Vector-valued Littlewood-Paley-Stein theory for semigroups
(after Q. Xu)

<u>Thm</u> 1.2 in Xu (baby version)
Let $p(x)$ be Poisson kernel. $P_t f = f * P_t$, $P_t(x) := p(x/t)\, 1/t^d$

$$\left\| \left(\int_0^\infty \left| t \frac{\partial}{\partial t} P_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim_d \max\{p^{1/2}, p'\} \, \|f\|_{L^p(\mathbb{R}^d)}$$

<u>Thm</u> (vector-valued version) Let $1 < p, q < \infty$.
Let X Banach space with cotype q.

$$\left\| \left(\int_0^\infty \left\| t \frac{\partial}{\partial t} P_t f \right\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \lesssim_d \max\{p^{1/q}, p'\} \, C_{q,X} \, \|f\|_{L^p(\mathbb{R}^d; X)}$$

$\forall f : \mathbb{R}^d \to X$, where $\|f\|_{L^p(\mathbb{R}^d, X)} := \left(\int_{\mathbb{R}^d} \|f\|_X^p \, dx \right)^{1/p}$.

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Let $\varepsilon, \delta > 0$. $\mathcal{H}_{\varepsilon,\delta} = \{ \varphi : \mathbb{R}^d \to \mathbb{R} \mid \int \varphi = 0,$

(size) $\quad |\varphi(x)| \leq \frac{c}{(1+|x|)^{d+\varepsilon}}$

(regularity) $\quad |\varphi(x) - \varphi(y)| \leq c \frac{|x-y|^\delta}{(1+\min\{|x|,|y|\})^{d+\varepsilon+\delta}} \}$

For $\varphi \in \mathcal{H}_{\varepsilon,\delta}$

Let $G_\varphi f := \left(\int_0^\infty |f * \varphi_t|^2 \frac{dt}{t} \right)^{1/2}$.

Thm 1.5 (Xu)

For $1<p<\infty$, $\varphi \in \mathcal{H}_{\varepsilon,\delta}$

$$\| G_\varphi f \|_{L^p(\mathbb{R}^d)} \lesssim_{d,\varepsilon,\delta} \max\{\sqrt{p}, p'\} \, \|f\|_{L^p(\mathbb{R}^d)}$$

Thm 1.5 (Xu)

For $1<p<\infty$, $\varphi \in \mathcal{H}_{\varepsilon,\delta}$

$$\| G_\varphi f \|_{L^p(\mathbb{R}^d)} \lesssim \max\{\sqrt{p}, p'\} \; \|f\|_{L^p(\mathbb{R}^d)}$$

A proof from "Operator-free sparse domination", Lerner, Lorist, Ombrosi 2021.

INGREDIENTS:

1. Weak (1,1) estimate:

for general Banach space X of cotype $q \in [2,\infty)$

$$\| G_\varphi f \|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim_{\varepsilon,d,\delta} C_{q,X} \; \|f\|_{L^1(\mathbb{R}^d)}$$

2. Localisation in scale: for $h>0$

$$G_\varphi^h f := \left(\int_0^h |f * \varphi_t|^2 \frac{dt}{t} \right)^{1/2}$$

3. Sparse domination:

$$G_\varphi^{\ell(Q)} f(x) \lesssim_{\varepsilon,\delta,d} \left(\sum_{P\in\mathcal{S}} \sum_{m=1}^{\infty} 2^{-m\varepsilon} \langle |f| \rangle_{2^m P}^2 \, \mathbb{1}_P(x) \right)^{1/2}$$

2. Localisation in scale: for $h > 0$

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3. Sparse domination: Q cube in $\mathbb{R}^d$, side length $\ell(Q)$.

Indicator function on P

$$G_\varphi^{\ell(Q)} f(x) \lesssim_{\varepsilon,\delta,d} \left(\sum_{P \in \mathcal{S}} \sum_{m=1}^{\infty} 2^{-m\varepsilon} \langle |f| \rangle_{2^m P}^2 \, \mathbb{1}_P(x) \right)^{1/2}, \quad \text{a.e. } x \in Q.$$

enlarged cube

Average on $2^m P$

$\langle |f| \rangle_P := \frac{1}{|P|} \int_P |f|$

Sparse family $\mathcal{S}$

"Collection of cubes in $\mathcal{D}(Q)$ that can shrink to a <u>disjoint</u> family"

$\forall P \in \mathcal{S}$ there exists $E_P \subseteq P$ such that $|P| \leq \frac{1}{2}|E_P|$ and $\{E_P\}_{P \in \mathcal{S}}$ disjoint.

