What is Sparse Domination and why is it so plentiful ?

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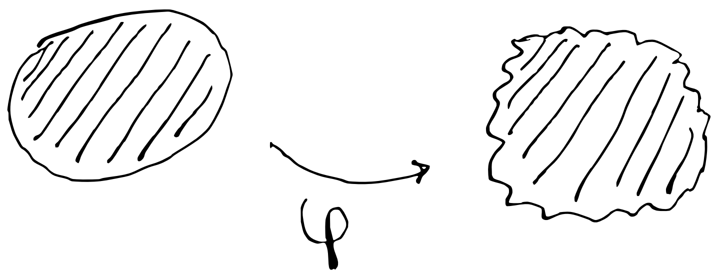
Weighted inequalities

$$\int_{\mathbb{B}} f(x)\,dx \qquad \int_{\varphi(\mathbb{B})} f(y)\,dy = \int_{\mathbb{B}} f(\varphi(x))\,|\varphi'(x)|\,dx$$



$$\int_{\mathbb{B}} F(x)\,w(x)\,dx$$

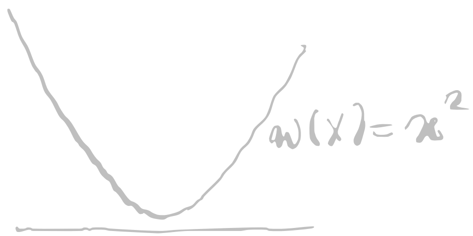
operator T acting on $L^p(\mathbb{R}) = \{ f \in C_c^\infty(\mathbb{R}) : \|f\|_{L^p} < +\infty \}$, $1 \le p \le \infty$

$$\|f\|_{L^p}^p := \int_{\mathbb{R}} |f|^p\,dx \quad , \quad \|f\|_{L^\infty(\mathbb{R})} := \sup_{x\in\mathbb{R}} |f(x)|$$

How does T "move" functions?

Consider $T : L^p(\mathbb{R}, dx) \longrightarrow L^p(\mathbb{R}, dx)$

$$\| Tf \|_{L^p} = \| \tau(Tf) \|_{L^p}$$



weight

Replace dx with $w(x)\,dx$

Lebesgue measure

Understanding $\|T\|$

$$\|Tf\|_{L^p(w)} \leq C(T,w)\, \|f\|_{L^p(w)}$$

best constant

$$\|T\| := \sup_{f \neq 0} \frac{\|Tf\|_{L^p(w)}}{\|f\|_{L^p(w)}} < +\infty$$

where $\|\cdot\|_{L^p(w)}^p = \int |\cdot|^p w(x)\,dx.$

- For which w ?
- Can we characterise them ?
- How do they depend on T and p ?

Benjamin Muckenhoupt (1972)

Maximal Operators

$$\mathcal{M}f(x) = \sup_{Q \ni x} \boxed{⨍_Q |f(y)|\,dy} \simeq \sup_{r>0} \frac{1}{r} \int_{B_r(x)} |f(u)|\,du$$

cube containing x

average on Q

$$\| \mathcal{M} : L^p(w) \longrightarrow L^p(w) \| < +\infty \iff w \in A_p$$

"Muckenhoupt weights"

[Eric Sawyer (1982)]

- characterisation of weights for which $M : L^p(w) \to L^p(v)$
- optimal dependence on $\| M \|_{L^p(w) \to L^p(w)} \lesssim [w]_p^{\alpha}$ $\quad \alpha(p) = \frac{1}{p-1}$

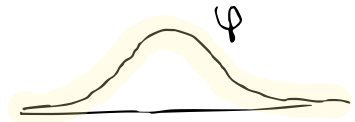
[Stephen Buckley (1993)]

Can we control $Tf(x) \lesssim \mathcal{M}f(x)$?

$$\varphi_t(\cdot) := \frac{1}{t}\varphi(\cdot/t)$$

Maximal operators

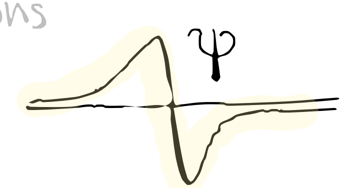
$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{r}\int_{B_r(x)} |f(u)|\,du = \sup_{t>0} |f * \varphi_t|(x)$$



$$\mathcal{H}f(x) := \text{p.v.}\left(f * \frac{1}{y}\right)(x) \quad , \quad Sf(x) = \left(\int_0^\infty |f * \psi_t|^2(x)\,\frac{dt}{t}\right)^{1/2}$$

Singular operators

square functions



For $1<p<\infty$:

$$\left.\begin{array}{l} \| \mathcal{M} : L^p(w) \longrightarrow L^p(w) \| < +\infty \\ \| \mathcal{H} : L^p(w) \longrightarrow L^p(w) \| < +\infty \\ \| S : L^p(w) \longrightarrow L^p(w) \| < +\infty \end{array}\right\} \iff w \in A_p$$

Can we control $Tf(x) \lesssim \mathcal{M}f(x)$?

