Fourth order Schrödinger equation and Strichartz estimates: an extreme adventure

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Schrödinger waves & Strichartz estimates

Figure 1: Erwin Schrödinger

The evolution of a quantum system is described by the solution of the Schrödinger equation $u(t,x)$:

$$\begin{cases} i\partial_t u(t,x) = \partial_x^2 u(t,x) & x,t \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

where $i^2 = -1$. The solution u is

$$u(t,x) = e^{-it\partial_x^2} u_0 = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x\xi + t\xi^2)} \widehat{u_0}(\xi)\,d\xi$$

where $\hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$ is the Fourier Transform.

- This is a *dispersive* equation: its solutions spread out in space as time evolves.

To measure dispersion we integrate in time and space, estimating a mixed norm:

$$\|e^{-it\partial_x^2} u_0\|_{L_t^q(\mathbb{R};L_x^p(\mathbb{R}))} \leqslant C\,\|u_0\|_{L^2(\mathbb{R})} \qquad \text{(Strichartz)}$$

where $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$

and $2 \leqslant p \leqslant \infty$.

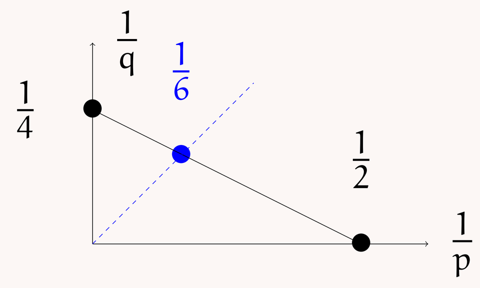


Figure 2: Admissible (p,q) for (Strichartz)

Figure 3: Robert Strichartz

These estimates are a fundamental tool in proving well-posedness of the Nonlinear Schrödinger Equation via fixed point theorems.

Extremizers

Definition. An *extremizing sequence* is a sequence $\{f_n\}_{n\in\mathbb{N}}$ in the unit ball of L^2 such that

$$\lim_{n\to\infty} \|e^{-it\partial_x^2} f_n\|_{L_t^q(\mathbb{R})L_x^p(\mathbb{R})} \to C.$$

Remark 1. Extremizing sequences may not converge!

Definition. An *extremizer* is a function $f \neq 0$ that realises equality in an inequality.

4^{th} order Schrödinger equation

Let's focus on the fourth order equation:

$$i\partial_t u(t,x) + \partial_x^4 u(t,x) = 0, \quad x,t \in \mathbb{R} \qquad (1)$$

The solution with initial datum $f \in L^2(\mathbb{R})$ is

$$S(t)f := e^{it\partial_x^4} f = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x\xi + t\xi^4)} \hat{f}(\xi)\,d\xi\,.$$

We have the Strichartz estimate:

$$\|\partial^{\frac{1}{3}} e^{it\partial^4} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant \mathbf{S}\,\|f\|_{L^2(\mathbb{R})}\,. \qquad (\star)$$

Where $\partial^{\frac{1}{3}} f(x) := \frac{1}{2\pi}\int e^{ix\xi}|\xi|^{\frac{1}{3}}\hat{f}(\xi)d\xi$, and $\mathbf{S}$ is the best constant.

Let $\mathbf{C}$ be the best constant in (Strichartz), when $q = p = 6$. By a result in [1], if $\mathbf{S} > \mathbf{C}$ then extremizers for $(\star)$ exist.

Using the Even Trick with $w(\xi) := |\xi|^{\frac{2}{3}}$ and $\rho := \delta(\tau - \xi^4)$ we have

$$\|\partial^{\frac{1}{3}} e^{it\partial^4} f\|^3_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} = (2\pi)^{-2}\|\hat{f}\sqrt{w}\rho * \hat{f}\sqrt{w}\rho * \hat{f}\sqrt{w}\rho\|_{L^2_{t,x}}.$$

Convolution of singular measures II

This time $\mathrm{supp}(\rho) \subset Q = \{(\tau,\xi) : \tau = \xi^4\}$, and the support of the convolution $\rho * \rho * \rho$ is

$$\{(\tau,\xi) \in \mathbb{R}^2 : 3^3\tau \geqslant \xi^4\}\,.$$

Moreover, its Radon-Nikodym derivative is radial and constant along branches of quartics $\tau = \alpha\xi^4$. Its value at a point depends only on the aperture α of the quartic through that point.

