The Bourgain–Milman (via Hörmander's) theorem

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after Fedor Nazarov[1]

Summer School on
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[1] after Bourgain and Milman

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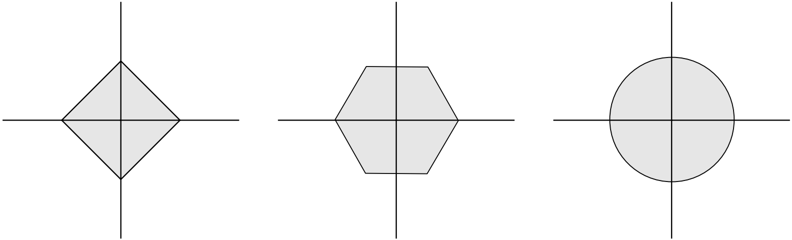

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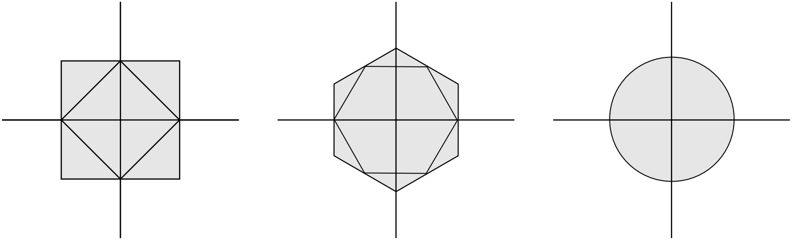

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$$\operatorname{vol} K \cdot \operatorname{vol} K^\circ \geqslant c^n \frac{4^n}{n!}$$

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- By Plancherel:

$$\|f\restriction_{\mathbb{R}^n}\|_{L^2}^2 = (2\pi)^n \|g\|_{L^2}^2$$

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Wanted function

Analytic on $\mathbb{C}^n$, fast decaying on $\mathbb{R}^n$. Not vanishing or too small at 0.

Let $g \in C_c^\infty$ be cut-off on Ω bounded domain in $\mathbb{C}^n$.

$$\omega \coloneqq \bar{\partial} g \in C_c^\infty, \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Suppose that exists h such that $\bar{\partial} h = -\omega$ in Ω, then

$$\bar{\partial} h + \omega = \bar{\partial}(h + g) = 0$$

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$f \coloneqq h + g$

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- ☐ decaying on $\mathbb{R}^n$?
- ☐ small $f(0) = h(0) + 1$?

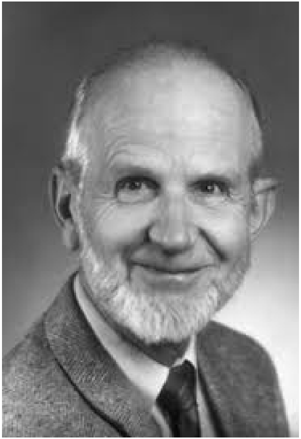

Figure: Lars Hörmander (1931-2012)

"leading figure in the theory of linear partial differential equations, his L^2 *estimates for the* $\bar{\partial}$ *equation became a revolutionary tool in complex analysis of several variables."*

Plurisubharmonic functions

Harmonic $\Delta f = 0$

Subharmonic $\Delta f \geqslant 0$

Plurisubharmonic $H \geqslant 0$ where $H = \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^{n}$

Definition

A $\varphi \colon \Omega \subset \mathbb{C}^n \to \mathbb{R}$ is *strictly plurisubharmonic* if $\exists \tau > 0$ such that

$$\langle H(z)w, w \rangle \geqslant \tau |w|^2, \quad \forall w \in \mathbb{C}^n, \forall z \in \Omega$$

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or, equivalently

$$\min_i \lambda_i \geqslant \tau > 0$$

Hörmander's Existence Theorem

Theorem (Hörmander)

Let $\Omega \subset \mathbb{C}^n$ *be an open, pseudoconvex domain,*
$\varphi\colon \Omega \to \mathbb{R}$ *strictly plurisubharmonic, for* $\tau > 0$*.*
For any $(0,1)$*-form* ω *on* Ω *with* $\bar\partial\omega = 0$*, there exists* h *such that* $\bar\partial h = \omega$ *in* Ω *satisfying*

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Remark

$\omega = \bar\partial g \, d\bar z \;\Rightarrow\; \bar\partial\omega = 0.$

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Let $T_K := \{x + iy : \; x \in \mathbb{R}^n, y \in K\}$. For any $t \in K^\circ$

$$z \mapsto \langle z, t\rangle = \zeta \mapsto \phi(\zeta)$$
$$T_K \to S \to D(0, 4/\pi)$$

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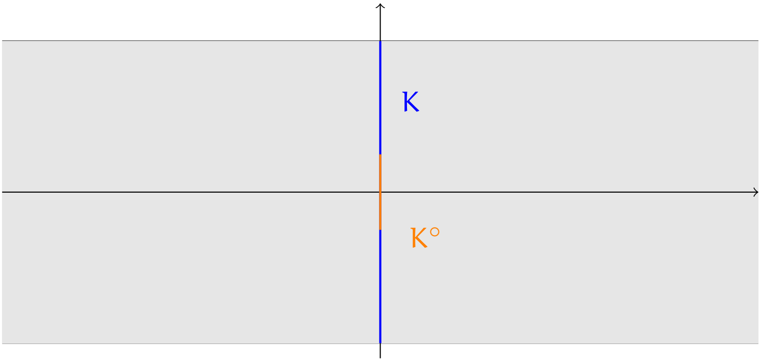
Remark

$|\phi(\zeta)| \leqslant \frac{4}{\pi}$ and $\phi(0) = 0$, $\phi'(0) = 1$, $\phi(\zeta) \sim |\zeta|$.

