The Stoïlow factorisation theorem

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after Astala, Iwaniec, and Martin

Summer School[a] on
Nodal Domains and Landscape Functions
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UNIVERSITY OF BIRMINGHAM

CHALMERS

GÖTEBORGS UNIVERSITET

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Wirtinger derivatives:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Chain rule: $g \circ f : \mathbb{C} \to \mathbb{C}$, $z = f(w)$

$$\partial_w g(f(w)) = g_z(f(w)) f_w(w) + g_{\bar{z}}(f(w)) \overline{f_{\overline{w}}(w)}$$
$$\partial_{\overline{w}} g(f(w)) = g_z(f(w)) f_{\bar{w}}(w) + g_{\bar{z}}(f(w)) \overline{f_w(w)}$$

Proof: use Wirtinger derivatives and classical chain rule in $\mathbb{R}^2$.

Complex analysis in the plane 2/2

Definition (Conformal map)
A function f on $\Omega \subset \mathbb{C}$ is conformal if it is a biholomorphism.

(Note: f conformal $\implies$ f is holomorphic and $f' \neq 0$.)

Theorem (Riemann mapping theorem ~~(1851)~~ (1900))
Any non-empty open simply connected proper $\Omega \subset \mathbb{C}$ admits a bijective conformal map to the open unit disk $\mathbb{D}$.

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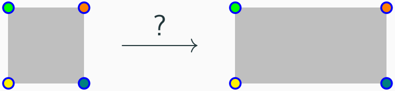

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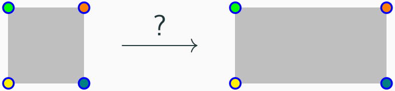

$$2 \leqslant \int_0^1 \int_0^1 |f'(x+iy)| dx dy$$
$$\leqslant \left(\iint_{[0,1]^2} |f'(x+iy)|^2 dx dy \right)^{\frac{1}{2}} \left(\iint_{[0,1]^2} dx dy \right)^{\frac{1}{2}} = \sqrt{2}.$$

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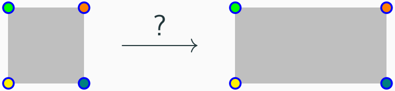

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Grötzsch (1928) asked for "approximate conformality".

Maps of bounded distorsion

Consider $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ such that $\exists K \geqslant 1$

$$\max_\alpha |\partial_\alpha f(z)| \leqslant K \min_\alpha |\partial_\alpha f(z)| \qquad \text{for a.e. } z$$
$$|f_z(z)| + |f_{\overline{z}}(z)| \leqslant K(|f_z(z)| - |f_{\overline{z}}(z)|)$$

We can rearrange them to get:

$$|f_{\overline{z}}(z)| \leqslant \frac{K-1}{K+1} |f_z(z)|$$

The Beltrami equation is $f_{\overline{z}}(z) = \mu(z) f_z(z)$ for $\|\mu\|_{L^\infty} \in (0,1)$.

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Linear algebra interpretation:

$$|Df(z)| = |f_z(z)| + |f_{\overline{z}}(z)| \leqslant K \frac{|f_z(z)|^2 - |f_{\overline{z}}(z)|^2}{|Df(z)|} = K \, \frac{J(f, z)}{|Df(z)|}$$

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What do these maps look like?

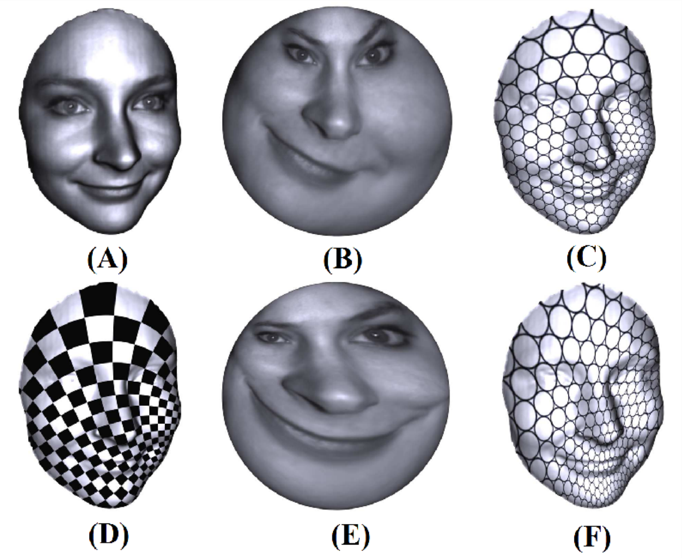

Figure 1: (A) Face. (B) Conformal Mapping. (C) Induced Circle Packing. (D) Conformal Checkerboard. (E) Quasiconformal Mapping. (F) Quasiconformal Circle Packing.
Source by Zeng, Lui, Luo, Liu, Chan, Yau, Gu arXiv:1005.464

Let $\mu \in L^\infty(\mathbb{C})$, with $\|\mu\|_\infty = \varepsilon \coloneqq \frac{K-1}{K+1}$.