if $f : \mathbb{R}^d \to X$ Banach space valued

we consider $\langle \|f\|_X \rangle_P$ in place of $\langle |f| \rangle_P$

Theorem (7.2 in Lerner-Lorist-Ombrosi)

Fix $\varphi \in \mathcal{H}_{\varepsilon,S}$. For any $f \in L^1(\mathbb{R}^d)$ and any cube Q in $\mathbb{R}^d$ there exists a $\frac{1}{2}$-sparse family $\mathcal{S} \subseteq \mathcal{D}(Q)$ such that we have the following Sparse domination:

$$G_\varphi^{\ell(Q)} f(x) \lesssim_{\varepsilon,S,d} \left(\sum_{P \in \mathcal{S}} \sum_{m=1}^{\infty} 2^{-m\varepsilon} \langle |f| \rangle_{2^m P}^2 \mathbb{1}_P(x) \right)^{1/2}, \quad \text{a.e. } x \in Q.$$

How does this imply $\| G_\varphi f \|_{L^p}^2 \lesssim \max\{\sqrt{p}, p'\}^2 \| f \|_{L^p}^2$? $(p>2)$

$\| G_\varphi f \|_{L^p}^2 = \| (G_\varphi f)^2 \|_{L^{p/2}}$

$|P| < \frac{1}{2} |E_P| = \frac{1}{2} \int_{E_P} dx$

$$\int_Q \left(G_\varphi^{\ell(Q)} f(x) \right)^2 g \lesssim_{\varepsilon,S,d} \sum_{P \in \mathcal{S}} \sum_{m=1}^{\infty} 2^{-m\varepsilon} \langle |f| \rangle_{2^m P}^2 \fint_P g \, |P|$$

$$\lesssim \sum_{P \in \mathcal{S}} \int_{E_P} (\mathcal{M} f)^2 \mathcal{M} g$$

$\mathcal{M} f(x) = \sup_{\substack{P \in \mathcal{D} \\ P \ni x}} \langle |f| \rangle_P$

$\| \mathcal{M} f \|_{L^p} \leq p' \| f \|_{L^p}$

$$\leq \| (\mathcal{M} f)^2 \|_{L^{p/2}} \| \mathcal{M} g \|_{L^{(p/2)'}}$$

$$\leq (p')^2 \cdot \frac{p}{2} \| f \|_{L^p}^2 \| g \|_{L^{(p/2)'}}$$

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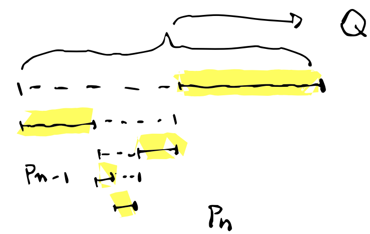
<u>Plan of the proof</u> (in L-L-O)

- Step 0: Localise f on $2Q$. Reduce to $G_\varphi^{\ell(Q)}(f \mathbb{1}_{2Q})$ using "size" of φ.
- Step 1: use "ℓ^2-sublinearity".

Let $P_n \subsetneq P_{n-1} \subsetneq \dots \subsetneq P_1 := Q$, $P_j \in \mathcal{D}(Q)$.

Let $f : \mathbb{R}^d \to \mathbb{R}$, $f_Q := f \mathbb{1}_Q$

$f_{Q \setminus P} := f(\mathbb{1}_Q - \mathbb{1}_P)$



$$\mathbb{1}_Q = \mathbb{1}_Q - \mathbb{1}_{P_1} + (\mathbb{1}_{P_1} - \mathbb{1}_{P_2}) + \dots$$

<u>Def</u> (ℓ^r-sublinearity)

There exists $C_r > 0$ such that

$$|f_Q(x)|^r = \left| \sum_{m=1}^{n-1} f_{P_m \setminus P_{m+1}}(x) + f_{P_n}(x) \right|^r \le (C_r)^r \sum_{m=1}^{n-1} |f_{P_m \setminus P_{m+1}}|^r + |f_{P_n}|^r$$

<u>Remark</u>: C_r independent of n.

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Fix $\varphi \in \mathcal{H}_{\varepsilon,S}$. For any $f \in L^1(\mathbb{R}^d)$ and any cube Q in $\mathbb{R}^d$ there exists a $\frac{1}{2}$-sparse family $\mathcal{S} \subseteq \mathcal{D}(Q)$ such that we have the following

Sparse domination:

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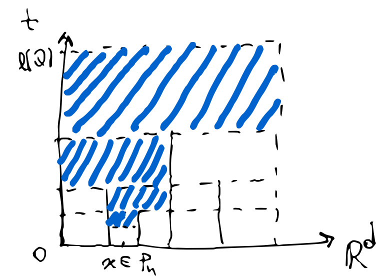
  For $x \in P_n$,

$$\left(G_\varphi^{\ell(Q)} f(x) \right)^2 = \int_0^{\ell(Q)} |f * \varphi_t|^2 \frac{dt}{t}$$

$$= \sum_{j=1}^{n-1} \int_{\ell(P_{j+1})}^{\ell(P_j)} |f * \varphi_t|^2 \frac{dt}{t} + \int_0^{\ell(P_n)} |f * \varphi_t|^2 \frac{dt}{t}$$



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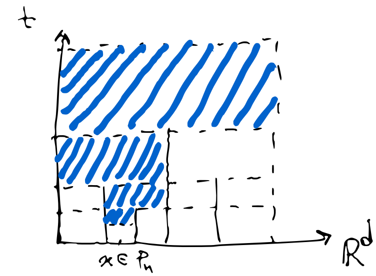

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- Step 2 : use sparse domination principle

Thm (3.2 in Lerner-Lorist-Ombrosi)

Let $\{G_Q, G_{P_j \setminus P_{j+1}}\}_{P_j \in \mathcal{D}(Q)}$ a sequence satisfying "ℓ^2-sublinearity".

Then for any Q cube in $\mathbb{R}^d$ there exists a $\frac{1}{2}$-sparse family $\mathcal{S} \subseteq \mathcal{D}(Q)$:

$$G_Q(x)^2 \lesssim \sum_{P \in \mathcal{S}} \gamma_P^2 \mathbb{1}_P(x) \quad , \quad \text{for a.e. } x \in Q$$

$$\gamma_P := \left[(G_P \mathbb{1}_P)^* + (m_P^\# G)^* \right] \left(\frac{|P|}{2^{d+3}} \right), \text{ and } \quad m_P^\# G(x) := \sup_{\substack{P \in \mathcal{D} \\ P \ni x}} \left(\fint\!\!\fint_{P \times P} |G(z) - G(y)|^2 \, dz\, dy \right)^{1/2}$$

where $\quad F^*(t) := \inf \{ \lambda > 0 : |\{x : |F(x)| > \lambda\}| \le t \}$

Plan of the proof (in L-L-O)

. Step 2 : use sparse domination principle

<u>Thm</u> (3.2 in Lerner-Lorist-Ombrosi)

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Then ETS that $\quad \gamma_P^2 \lesssim \sum_{m=1}^{\infty} 2^{-m\varepsilon} \langle |f| \rangle_{2^m P}^2 \,.$

<u>Example</u>

$$G_P := G_\varphi^{\ell(P)} f \leq G_\varphi^{\ell(P)}(f \mathbb{1}_{2P}) + G_\varphi^{\ell(P)}(f \mathbb{1}_{(2P)^c})$$

use decay of φ here

<u>Fact</u> : If $\quad T : L^1 \to L^{1,\infty}$ then

$$\left(T(f \mathbb{1}_{2P}) \mathbb{1}_P \right)^* (t) \lesssim \frac{1}{t} \| f \mathbb{1}_{2P} \|_{L^1}$$

if $t = \frac{|P|}{2^{d+3}}$

we get $\lesssim_d \langle |f| \rangle_{2P}$

Plan of the proof (in L-L-O)

Let $\quad F^*(t) := \inf \{ \lambda > 0 : |\{x : |F(x)| > \lambda\}| \leq t \}$.

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Fact : If $T : L^1 \to L^{1,\infty}$ then

$$\left(T(f \mathbb{1}_{2P}) \mathbb{1}_P \right)^*(t) \lesssim \frac{1}{t} \| f \mathbb{1}_{2P} \|_{L^1}$$

by weak(1,1) : $\forall \lambda > 0$

$$\frac{C}{\lambda} \|f\|_{L^1} \geq |\{x : |Tf(x)| > \lambda\}|$$

$$\inf \{ \lambda > 0 : \frac{C}{\lambda} \|f\|_{L^1} < t \} \geq \inf \{ \lambda > 0 : |\{x : |Tf(x)| > \lambda\}| < t \}$$

Plan of the proof (in L-L-O)

. Step 1 : use "ℓ^2-sublinearity".