Usually NO :(

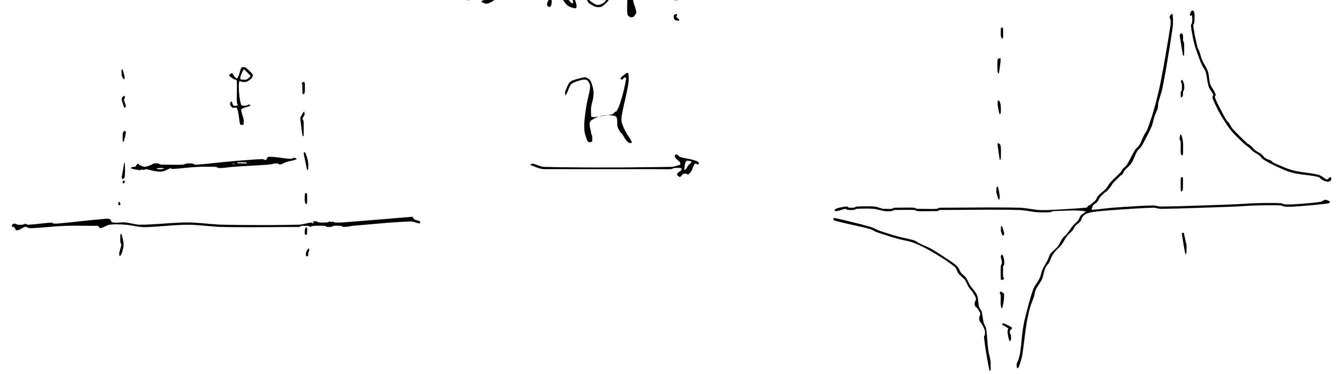
$\mathcal{M}$ is bounded on bounded f : $\|\mathcal{M}f\|_{L^\infty(\mathbb{R})} < +\infty$

but

The Hilbert transform

$$\mathcal{H}f(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x-y)\, \frac{dy}{y} \quad , \quad f \in C_c^\infty(\mathbb{R})$$

is NOT!



Can we (still) understand T in terms of averages?

$$|Tf(x)| \lesssim \sum_{I \in \mathcal{S}} \fint_I |f(y)|\,dy \; \mathbb{1}_I(x)$$

collection of cubes (it depends on f!)

if $\mathcal{S}$ disjoint:

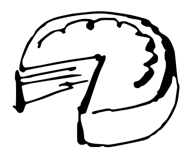
$$\|Tf\|_{L^\infty} \lesssim \sup_I \fint_I |f| \lesssim \|f\|_{L^\infty}$$

like the M!

if cubes in $\mathcal{S}$ overlap (∞-many times):

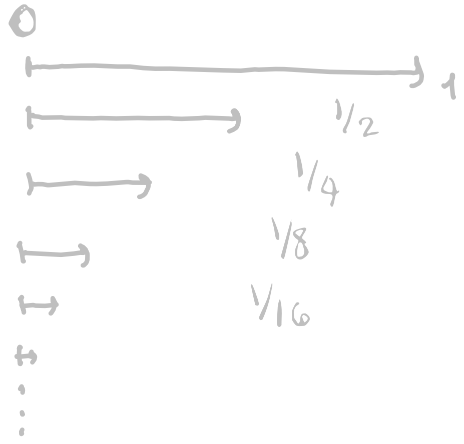
$$\|Tf\|_{L^\infty} \lesssim \sum_{I \in \mathcal{S}} \fint_I |f| = \infty$$

Collections with comparable & disjoint subfamilies



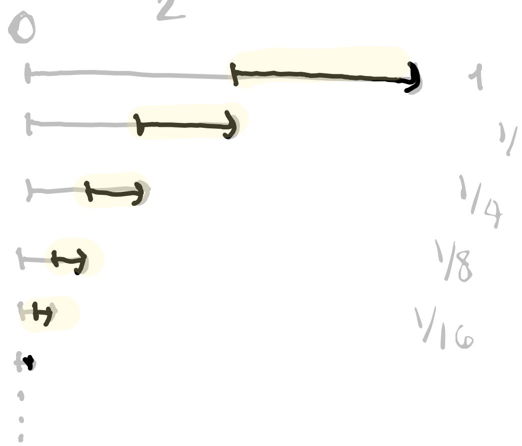
"Sparse"

$[0, 2^{-n}) =: I_n$



∞ - overlapping

$[\frac{2^{-n}}{2}, 2^{-n}) =: E_n$



disjoint "comparable" subfamily

$$|I_n| \leq 2|E_n| \quad \forall n \in \mathbb{N}$$

$$\sum \left(\fint_{I_n} |f| \right) |I_n| \leq 2 \left(\fint_{I_n} |f| \right) |E_n| = 2 \int_{E_n} \left(\fint_{I_n} |f| \right) dx \leq 2 \int_{E_n} \mathcal{M}f(x)\, dx$$

Thank you!


SPARSE
domination
11th Dec 2020
LMS Harmonic Analysis
& PDEs meeting