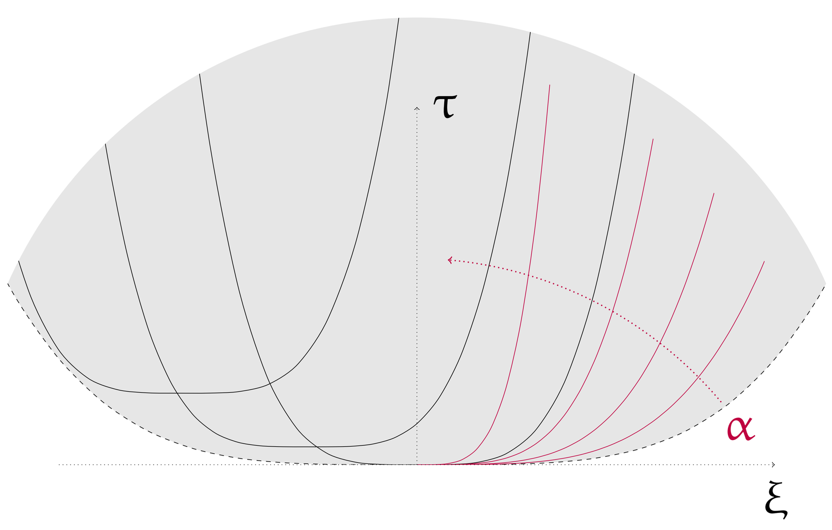


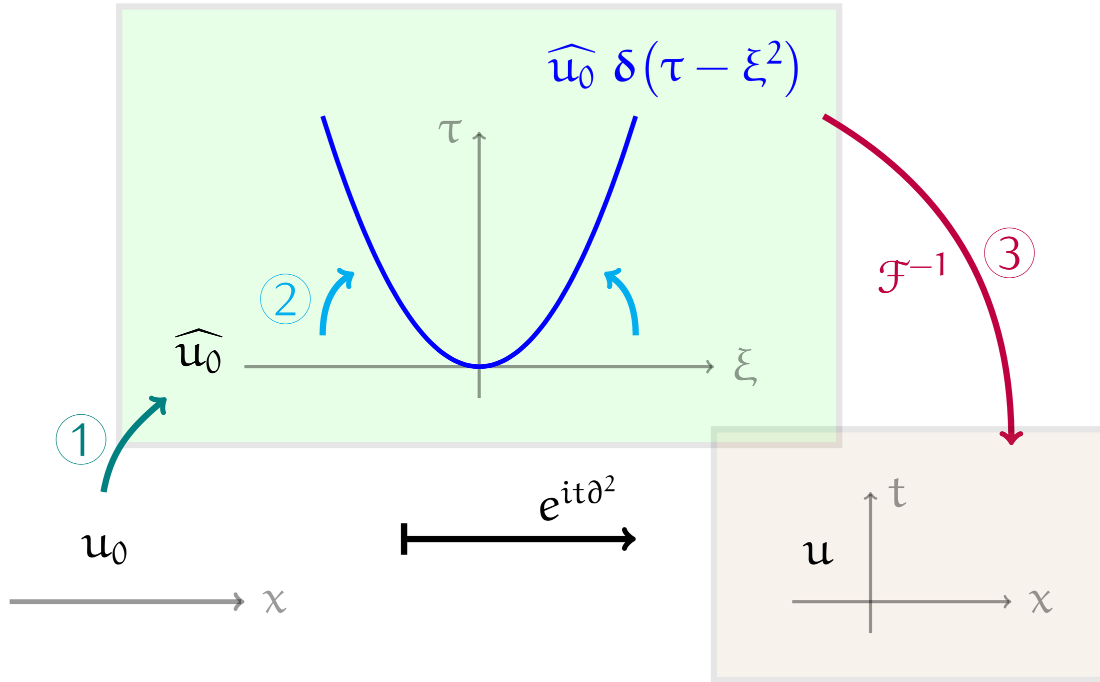
Figure 4: Support of the measure $\rho * \rho * \rho$

Restriction* theory & Even Trick

The space-time Fourier transform of u

$$\mathcal{F}(u)(\tau,\xi) = \int_{\mathbb{R}\times\mathbb{R}} e^{-i(x\xi + \tau t)} u(t,x)\,dxdt$$

is supported on the parabola $P = \{(\tau,\xi) \in \mathbb{R}^2, \tau = \xi^2\}$.



In fact, we can write $u(t,x)$ as the inverse space-time Fourier transform of a measure supported on P, like $\sigma := \delta(\tau - \xi^2)$.

This operation is called *Fourier extension*. It is the adjoint operator of the *Fourier Restriction*.

