Superexponential L^2 decay via Batesian mimicry

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Batesian mimicry

> Batesian mimicry is a form of mimicry where a harmless species has evolved to imitate the warning signals of a harmful species. [...] It is named after the English naturalist *Henry Walter Bates*, after his work on butterflies in the rainforests of Brazil.
>
> \- Wikipedia

Figure 1: Below, an example of Batesian mimicry between species. Papilio polytes (left) resembles the inedible Pachliopta aristolochiae (right).

Figure 2: Papilio polytes.

Figure 3: Pachliopta aristolochiae.

Sources.
Papilio : Photo by Jeevan Jose, Kerala, India. Attribution: © 2016 Jee & Rani Nature Photography.
Pachliopta : Photo by J.M.Garg, India.

Maximisers for an extension inequality

For any real $p > 1$, consider the linear operator

$$\mathcal{E}_p(f)(x,t) = \int_{\mathbb{R}} e^{ixy} e^{it|y|^p} |y|^{\frac{p-2}{6}} f(y)\,dy.$$

The operator $\mathcal{E}_p(f)$ is a Fourier extension operator $\mathcal{F}(f\sigma_p)(-\cdot)$ where

$$\mathcal{F}(g)(x,t) = \iint_{\mathbb{R}^2} e^{-i(xy+ts)} g(y,s)\,dy\,ds$$

is the Fourier transform on $\mathbb{R}^2$ and

$$d\sigma_p(y,s) = \delta(s - |y|^p)|y|^{\frac{p-2}{6}}\,dy\,ds$$

is a singular measure supported on the curve $s = |y|^p$.

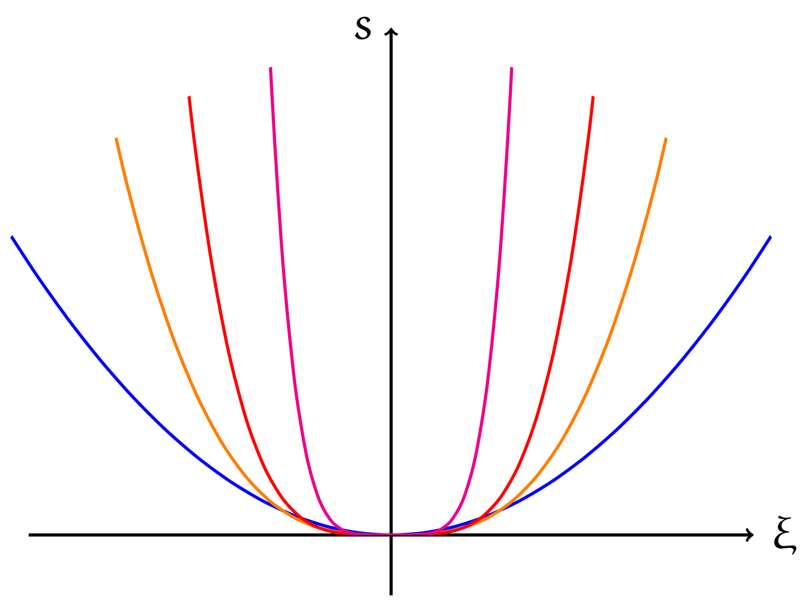


Figure 4: The curves $s = |\xi|^p$, for $p = 2, 3, 4$ and 5. The operator makes sense for any real $p > 1$.

The operator $\mathcal{E}_p$ is bounded from $L^2 \to L^6$ and satisfies

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)} \leqslant \mathbf{E}_p \|f\|_{L^2(\mathbb{R})} \tag{1}$$

where $\mathbf{E}_p$ is the best constant.

Definition *A* maximiser *for* (1) *is a function $f \neq 0$ that satisfies*

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)} = \mathbf{E}_p\|f\|_{L^2(\mathbb{R})}.$$

The existence of maximisers for (1) for $1 \leqslant p \leqslant 5$ has been proved in [1]. Even without knowing what they are, we can still claim that

Theorem 1 *Any maximiser of* (1) *decays superexponentially fast.*

In particular, we have the following result.

Theorem *If f is a maximiser of* (1)*, there exists $\mu > 0$ such that*

$$x \mapsto e^{\mu|x|^p} f(x) \in L^2(\mathbb{R})$$

and its Fourier transform $\hat{f}$ can be extended to an entire function on $\mathbb{C}$.

Strategy of the proof

1. Every maximiser f of (1) satisfies an **Euler-Lagrange equation**. This helps us to neutralise the exponential weight splitting f in pieces.
2. Using **bilinear estimates** we can get decay from faraway pieces, in term of their distance.
3. Acting as in a Batesian mimicry, the weight evolves into a harmless exponential that can be controlled.

