Sharp Strichartz Inequalities

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based on a joint work with
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Fix $p > 1$ and consider

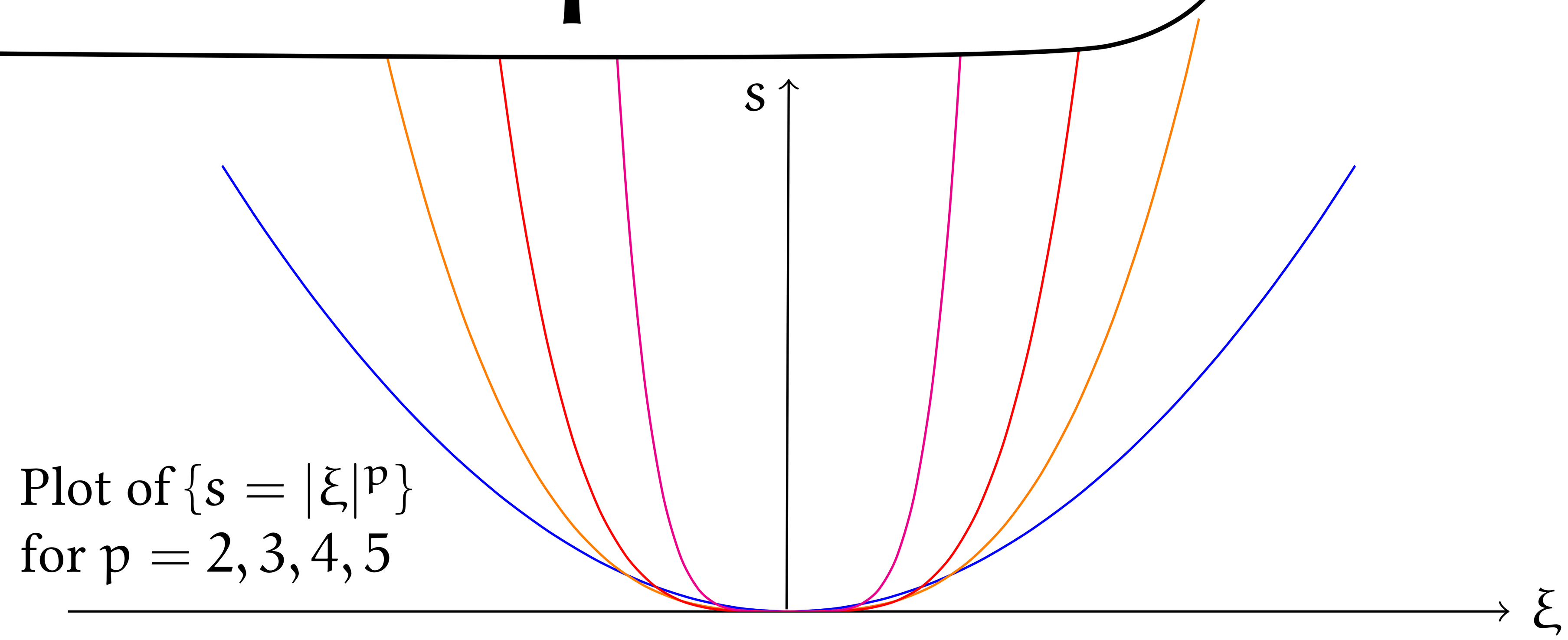
$$\begin{cases} i\partial_t u(t,x) = \partial_x^p u(t,x) & x, t \in \mathbb{R} \\ u(0,x) = f(x) \end{cases}$$

The solution is

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x\xi + t|\xi|^p)} \hat{f}(\xi)\, d\xi$$

We have the dispersive estimate with smoothing [1]:

$$\|\partial_x^{\frac{p-2}{6}} u\|_{L^6_{x,t}(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R})}$$



Plot of $\{s = |\xi|^p\}$ for $p = 2, 3, 4, 5$

Consider the singular weighted measure

$$d\sigma_p(s,\xi) = \delta(s - |\xi|^p)\, |\xi|^{\frac{p-2}{6}}\, d\xi ds$$

supported on one of the curves $s = |\xi|^p$ above.
The estimate can be written as a Fourier extension inequality:

$$\|\mathcal{E}_p f\|_{L^6_{t,x}(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R})}, \text{ with } \mathcal{E}_p f = \mathcal{F}(f\sigma_p)(-\cdot)$$

We focus on the sharp inequality in convolution form:

$$\|f\sigma_p * f\sigma_p * f\sigma_p\|_{L^2(\mathbb{R}^2)} \leq \mathbf{C}_p^3 \|f\|^3_{L^2(\mathbb{R})}$$

Do extremizers exist? And how do extremizing sequences behave?