The propagator $e^{-it\partial^2}$ is the Fourier extension (from P) of the measure $\widehat{u_0}\sigma$

$$u_0 \xmapsto{\ e^{-it\partial^2}\ } u(t,x)$$

$$u_0 \overset{1}{\longmapsto} \widehat{u_0} \overset{2}{\longmapsto} \widehat{u_0}\,\sigma \overset{3}{\longmapsto} \mathcal{F}^{-1}(\widehat{u_0}\,\sigma) = u(t,x)$$

Reduce the L^6-norm to L^2-norm Raise the norm to power 3

$$\|u\|^3_{L_t^6(\mathbb{R})L_x^6(\mathbb{R})} = \||u|^3\|_{L_t^2(\mathbb{R})L_x^2(\mathbb{R})}$$

then apply Plancherel using the space-time Fourier Transform:

$$(2\pi)^3\,\|u\cdot u\cdot u\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x)} = \|\mathcal{F}(u)*\mathcal{F}(u)*\mathcal{F}(u)\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)} = \|\widehat{u_0}\sigma*\widehat{u_0}\sigma*\widehat{u_0}\sigma\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)}.$$

The problem reduces to estimating the L^2-norm of (3-fold) convolutions of the weighted measure $\widehat{u_0}\sigma$ with itself.

Convolution of singular measures

The support of the (3-fold) convolution $\sigma * \sigma * \sigma$ is

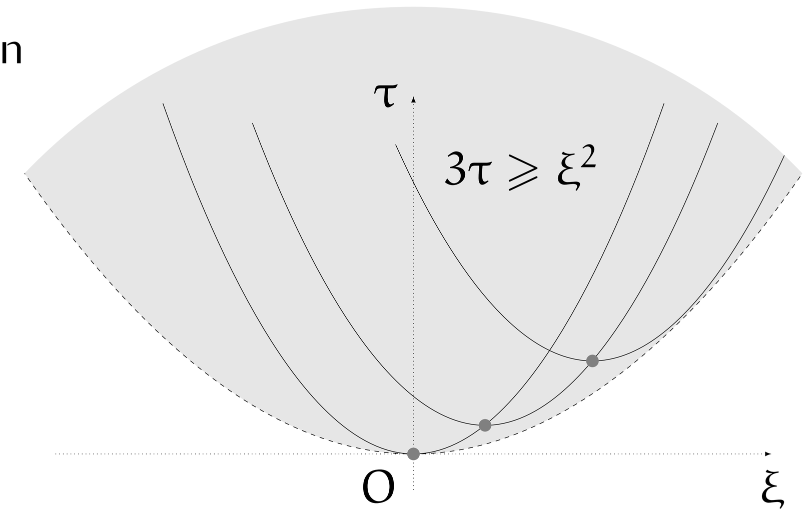
$$\overline{P+P+P} = \left\{(\tau,\xi)\ :\ 3\tau \geqslant \xi^2\right\}.$$

This is because $\mathrm{supp}(\sigma) \subset P$ and

$$\mathrm{supp}(\sigma * \sigma) \subseteq \overline{\mathrm{supp}(\sigma) + \mathrm{supp}(\sigma)},$$

and so

$$\mathrm{supp}(\sigma * \sigma * \sigma) \subseteq \overline{P+P+P}.$$



Existence of Extremizers

Theorem. *Extremizers for the Strichartz inequality $(\star)$ exist.*

Consider $f(x) = e^{-x^4}\sqrt{w(x)}$. Then

$$\|f\sqrt{wv} * f\sqrt{wv} * f\sqrt{wv}\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{-2\tau}(wv * wv * wv)^2(\xi,\tau)\,d\xi d\tau,$$

We can also compute the $\|f\|_2$ explicitly. Changing variables and exploiting the homogeneity we obtain

$$2\pi\,\mathbf{S}^6 \geqslant \frac{\|f\sqrt{wv} * f\sqrt{wv} * f\sqrt{wv}\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}} = c_0 \int_{-1}^{1} g^2(t)\,dt.$$

where $g(t) = (wv * wv * wv)(1, 3^{-3}t^{-4})$ and $c_0 = \frac{2^3 3^{\frac{3}{4}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{12}\right)^3}$.

Writing g in the basis of Legendre polynomials $g = \sum c_n L_n$ gives

$$\|g\|^2_{L^2} = \sum_{n\geqslant 0} c_{2n}^2 \geqslant c_0^2 + c_2^2 + c_4^2 \approx 0.306879 > \frac{\pi}{6\sqrt{3}}.$$

This lower bound for $\mathbf{S}$ is good enough to ensure that $\mathbf{S} > \mathbf{C}$.

References

[1] Jin-Cheng Jiang, Benoit Pausader, and Shuanglin Shao. "The linear profile decomposition for the fourth order Schrödinger equation". In: *Journal of Differential Equations* 249.10 (2010), pp. 2521–2547.