Definition (Quasiconformal mappings)
Functions f on $\Omega \subset \mathbb{C}$ that are $W^{1,2}_{\text{loc}}$ solutions to

$$\frac{\partial}{\partial \bar{z}} f(z) = \mu(z) \frac{\partial}{\partial z} f(z) \qquad \text{a.e. } z \in \Omega \subset \mathbb{C} \tag{B}$$

and that are *homeomorphisms*: continuous and open.

Are two solutions of (B) related?

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Theorem (Stoïlow factorization)

Let $f, g \in W^{1,2}_{loc}(\Omega)$ *be two solutions to the same equation* (B), *and let* f *be quasiconformal.*
Then there exists a holomorphic map Φ *on* $f(\Omega)$ *such that*

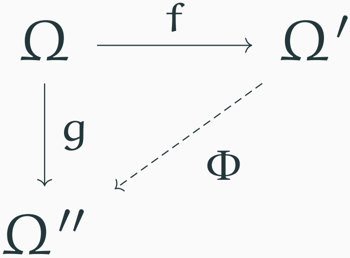
$$g(z) = \Phi(f(z)) \qquad \textit{for all } z \in \Omega.$$

Moreover, for any holomorphic function Φ *on* $f(\Omega)$*, the map* $\Phi \circ f$ *is a solution of* (B).

Recall the Beltrami equation:

$$\frac{\partial}{\partial \overline{z}} f(z) = \mu(z) \frac{\partial}{\partial z} f(z)$$

The Stoïlow factorisation theorem says that two different solutions to the Beltrami equation are related by a holomorphic function.



Stoïlow factorisation

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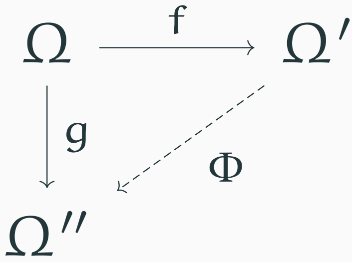

How can this be used?

Applications to PDEs

Let A be symmetric and elliptic on $\mathbb{C}$

$$\frac{1}{K}|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant K|\xi|^2$$

Let $u \in W^{1,2}_{\text{loc}}(\Omega)$ be a solution of $\text{div} A(z)\nabla u = 0$.

1. Does $u \in W^{1,p}_{\text{loc}}(\Omega)$ for other (higher) p ?
2. What is the regularity of u?

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Idea: construct a K-quasiconformal map from u.

Definition (A-harmonic conjugate)

A function $v \in W^{1,2}_{\text{loc}}(\Omega)$ such that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A(z)\nabla u = \nabla v$$

Then $f = u + iv$ is K-quasiconformal. By Stoïlow factorisation $\exists \Phi$ holomorphic and F K-quasiconformal such that $f = \Phi \circ F$. Then

$$u = \Re(f) = \Re(\Phi) \circ F.$$

Applications to PDEs

Theorem
Let $u \in W^{1,2}_{loc}(\Omega)$ *be a solution of*

$$\mathrm{div} A(z) \nabla u = 0$$

Then, by Stoïlow factorisation, we have the following:

1. *Improved integrability:* $u \in W^{1,p}_{loc}(\Omega)$ *for* $p \in [2, \frac{2K}{K-1})$.
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Stoïlow factorisation successfully used by (at least) one participant in this school: Gallegos, Josep M. *"Size of the zero set of solutions of elliptic PDEs near the boundary of Lipschitz domains with small Lipschitz constant."* arXiv:2201.12307, check it out!

Corollary (Uniqueness of normalised solution)
Let $f, g \in W^{1,2}_{loc}(\mathbb{C})$ *be two homeomorphic solutions to* (B) *on* $\mathbb{C}$*.*
If f *and* g *fix the points* 0 *and* 1*, then* $f = g$*.*

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Proof of the Corollary. By Stoïlow factorisation,

- $\exists$ an entire function Φ such that $g = \Phi \circ f$.
- f and g are homeomorphisms $\implies$ Φ injective, so Φ is conformal.
- Entire conformal maps are similarities: $\forall \zeta, z, w \in \mathbb{C}$

$$\frac{|\Phi(\zeta) - \Phi(w)|}{|\Phi(z) - \Phi(w)|} = \frac{|\zeta - w|}{|z - w|}$$

- a similarity which fixes 0 and 1 (and $\Phi(\infty) = \infty$) is the identity. $\square$

Remark. The map

$$f(z) = z + \varepsilon \prod_{j=1}^{N} (z - \zeta_j)$$

fixes $\{\zeta_j, j = 1, \ldots, N\}$, is holomorphic and locally injective ($f' \neq 0$ for small ε), but it is not the identity.