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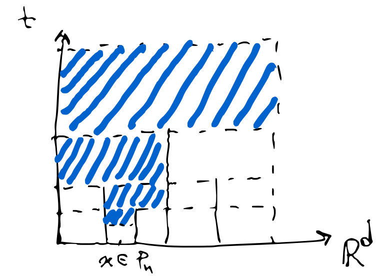

$$\underbrace{\left(G_\varphi^{\ell(Q)} f(x)\right)^2}_{=: G_Q} = \sum_{j=1}^{n-1} \underbrace{\int_{\ell(P_{j+1})}^{\ell(P_j)} |f * \varphi_t|^2 \frac{dt}{t}}_{=: G_{P_j \setminus P_{j+1}}} + \underbrace{\int_0^{\ell(P_n)} |f * \varphi_t|^2 \frac{dt}{t}}_{= G_{P_n \setminus \emptyset}}$$

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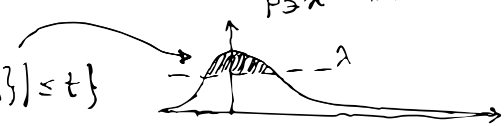
where $\quad F^*(t) := \inf \{ \lambda > 0 : |\{x : |F(x)| > \lambda\}| \le t \}$



Plan of the proof (in L-L-O)

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For $x \in P_n$,

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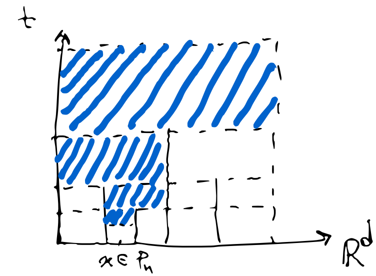
out in

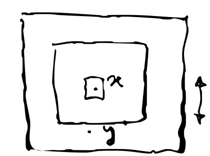


$$f = f\,\mathbb{1}_{2P} + f\,\mathbb{1}_{(2P)^c}$$

. Step 2 : use sparse domination principle OR estimate

For $x \in P$,

$$\left|f_{(2P)^c} * \varphi_t(x)\right| \le \int_{\mathbb{R}^d \setminus 2P} |f(y)\,\varphi_t(x-y)|\,dy = \sum_{m=1}^{\infty} \int_{2^{m+1}P \setminus 2^m P} |f(y)\,\varphi_t(x-y)|\,dy$$



$$\left[\, |\varphi_t(x-y)| \le \frac{t^\varepsilon}{(t+|x-y|)^{d+\varepsilon}} \le \frac{t^\varepsilon}{|x-y|^{d+\varepsilon}} \,\right]$$

(decay in $|x-y|$)

$$\lesssim \sum_{m=2}^{\infty} \int_{2^m P \setminus 2^{m-1}P} |f(y)| \frac{dy}{|x-y|^{d+\varepsilon}}\, t^\varepsilon$$

$y \in 2^{m+1}P \setminus 2^m P$

$|x-y| \ge \frac{1}{4}\ell(2^m P)$

$$\int_0^{\ell(P)} \left|f_{(2P)^c} * \varphi_t(x)\right|^2 \frac{dt}{t} \lesssim \left(\sum_{m=2}^{\infty} 2^{-m\varepsilon} \frac{\cancel{\ell(P)^{-\varepsilon}}}{|2^m P|} \int_{2^m P} |f(y)|\,dy \right)^2 \underbrace{\int_0^{\ell(P)} \frac{dt}{t^{1-2\varepsilon}}}_{\approx\ \cancel{\ell(P)^{2\varepsilon}}}$$

Plan of the proof (in L-L-O)

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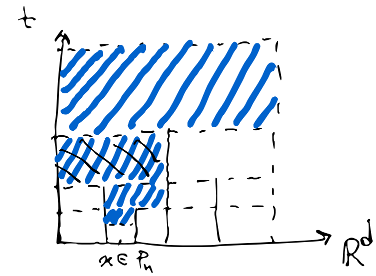
out "loc" in out in



For $R, P \in \mathcal{D}$, $R \subset P$

$$f = f_{(2P)^c} + f_{2P \setminus 2R} + f_{2R}$$

loc : $\displaystyle\int_{\ell(R)}^{\ell(P)} \left| f_{2P\setminus 2R} * \varphi_t(x) - f_{2P\setminus 2R} * \varphi_t(y)\right|^2 \frac{dt}{t}$ (exploit regularity of φ)

in : $\displaystyle\int_{\ell(R)}^{\ell(P)} |f_{2R} * \varphi_t(x)|^2 \frac{dt}{t} \lesssim \int_{\ell(R)}^{\ell(P)} \left(\int_{2R} \frac{|f(y)|\, t^\varepsilon}{(t+|x-y|)^{d+\varepsilon}} dy \right)^2 \frac{dt}{t} \leq \langle |f| \rangle_{2R}^2 \int_{\ell(R)}^{\ell(P)} \frac{dt}{t^{1-2\varepsilon}}$

$$\boxed{|\varphi_t(x-y)| \leq \frac{t^\varepsilon}{(t+|x-y|)^{d+\varepsilon}} \leq \frac{t^\varepsilon}{|x-y|^{d+\varepsilon}}}$$

$t > \ell(R)$

$\frac{1}{t^d} \lesssim \frac{1}{|2R|}$