Euler-Lagrange equation

Idea: Any maximiser of (1) is a critical point of a functional $\mathfrak{L}\colon L^2(\mathbb{R}) \to \mathbb{R}_+$.

Consider the functional $\mathfrak{L}$ given by

$$\mathfrak{L}(f) := \frac{\|\mathcal{E}_p(f)\|^6_{L^6_{t,x}(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}}.$$

Imposing the condition

$$\frac{\partial}{\partial\tau}\mathfrak{L}(f + \tau v)\restriction_{\tau=0} = 0, \quad \forall v \in L^2(\mathbb{R})$$

one can see that any maximiser of (1) is a solution of the following equation:

$$\mathcal{E}_p^\star\left(|\mathcal{E}_p(f)(\cdot,t)|^4 \mathcal{E}_p(f)(\cdot,t)\right) = \lambda f. \tag{E-L}$$

We introduce the 6-linear form

$$Q(f_1,f_2,f_3,f_4,f_5,f_6) := \int_{\mathbb{R}^2} \Pi_{j=1}^3 \mathcal{E}_p(f_j)(x,t)\overline{\mathcal{E}_p(f_{j+3})(x,t)}\,dx\,dt.$$

Definition *A function $f \in L^2(\mathbb{R})$ is a* weak solution *of Equation* (E-L) *if there exists $\lambda > 0$ such that*

$$Q(g,f,f,f,f,f) = \lambda\langle g, f\rangle, \quad \textit{for every } g \in L^2(\mathbb{R}). \tag{2}$$

Bilinear estimates

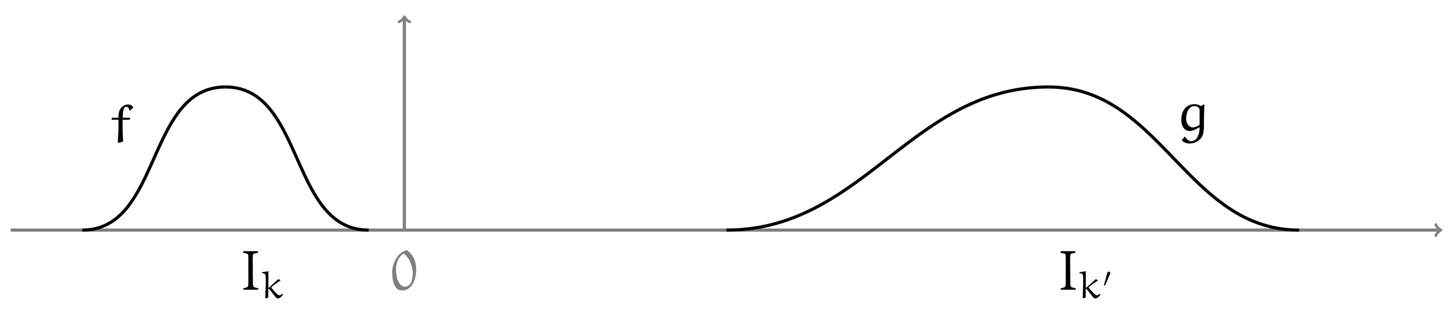
Idea: Gain decay if functions have disjoint support.

Let $I_k := (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1})$, for $k \in \mathbb{Z}$.

Proposition *Let $k, k' \in \mathbb{Z}$. For every $f, g \in L^2(\mathbb{R})$ we have that*

$$\|\mathcal{E}_p(f)\,\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} \lesssim_p 2^{-|k-k'|\frac{p-2}{6}} \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}$$

whenever $\operatorname{supp} f \subset I_k$ *and* $\operatorname{supp} g \subset I_{k'}$.



Batesian mimicry in action

Idea: Introduce an exponential weight, uniformly controlled.

The function

$$t \mapsto \frac{\mu t}{1 + \varepsilon t}$$

is increasing on $\mathbb{R}_+$ for every positive μ, ε.