Existence vs Concentration

Up to *rescaling* and *normalizing*, any extremizing sequence $\{f_n\}_n$ in $L^2(\mathbb{R})$ has a subsequence $\{f_{n_k}\}$ that satisfies one of the following:

(i) There exists $f \in L^2(\mathbb{R})$ such that $f_{n_k} \xrightarrow{k\to\infty} f$ in L^2, or

(ii) $\{f_{n_k}\}$ concentrates at $\xi_0 = 1$.

Main ingredients for the proof:

- Refined Strichartz inequality
- Lions' concentration-compactness
- Revised Brézis–Lieb lemma

Resolve the dichotomy

If $\{f_n\}$ concentrates at $\xi_0 \neq 0$, then

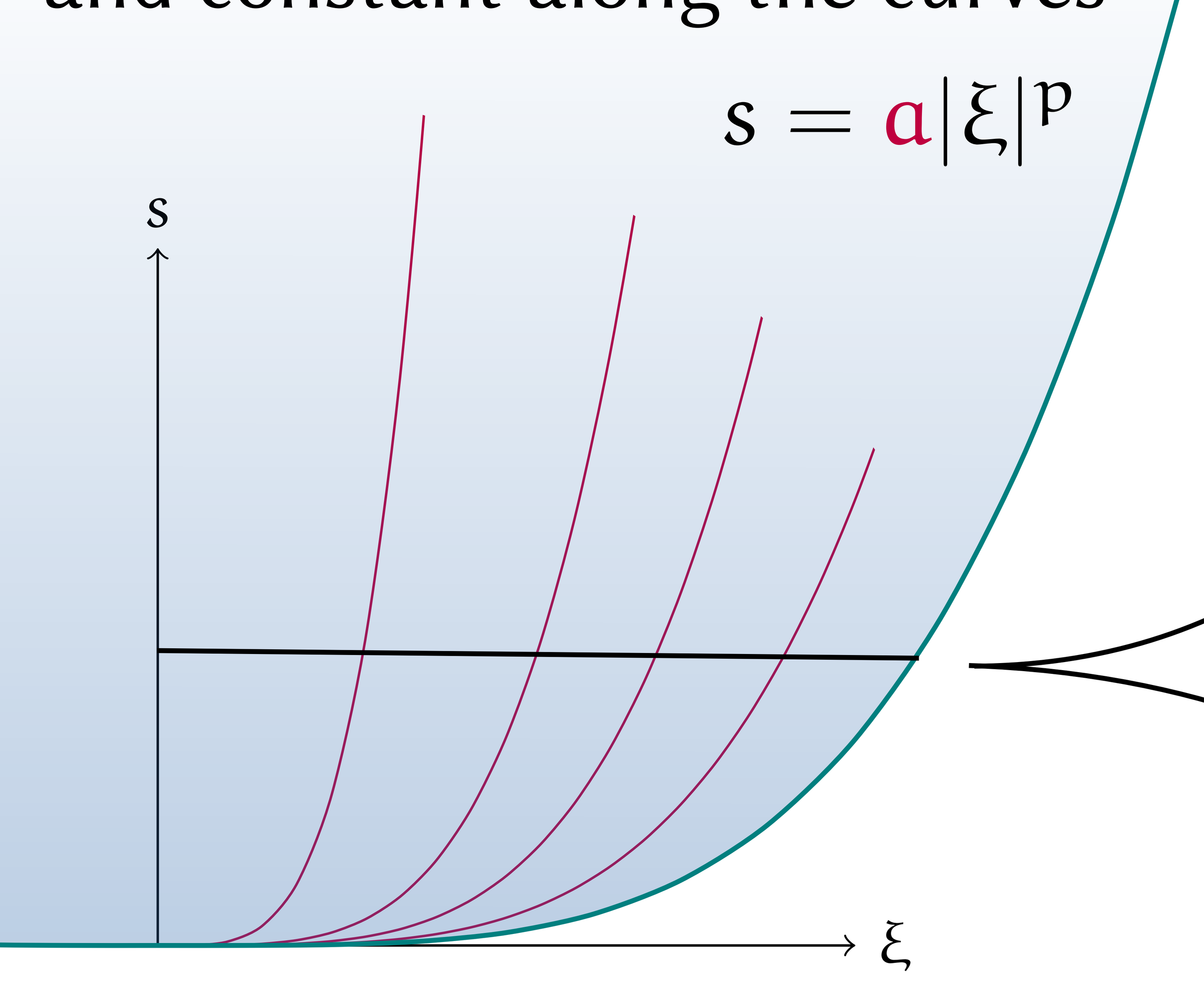
$$\limsup_{n\to\infty} \frac{\|f_n\sigma_p * f_n\sigma_p * f_n\sigma_p\|^2_{L^2}}{\|f_n\|^6_{L^2}} \leq \frac{2\pi}{\sqrt{3p(p-1)}}$$

Consider $g(\xi) = e^{-|\xi|^p} w(\xi)$, where $w(\xi) = |\xi|^{\frac{p-2}{6}}$, then