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Proof of the factorisation

$$\begin{array}{ccc} \Omega & \xrightarrow{f} & \Omega' \\ \downarrow g & \swarrow \Phi & \\ \Omega'' & & \end{array}$$

Goal: show that the map $\Phi \coloneqq g \circ f^{-1}$ is holomorphic.

1. Assume g continuous (and in $W^{1,2}_{\text{loc}}$). Let $f(z) = w$.
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$$\partial_z (h \circ f)(z) = h_w(w) f_z(z) + h_{\bar{w}}(w) \overline{f_{\bar{z}}(z)} = 1$$

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It follows that h satisfies

$$h_{\bar{w}} = -\mu(h(w)) \overline{h_w}$$

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3. Use Weyl's lemma: weak solutions to $\bar{\partial}$ in $L^1_{\mathrm{loc}}(\mathbb{C})$ are analytic. $\square$

Two operators

The solid Cauchy transform is

$$\mathcal{C}f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{z-\zeta}d\zeta$$

The Beurling transform S is given by

$$Su(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u(\zeta)}{(z-\zeta)^2}d\zeta$$

Note:

- $\mathcal{C} := (\partial_{\bar{z}})^{-1}$, mapping $\mathcal{C}\colon L^p(\mathbb{C}) \to W^{1,p}(\mathbb{C})$ for $p > 2$.
- S is bounded on $L^p(\mathbb{C})$, and $S(\partial_{\bar{z}} f) = \partial_z f$ for $f \in W^{1,p}(\mathbb{C})$. Also

$$\|S\|_{p\to p} \leqslant \frac{1}{\varepsilon} \quad \text{for } p \in (1+\varepsilon, (1+\varepsilon)')$$

Solving the Beltrami equation

Consider the inhomogeneous equation for $\varphi \in L^p_{\text{comp}}(\mathbb{C})$.

$$\partial_{\bar{z}}\sigma = \mu(z)\partial_z\sigma + \varphi \qquad (\tilde{B})$$

If σ solves $(\tilde{B})$ with $\varphi = \mu$, then $f = z + \sigma$ solve (B).

Rewrite using $\partial_z = S\partial_{\bar{z}}$:

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A solution is given by

$$\sigma = \partial_{\bar z}^{-1}(I - \mu(z)S)^{-1}\varphi = \partial_{\bar z}^{-1}\sum_{k\geqslant 0}(\mu(z)S)^k\varphi$$

The Neumann series converges for $\|\mu S\| \leqslant \|\mu\|_\infty\|S\| < 1$.
If $\|\mu\|_\infty = \varepsilon$, we need $\|S\| < 1/\varepsilon$, "not too large".

Why are weak $W^{1,2}$ solution continuous? 1/2

Recall $I_\varepsilon \coloneqq (1+\varepsilon, 1+1/\varepsilon)$.

Theorem

Let $\Omega \subset \mathbb{C}$. *If* $f \in W^{1,2}_{loc}(\Omega)$ *solves* (B) *with* $\|\mu\|_\infty = \varepsilon < 1$, *then* $f \in W^{1,p}_{loc}(\Omega)$ *for all* $p \in I_\varepsilon$.

Remark. In particular, $f \in W^{1,2+s}_{\text{loc}}(\Omega)$ for some $s > 0$, so by the Sobolev embedding f is continuous.

Sketch of the proof. Consider $F \coloneqq \psi f$, for $\psi \in C^\infty_c(\Omega)$. Since f is a solution to the Beltrami equation, by the chain rule

$$(\psi f)_{\bar{z}} - \mu(\psi f)_z = f \cdot (\psi_{\bar{z}} - \mu\psi_z) \eqqcolon \varphi.$$

Then $F = \psi f$ solves the inhomogeneous Beltrami equation

$$F_{\bar{z}} = \mu F_z + \varphi$$

Why are weak $W^{1,2}$ solution continuous? 2/2

We find expressions for the weak derivative of F, that are

$$F_{\bar{z}} = (I - \mu S)^{-1}\varphi$$
$$F_z = S(F_{\bar{z}}) = S \circ (I - \mu S)^{-1}\varphi$$

where $\varphi \coloneqq f \cdot (\psi_{\bar{z}} - \mu\psi_z)$, and $F = \psi f$, $\psi \in C_0^\infty$.

- For $p \in I_\varepsilon$, we control $\|S\|_p$, $\|(I - \mu S)^{-1}\|_p$
- $F_{\bar{z}}, F_z$ are in L^p since

$$\|DF\|_p \leqslant \left(\|(I - \mu S)^{-1}\|_p + \|S\|_p\|(I - \mu S)^{-1}\|_p\right)\|\varphi\|_p$$
$$\lesssim_p \|\psi_{\bar{z}} - \mu\psi_z\|_\infty\|f\|_p$$

which holds for $p > 2$.

□

Thank you