Example

$$K = [-2, 2],\ K^\circ = \left[-\tfrac{1}{2}, \tfrac{1}{2}\right],\ T_K := \{x + iy :\ x \in \mathbb{R}^n, y \in K\}$$



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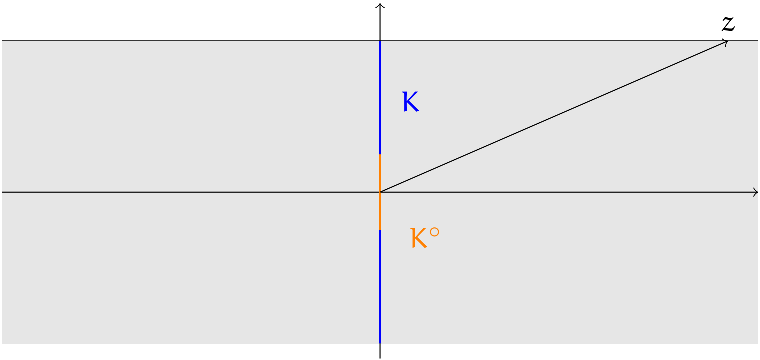

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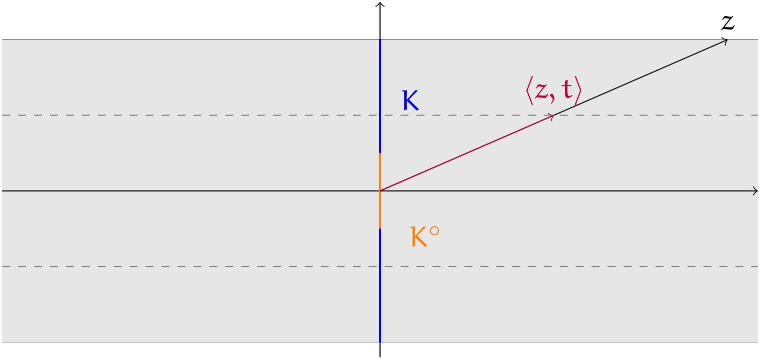

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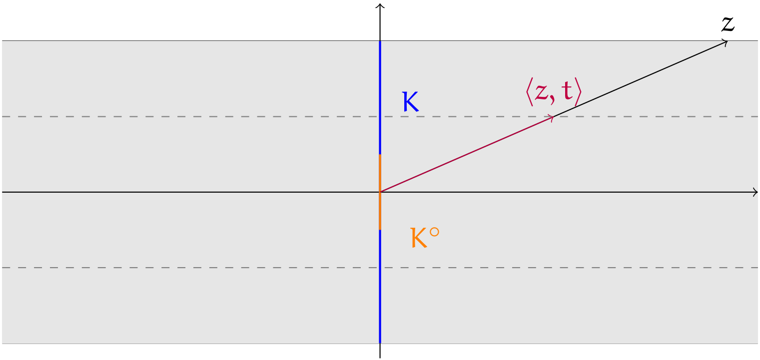

Consider

$$K_{\mathbb{C}} := \{z \in \mathbb{C}^n \ :\ |\langle z, t\rangle| \leqslant 1, \forall t \in K^\circ\} \subset T_K$$

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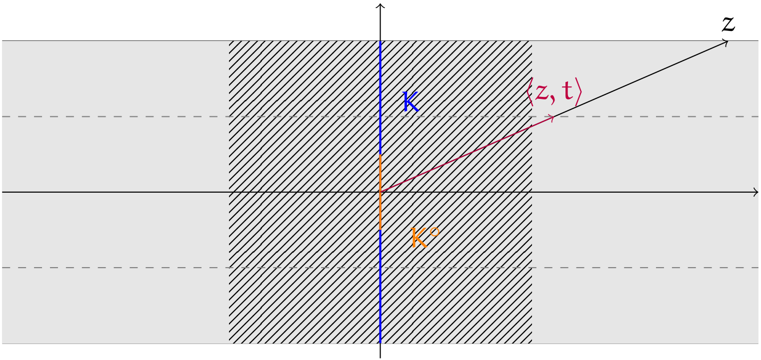

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Since $|\phi(\zeta)| \leqslant \frac{4}{\pi}$

$$\varphi(z) \leqslant 1 + \log\left(\frac{4}{\pi}\right)^{2n} \qquad \text{for } z \in K_{\mathbb{C}}$$

In particular $\|e^{\varphi}\|_\infty \lesssim \left(\frac{4}{\pi}\right)^{2n}$.