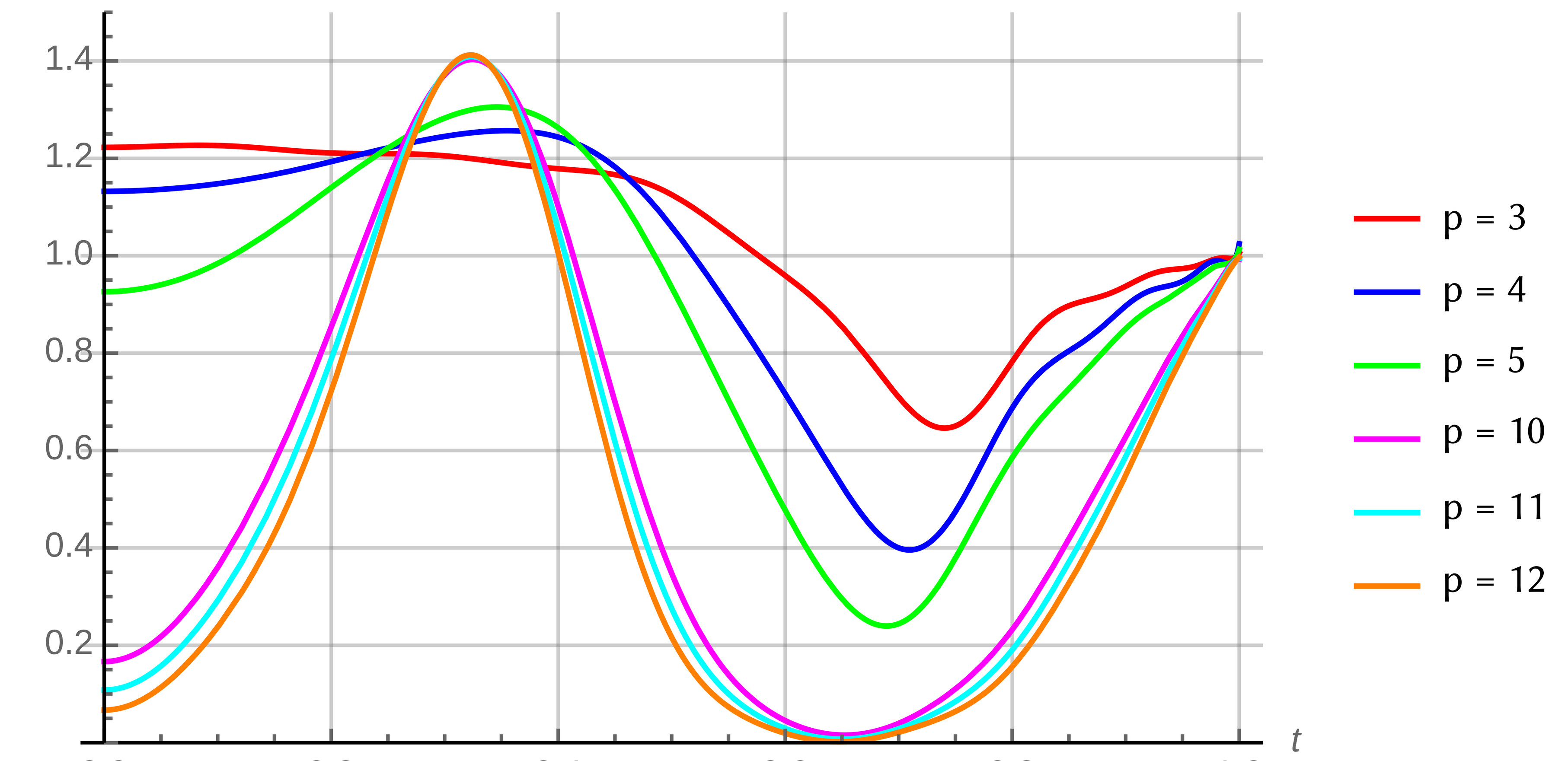
$$< \frac{\|g\sigma_p * g\sigma_p * g\sigma_p\|^2_{L^2}}{\|g\|^6_{L^2}} \approx \|w\sigma_p * w\sigma_p * w\sigma_p\|^2_{L^2}$$

For $1 < p \leq p_0 \approx 4.803$ concentrating sequences *cannot* extremize! With a different trial function we can get $p_0 \approx 5.1$

The measure $\sigma_p * \sigma_p * \sigma_p$ is even and constant along the curves $s = a|\xi|^p$



Approximation of $(w\sigma_p * w\sigma_p * w\sigma_p)(3^{1-\frac{1}{p}}t, 1)$, for $t \in [0,1]$:



References

Brocchi, Oliveira e Silva, and Quilodrán, *Sharp Strichartz estimates for fractional Schrödinger equations*, arXiv:1804.11291.

[1] Kenig, Ponce, and Vega. "Oscillatory integrals and regularity of dispersive equations." Indiana University Mathematics Journal 40.1