langle A, A_1\rangle, \dots, \langle A, A_n\rangle) \end{aligned}$$

consider $\phi(\mathfrak{g})^\perp = \{ w \in V : \langle w, \phi(A)\rangle = 0 \ \forall A \in \mathfrak{g} \}$. For $w \in \phi(\mathfrak{g})^\perp$, let

$$f_w(v) := \prod_{i=1}^{n} v_i^{w_i}$$

Lemma Given $v \notin N_G$ there exists $w \in \phi(\mathfrak{g})^\perp$ such that

$$f_w(v) \neq f_w(0).$$

Proof: suppose $f_w(v) = f_w(u)$ $\forall w \in \phi(\mathfrak{g})^\perp$, where $u \notin G \cdot v$.

Then $\log f_w(v) = \sum_{i=1}^{n} w_i \log v_i = \sum_{i=1}^{n} w_i \log u_i = \log f_w(u)$

$\Rightarrow \quad \langle w, \log v - \log u \rangle = 0 \ \forall w \in \phi(\mathfrak{g})^\perp \Rightarrow \log u = \phi(A) + \log v$

$\Rightarrow \quad u = e^{\phi(A)} \cdot v \quad \lightning$

"A bit of Algebra never hurts"
— Lukas M.

Thank you

Schön, dass Sie da sind

Thanks to Rajula, Linda Ness and Ramiro Lafuente for the exciting maths discussions.

Don't believe that f_w is invariant?

$$f_w(\sigma) := \prod_{i=1}^{n} \sigma_i^{w_i}$$

$$f(e^A \sigma) = f\left(e^{\langle A, A_1\rangle} \sigma_1, \dots, e^{\langle A, A_n\rangle} \sigma_n\right) = e^{\sum_j \langle A, A_j\rangle \theta_j} \prod_j \sigma_j^{\theta_j} = e^{\langle A, \sum \theta_j A_j\rangle} \prod_j \sigma_j^{\theta_j} = f(\sigma)$$

Look at critical points of

$$g \longmapsto \|g \cdot v\|^2$$

Which $g \in G$ the map is constant?

Let $K \subseteq G$ such that $\|k \cdot v\| = \|v\| \quad \forall v \in V$.

<u>Example</u>: $\mathbb{C}^*$ acting on $(\mathbb{C}, |\cdot|^2)$.

$$\mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{Then for } e^{i\theta} \in U(1) = \{z \in \mathbb{C}^* : |z| = 1\}$$

$$(z, \xi) \longmapsto z\xi \qquad |e^{i\theta}\xi| = |\xi|$$

Let $\mathrm{Stab}(v) = \{g \in G : g \cdot v = v\}$. Then for $s \in \mathrm{Stab}(v)$

$$\|s \cdot v\| = \|v\|$$

▷ Reduce to work in $G/\mathrm{Stab}(v) \simeq G \cdot v$

where $G \cdot v = \{g \cdot v \mid g \in G\}$ is the orbit of v in G.