The Bergman space $\mathcal{A}^2$

Any $f \in \mathcal{A}^2(T_K) \coloneqq L^2 \cap \mathscr{H}(T_K)$,

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Cauchy–Schwarz:

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$$= \mathcal{K}(0, 0) \, \| f \|^2_{A^2(T_K)}$$

$$\frac{|f(0)|^2}{\| f \|^2_{A^2}} \leqslant \mathcal{K}(0, 0) = \int_{\mathbb{R}^n} \frac{1}{\int_K e^{-2\langle x, v \rangle} dx} \frac{dv}{(2\pi)^n} \leqslant \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}.$$

using convexity of $x \mapsto e^{-\langle x, v \rangle}$ and optimising in v.

Fix δ small. Let g cut-off on $\delta K_{\mathbb{C}}$. Let $f := h + g$ Hörmander holomorphic extension of g.

- ✓ analytic on $\mathbb{C}^n$
- ✓ L^2 decay on $\mathbb{R}^n$
- ✓ $f(0) = h(0) + 1 = 1$

- $\|h\|_2 \leqslant \|e^{\varphi}\|_\infty \| |h|^2 e^{-\varphi}\|_1 \leqslant \tau^{-1} \|e^{\varphi}\|_\infty \| |\bar{\partial} g|^2 e^{-\varphi}\|_1$

$$\begin{aligned}\|f\|^2_{A^2(T_K)} &\lesssim \|h\|^2_{L^2(T_K)} + \|g\|^2_{L^2(T_K)} \\ &\leqslant \left(\frac{4}{\pi}\right)^{2n} \frac{R^2}{\delta^2} \|\bar{\partial} g\|^2_{L^2(e^{-\varphi})} + \|g\|^2_{L^2}.\end{aligned}$$

As $\delta \to 0$:

$$\|f\|^2_{A^2(T_K)} \leqslant \left(\frac{4}{\pi}\right)^{2n} e^{o(n)} (\operatorname{vol} K)^2$$

Conclusion

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Recall

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$$\frac{4^n}{n!} \left(\frac{\pi}{4}\right)^{2n} \leqslant \frac{4^n}{n!} \frac{n!}{\pi^n} \mathrm{vol}\, K^\circ \, \mathrm{vol}\, K.$$

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$$\frac{4^n}{n!} \left(\frac{\pi}{4}\right)^{2n} \leqslant \frac{4^n}{n!} \frac{n!}{\pi^n} \operatorname{vol} K^\circ \operatorname{vol} K.$$

$$\frac{4^n}{n!} \left(\frac{\pi}{4}\right)^{3n} \leqslant \operatorname{vol} K^\circ \operatorname{vol} K.$$

Call $\nu(K) = \operatorname{vol} K \cdot \operatorname{vol} K^{\circ}$, so $\nu(C_n) = \frac{4^n}{n!}$.

$$\nu(K) \overset{?}{\geqslant} \nu(C_n) \overset{(1)}{\geqslant} \left(\frac{\pi}{4}\right)^n \nu(C_n) \overset{(2)}{\geqslant} \left(\frac{\pi}{4}\right)^{3n} \nu(C_n)$$

1. Kuperberg, Greg.
   (2008) *"From the Mahler conjecture to Gauss linking integrals."*
2. Nazarov, Fedor.
   (2012) *"The Hörmander Proof of the Bourgain–Milman Theorem."*

Thank you for listening

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and thanks for all the beef

References

- Bourgain, Jean, and Vitaly D. Milman. "New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$." *Invent. Math.* 88, no. 2 (1987): 319–340.
- Kuperberg, Greg. "From the Mahler conjecture to Gauss linking integrals." *Geom. Funct. Anal.* 18, no. 3 (2008): 870–892.
- Mahler, Kurt. "Ein Übertragungsprinzip für konvexe Körper." *Časopis Pěst. Mat. Fys.* 68, no. 3–4 (1939): 93–102.
- Nazarov, Fedor. "The Hörmander Proof of the Bourgain–Milman Theorem." In *Geometric Aspects of Functional Analysis*, pp. 335–343. Springer, Berlin, Heidelberg, 2012.
- Santaló, Luis A. "Un invariante afin para los cuerpos convexos del espacio de n dimensiones." *Portugaliae Math.* 8, no. 4 (1949): 155–161.

Open questions

- Is C_n (or O_n) a *global* minimiser of $\nu(K)$ in every dimension?
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- Where is the beef?

$$\mathcal{K}(z,w) = \int_{\mathbb{R}^n} \frac{e^{i\langle z-\bar{w},v\rangle}}{\int_K e^{-2\langle x,v\rangle}dx} \frac{dv}{(2\pi)^n}$$

Remark

Consider the Minkowski functional of K°:

$$\|y\|_{K^\circ} = \inf\{\lambda > 0\colon \lambda K^\circ \ni y\}$$

By convexity of $x \mapsto e^{-\langle x,v\rangle}$

$$\int_K e^{-2\langle x,y\rangle}dx \geqslant 2^{-n}\,\mathrm{vol}\,K\, e^{\|-y\|_{K^\circ}}$$