Consider the function

$$G_{\mu,\varepsilon}(x) = \frac{\mu|x|^p}{1 + \varepsilon|x|^p}$$

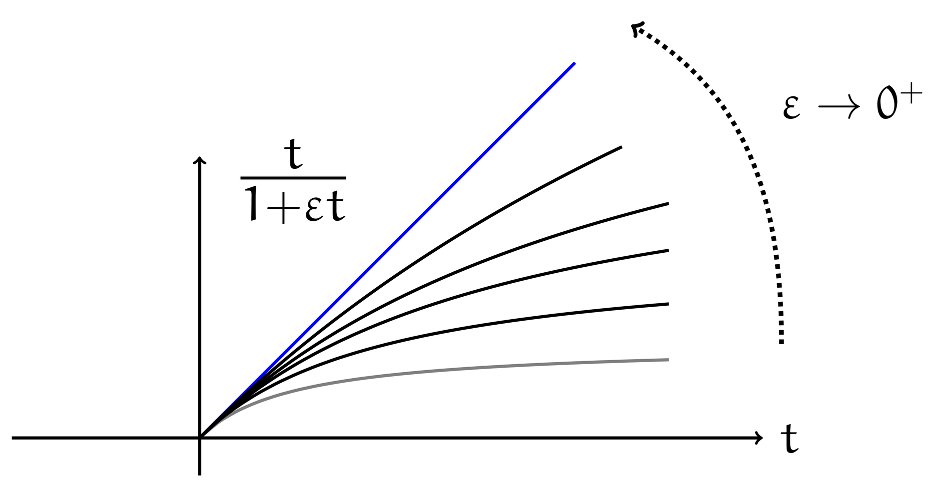


Figure 5: Plot of the functions $t \mapsto \frac{t}{1+\varepsilon t}$ for different values of $\varepsilon \in (0,1)$.

Reduce the weighted L^2 norm to the 6-linear form Q:

$$\lambda\|e^G f\|^2_{L^2} = \lambda\langle e^{2G} f, f\rangle = Q(e^{2G}f, f, f, f, f, f)$$

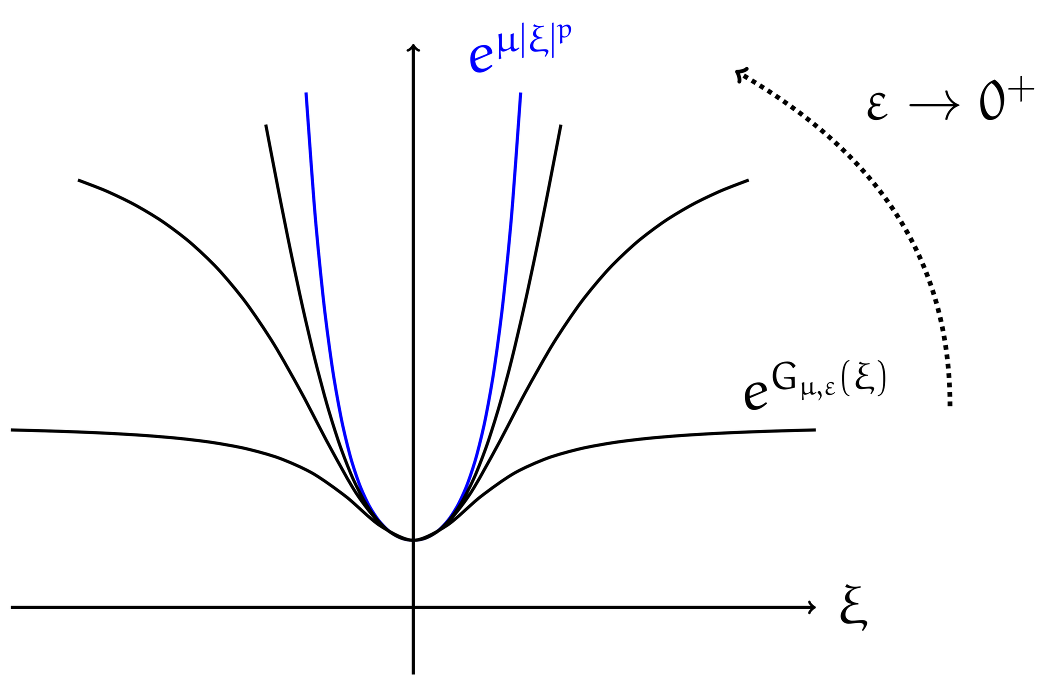


Figure 6: The functions $e^{G_{\mu,\varepsilon}}$, as ε approaches zero.

$G_{\mu,\varepsilon}(x) \to \mu|x|^p$ as $\varepsilon \to 0^+$. We can choose μ such that $\|e^{G_{\mu,\varepsilon}} f\|_2$ is *uniformly* bounded in ε.

It is enough to control f outside a compact interval, since for any $a \in \mathbb{R}$ one has $e^{a|x|} f(x) = e^{a|x| - \mu|x|^p} \cdot e^{\mu|x|^p} f(x)$. The second factor is in L^2, while the first is bounded.

References

[1] G. Brocchi, D. O. e Silva, and R. Quilodrán. *Sharp Strichartz inequalities for fractional and higher order Schröodinger equations.* To appear in *Analysis and PDE*.

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